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COMPACT PERTURBATIONS OF DRAZIN INVERTIBLE OPERATORS

BOTING JIA* AND YOULING FENG

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Abstract. Necessary and sufficient conditions for the invariance of Drazin invertibility under compact perturbations are established. This proves a conjecture of Kaifan Yang and Hongke Du.

1. Introduction

Throughout this note, $\mathscr{H}$ is a complex separable infinite dimensional Hilbert space. We denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$, and by $\mathscr{K}(\mathscr{H})$ the ideal of all compact operators acting on $\mathscr{H}$. Recall that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *Drazin invertible* [6] if there exist a nonnegative integer n and $S \in \mathscr{B}(\mathscr{H})$ such that

$$TS = ST, \quad TS^2 = S, \quad T^{n+1}S = T^n. \tag{1}$$

In this case, the smallest nonnegative integer n such that the equalities (1) hold is called the *Drazin index* of T. Note that invertible operators are always Drazin invertible. It is well known that T is Drazin invertible if and only if T has finite ascent and descent (see [12, Chapter 3, Theorem 10] or [1]), that is, there is a nonnegative integer n such that

$$\mathscr{N}(T^n) = \mathscr{N}(T^{n+1}), \quad \mathscr{R}(T^n) = \mathscr{R}(T^{n+1}).$$

Here and in what follows, $\mathscr{N}(T)$ denotes the kernel of T and $\mathscr{R}(T)$ denotes the range of T.

In 1996, Koliha [11] introduced a generalization of Drazin invertibility. An operator T on $\mathscr{H}$ is said to be *generalized Drazin invertible* if there exists $R \in \mathscr{B}(\mathscr{H})$ such that $TR = RT$, $TR^2 = R$ and $T^2R - T$ is quasinilpotent. Recall that an operator $X \in \mathscr{B}(\mathscr{H})$ is called *quasinilpotent* if $\sigma(X) = \{0\}$.

In general the Drazin inverse is unstable under small perturbations. Many of previous papers are devoted to the perturbation theory of the Drazin inverse (see [2, 3, 5, 14, 15]). We remark that Drazin invertibility is also unstable under perturbations (see [16] for counterexamples). In their paper [16], K. Yang and H. Du investigated the stability of Drazin invertibility under small perturbations and finite-rank perturbations. A necessary and sufficient condition is given for the invariance of Drazin invertibility under small perturbations (see [16, Theorem 1.2]). In addition, K. Yang and H. Du raised the following conjecture.

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* Corresponding author.

CONJECTURE 1.1. ([16], Conjecture 3.5) Let $A \in \mathscr{B}(\mathscr{H})$ be Drazin invertible with Drazin index k. If 0 is in the unbounded component of $\rho(A|_{\mathscr{R}(A^k)})$ and $\dim \mathscr{R}(A^k)^{\perp} < \infty$, then $A+K$ is Drazin invertible for any compact operator $K \in \mathscr{B}(\mathscr{H})$.

In this note it is completely determined when an operator T satisfies that all compact perturbations of T are Drazin invertible. In particular, we give a positive answer to Conjecture 1.1.

Before we state the main result, we first introduce some terminology.

Let $T \in \mathscr{B}(\mathscr{H})$. We denote by $\sigma(T)$ and $\rho(T)$ the spectrum of T and the resolvent set of T respectively. T is called a *semi-Fredholm* operator, if $\mathscr{R}(T)$ is closed and either $\dim \mathscr{N}(T)$ or $\dim \mathscr{N}(T^*)$ is finite. In this case, $\operatorname{ind} T = \dim \mathscr{N}(T) - \dim \mathscr{N}(T^*)$ is called the *index* of T. In particular, if $-\infty < \operatorname{ind} T < \infty$, then T is called a *Fredholm operator*. The *Wolf spectrum* $\sigma_{lre}(T)$ is defined by

$$\sigma_{lre}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}.$$

The set $\rho_{s-F}(T) = \mathbb{C} \setminus \sigma_{lre}(T)$ is called the *semi-Fredholm domain* of T. Note that $\rho_{s-F}(T)$ is an open subset of $\mathbb{C}$, $\rho(T) \subset \rho_{s-F}(T)$ and $\rho_{s-F}(T+K) = \rho_{s-F}(T)$ for all $K \in \mathscr{K}(\mathscr{H})$.

For convenience we will use the term "D-invertible" for "Drazin invertible", and use the term "d-invertible" for "generalized Drazin invertible".

The main result of this note is the following theorem.

THEOREM 1.2. *For $T \in \mathscr{B}(\mathscr{H})$, the following statements are equivalent.*

(i) $T+K$ is D-invertible for all $K \in \mathscr{K}(\mathscr{H})$.

(ii) $T+K$ is d-invertible for all $K \in \mathscr{K}(\mathscr{H})$.

(iii) 0 lies in the unbounded component of $\rho_{s-F}(T)$.

We shall prove in Section 2 the following lemma, which shows that the result stated in Conjecture 1.1 is a consequence of Theorem 1.2.

LEMMA 1.3. *Let $T \in \mathscr{B}(\mathscr{H})$ be Drazin invertible with Drazin index n. Then the following two statements are equivalent.*

(i) 0 lies in the unbounded component of $\rho(T|_{\mathscr{R}(T^n)})$ and $\dim \mathscr{R}(T^n)^{\perp} < \infty$.

(ii) 0 lies in the unbounded component of $\rho_{s-F}(T)$.

2. Proof of theorem 1.2

We first gather some known results we will rely on.

Given a subset Δ of $\mathbb{C}$, we denote by $\operatorname{iso} \Delta$ the set of all isolated points of Δ, and by $\operatorname{inte} \Delta$ the interior of Δ.

Let $T \in \mathscr{B}(\mathscr{H})$. For $-\infty \leqslant n \leqslant \infty$, denote

$$\rho_{s-F}^{(n)}(T) = \{\lambda \in \rho_{s-F}(T) : \text{ind}\,(\lambda - T) = n\}.$$

By the continuity of index function, $\{\rho_{s-F}^{(n)}(T) : -\infty \leqslant n \leqslant \infty\}$ are pairwise disjoint open subsets of $\mathbb{C}$.

If Δ is a nonempty clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\Delta \subset \Omega$ and $[\sigma(T) \setminus \Delta] \cap \overline{\Omega} = \emptyset$. We let $E(\Delta;T)$ denote the *Riesz idempotent* of T corresponding to Δ, that is,

$$E(\Delta;T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\lambda - T)^{-1} \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. By the Riesz Decomposition Theorem ([13, Theorem 2.10]), $\mathscr{H}(\Delta;T) := \mathscr{R}(E(\Delta;T))$ is an invariant subspace of T and $\sigma(T|_{\mathscr{H}(\Delta;T)}) = \Delta$. If $\lambda \in \text{iso}\,\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$; if, in addition, $\dim \mathscr{H}(\{\lambda\};T) < \infty$, then λ is called a *normal eigenvalue* of T. We denote by $\sigma_0(T)$ the set of all normal eigenvalues of T. The reader is referred to [4, Chapter XI] or [8, Chapter 1] for more details.

LEMMA 2.1. ([4], page 366) *Let $T \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \text{iso}\,\sigma(T)$. Then the following statements are equivalent.*

(i) $\lambda \in \sigma_0(T)$.

(ii) $\lambda \in \rho_{s-F}^{(0)}(T)$.

(iii) $\lambda \in \rho_{s-F}(T)$.

Let $T \in \mathscr{B}(\mathscr{H})$. For $\lambda \in \rho_{s-F}(T)$, the *minimal index* of $\lambda - T$ is defined by

$$\min\text{ind}\,(\lambda - T) = \min\{\dim \mathscr{N}(\lambda - T), \dim \mathscr{N}(\lambda - T)^*\}.$$

LEMMA 2.2. ([8], Corollary 1.14) *Let $T \in \mathscr{B}(\mathscr{H})$. Then*

(i) *$\rho_{s-F}^{(0)}(T)$ includes the resolvent set $\rho(T)$ of T, and $\sigma_0(T)$;*

(ii) *if $n \neq 0$, then $\rho_{s-F}^{(n)}(T)$ is a bounded set;*

(iii) *the function $\lambda \mapsto \min\text{ind}\,(\lambda - T)$ is constant on every component of $\rho_{s-F}(T)$ except for an at most denumerable subset $\rho_{s-F}^{s}(T)$ of $\rho_{s-F}(T)$ without limit points in $\rho_{s-F}(T)$.*

COROLLARY 2.3. *Let $T \in \mathscr{B}(\mathscr{H})$. If λ lies in the unbounded component of $\rho_{s-F}(T)$, then either $\lambda \in \rho(T)$ or $\lambda \in \sigma_0(T)$.*

Proof. Denote by Ω the unbounded component of $\rho_{s-F}(T)$ containing λ. In view of Lemma 2.2 (iii), there exist a nonnegative integer n and an at most denumerable subset Γ of Ω (which has no limit points in Ω) such that $\min\operatorname{ind}(z-T) = n$ for $z \in \Omega \setminus \Gamma$.

Claim. $n = 0$ and $\Omega \subset \rho_{s-F}^{(0)}(T)$.

Denote by G the unbounded component of $\rho(T)$. It is easy to see that $G \subset \Omega$. Since $\operatorname{ind}(z-T) = 0$ for $z \in G$ and $\operatorname{ind}(\cdot)$ is constant on Ω, we obtain $\Omega \subset \rho_{s-F}^{(0)}(T)$. On the other hand, note that $\min\operatorname{ind}(z-T) = n$ for $z \in G \setminus \Gamma$. Since $G \setminus \Gamma \neq \emptyset$ and $\min\operatorname{ind}(z-T) = 0$ for $z \in G$, we deduce that $n = 0$. This proves the claim.

Now assume that $\lambda \in \sigma(T)$. It suffices to prove that $\lambda \in \sigma_0(T)$. In view of Claim, we have $[\Omega \setminus \Gamma] \subset \rho(T)$. It follows that $\lambda \in \Gamma$. Since Γ has no limits in Ω, we deduce that $\lambda \in \operatorname{iso}\sigma(T)$. By Lemma 2.1, it follows that $\lambda \in \sigma_0(T)$. $\square$

COROLLARY 2.4. *Let $T \in \mathscr{B}(\mathscr{H})$. If Ω is the unbounded component of $\rho_{s-F}(T)$ and G is the unbounded component of $\rho(T)$, then $\Omega \setminus \sigma_0(T) = G$.*

Proof. By definitions, the inclusion "$\supset$" is clear.

"$\subset$". Note that Ω is an unbounded connected open subset of $\mathbb{C}$ and $\sigma_0(T)$ consists of some isolated points of $\sigma(T)$. It follows that $\Omega \setminus \sigma_0(T)$ is still unbounded, connected and open. From Corollary 2.3, one can see the inclusion $[\Omega \setminus \sigma_0(T)] \subset \rho(T)$. Therefore $[\Omega \setminus \sigma_0(T)] \subset G$. Thus we are done. $\square$

LEMMA 2.5. ([10], Lemma 3.2.6) *Let $T \in \mathscr{B}(\mathscr{H})$ and suppose that $\emptyset \neq \Delta \subset \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that*

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix},$$

where $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, N is a diagonal normal operator of uniformly infinite multiplicity, $\sigma(N) = \sigma_{lre}(N) = \overline{\Delta}$ and $\sigma(T) = \sigma(A)$.

The point spectrum of an operator T is denoted by $\sigma_p(T)$.

LEMMA 2.6. ([9], Lemma 5.3) *Let $T \in \mathscr{B}(\mathscr{H})$ and let Φ be a union of bounded components of $\rho_{s-F}(T)]$ such that $\Phi \cap \operatorname{inte}\sigma_p(T) = \emptyset$. Then there exists $K \in \mathscr{K}(\mathscr{H})$ such that $\Phi \subset \sigma(T+K)$.*

LEMMA 2.7. ([11]) *Let $A \in \mathscr{B}(\mathscr{H})$. Then A is d-invertible if and only if 0 is not an accumulation point of $\sigma(A)$.*

LEMMA 2.8. ([7], Theorem 2.5) *An operator $T \in \mathscr{B}(\mathscr{H})$ is D-invertible with Drazin index n if and only if $\mathscr{R}(T^n)$ is closed and*

$$T = \begin{bmatrix} A & E \\ 0 & B \end{bmatrix}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(T^n) \oplus \mathscr{R}(T^n)^\perp$, where A is invertible and B is nilpotent of order n.

By Lemma 2.8, if T is D-invertible, then either $0 \in \rho(T)$ or $0 \in \operatorname{iso} \sigma(T)$. Hence D-invertible operators are always d-invertible.

LEMMA 2.9. *Let $T \in \mathscr{B}(\mathscr{H})$. If $0 \in \sigma_0(T)$, then T is D-invertible.*

Proof. Denote $\Delta = \sigma(T) \setminus \{0\}$. We claim that $\Delta \neq \emptyset$. In fact, if not, then $\sigma(T) = \{0\}$ and, by definition, $\mathscr{H} = \mathscr{H}(\{0\};T) < \infty$, a contradiction. Then both Δ and $\{0\}$ are nonempty clopen subsets of $\sigma(T)$.

Denote $\mathscr{H}_0 = \mathscr{H}(\{0\};T)$ and $\mathscr{H}_1 = \mathscr{H}(\Delta;T)$. Then, by the Riesz Decomposition Theorem (see [13, Theorem 2.10]), $\mathscr{H}_0, \mathscr{H}_1$ are complementary invariant subspaces of T and

$$\sigma(T|_{\mathscr{H}_0}) = \{0\}, \quad \sigma(T|_{\mathscr{H}_1}) = \Delta.$$

Since $\dim \mathscr{H}_0 < \infty$, it is easy to see that $T|_{\mathscr{H}_0}$ is nilpotent (say, of order n). Note that $T|_{\mathscr{H}_1}$ is invertible. Then

$$\mathscr{N}(T^n) = \mathscr{N}(T^{n+1}) = \mathscr{H}_0, \quad \mathscr{R}(T^n) = \mathscr{R}(T^{n+1}) = \mathscr{H}_1.$$

Therefore T is D-invertible. $\square$

Now we are going to give the proof of theorem 1.2.

Proof of theorem 1.2. "(iii)$\Longrightarrow$(i)". Assume that 0 lies in the unbounded component of $\rho_{s-F}(T)$. Let K be a compact operator on $\mathscr{H}$. Since semi-Fredholm domains are invariant under compact perturbations, we deduce that 0 still lies in the unbounded component of $\rho_{s-F}(T+K)$. By Corollary 2.3, we have either $0 \in \rho(T+K)$ or $0 \in \sigma_0(T+K)$. Then, by Lemma 2.9, $T+K$ is D-invertible.

"(i)$\Longrightarrow$(ii)". This is obvious, since D-invertible operators are always d-invertible.

"(ii)$\Longrightarrow$(iii)". Assume that $T+K$ is d-invertible for all $K \in \mathscr{K}(\mathscr{H})$. For a proof by contradiction, we assume that 0 does not lie in the unbounded component of $\sigma_{lre}(T)$. Then either $0 \in \sigma_{lre}(T)$ or 0 lies in a bounded component of $\rho_{s-F}(T)$. So the rest of the proof is divided into two cases.

Case 1. $0 \in \sigma_{lre}(T)$.

By Lemma 2.5, there exists $K_1 \in \mathscr{K}(\mathscr{H})$ such that

$$T + K_1 = \begin{bmatrix} 0 & E \\ 0 & A \end{bmatrix}$$

relative to some space decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, where $\dim \mathscr{H}_1 = \infty$. Then we can choose an orthonormal basis $\{e_i : i = 1,2,3,\cdots\}$ of $\mathscr{H}_1$. For $x \in \mathscr{H}$, define

$$K_2 x = \sum_{n=1}^{\infty} \frac{\langle x, e_n \rangle e_n}{n}.$$

Then $K_2 \in \mathscr{K}(\mathscr{H})$. Set $C = K_1 + K_2$. It is easy to see that $C \in \mathscr{K}(\mathscr{H})$ and

$$(T+C)e_n = K_2 e_n = e_n/n, \quad \forall n \geqslant 1.$$

Thus $\{1/n : n \geqslant 1\} \subset \sigma_p(T+C)$. Hence 0 is an accumulation point of $\sigma(T+C)$. By Lemma 2.7, $T+C$ is not d-invertible, a contradiction.

Case 2. 0 lies in a bounded component Ω of $\rho_{s-F}(T)$.

In view of Lemma 2.2 (iii), there exist a nonnegative integer n and an at most denumerable subset Γ of Ω (which has no limit points in Ω) such that $\min\mathrm{ind}\,(z - T) = n$ for $z \in \Omega \setminus \Gamma$.

Note that T is d-invertible. Then, by Lemma 2.7, we have either $0 \in \rho(T)$ or $0 \in \mathrm{iso}\,\sigma(T)$. Then there exists $\delta > 0$ such that $[B(0,\delta) \setminus \{0\}] \subset \rho(T)$. It follows that $[B(0,\delta) \setminus \{0\}] \subset \Omega$ and $\min\mathrm{ind}\,(T-z) = \mathrm{ind}\,(T-z) = 0$ for $z \in [B(0,\delta) \setminus \{0\}]$. Thus $\Omega \subset \rho_{s-F}^{(0)}(T)$ and, by Lemma 2.2 (iii), $n = 0$. Thus we have proved that $\mathrm{ind}\,(T-z) = 0$ for $z \in \Omega$ and $\min\mathrm{ind}\,(T-z) = 0$ for $z \in \Omega \setminus \Gamma$. It follows that $[\Omega \setminus \Gamma] \subset \rho(T)$. So $\Omega \cap \mathrm{inte}\,\sigma_p(T) = \emptyset$. By Lemma 2.6, there exists $K \in \mathscr{K}(\mathscr{H})$ such that $\Omega \subset \sigma(T + K)$. Thus 0 is an accumulation point of $\sigma(T+K)$. By Lemma 2.7, $T+K$ is not d-invertible, a contradiction. Thus the proof is complete. $\square$

Proof of lemma 1.3. Since T is D-invertible, by Lemma 2.8, $\mathscr{R}(T^n)$ is closed and T can be represented as

$$T = \begin{bmatrix} A & E \\ 0 & B \end{bmatrix}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(T^n) \oplus \mathscr{R}(T^n)^{\perp}$, where A is invertible and B is nilpotent of order n. Thus $A = T|_{\mathscr{R}(T^n)}$.

"(i)$\Longrightarrow$(ii)". Since 0 lies in the unbounded component of $\rho(A)$, it follows that 0 lies in the unbounded component of $\rho_{s-F}(A)$. On the other hand, noting that $\dim \mathscr{R}(T^n)^{\perp} < \infty$, we deduce that B, E are both compact and $\rho_{s-F}(A) = \rho_{s-F}(T)$. This proves "(i)$\Longrightarrow$(ii)".

"(ii)$\Longrightarrow$(i)". Note that $\sigma(B) = \{0\}$ and $0 \notin \sigma(A)$. By [8, Corollary 3.22], T is similar to $A \oplus B$, that is, $TX = X(A \oplus B)$ for some invertible $X \in \mathscr{B}(\mathscr{H})$.

Claim. $\dim \mathscr{R}(T^n)^{\perp} < \infty$.

In fact, if $\dim \mathscr{R}(T^n)^{\perp} = \infty$, then $\dim \mathscr{N}(B) = \infty$ (since B is nilpotent). Thus $\dim \mathscr{N}(T) = \dim \mathscr{N}(A \oplus B) = \infty$. Also, B^* is nilpotent acting on $\mathscr{R}(T^n)^{\perp}$. Thus we also have $\dim \mathscr{N}(T^*) = \dim \mathscr{N}(A^* \oplus B^*) = \infty$. So T is not semi-Fredholm, contradicting the hypothesis that $0 \in \rho_{s-F}(T)$. This proves the claim.

By Claim, we have $\rho_{s-F}(A) = \rho_{s-F}(T)$. It follows that 0 lies in the unbounded component Ω of $\rho_{s-F}(A)$. Since $0 \in \rho(A)$, we have $0 \in [\Omega \setminus \sigma_0(A)]$. In view of Corollary 2.4, 0 lies in the unbounded component of $\rho(A)$. $\square$

We conclude this note with an example.

EXAMPLE 2.10. Assume that $\{e_i\}_{i=-\infty}^{\infty}$ is an orthonormal basis of $\mathscr{H}$ and define $T \in \mathscr{B}(\mathscr{H})$ as

$$Te_i = e_{i+1}, \quad \forall i \in \mathbb{Z}.$$

Here $\mathbb{Z}$ denotes the set of integers. Then T is a bilateral shift and

$$\sigma(T) = \sigma_{lre}(T) = \{z \in \mathbb{C} : |z| = 1\}.$$

So 0 lies in the unique bounded component of $\rho_{s-F}(T)$. By Theorem 1.2, there exists $K_0 \in \mathscr{K}(\mathscr{H})$ such that $T + K_0$ is not d-invertible.

Set $R = 2I + T$. Then

$$\sigma(R) = \sigma_{lre}(R) = \{z \in \mathbb{C} : |z-2| = 1\}.$$

So 0 lies in the unbounded component of $\rho_{s-F}(R)$. By Theorem 1.2, $R+K$ is D-invertible for all $K \in \mathscr{K}(\mathscr{H})$.

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Boting Jia
School of Statistics
Jilin University of Finance and Economics
Changchun 130117, P. R. China
e-mail: botingjia@163.com

Youling Feng
School of Management Science and Information Engineering
Jilin University of Finance and Economics
Changchun 130117, P.R. China
e-mail: fengyouling79@163.com