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SIMILARITY JORDAN MULTIPLICATIVE MAPS

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Abstract. We characterize bijections $\phi : B(X) \to B(X)$ satisfying that $\phi(AB+BA)$ and $\phi(A)\phi(B)+\phi(B)\phi(A)$ are similar for all $A,B \in B(X)$.

1. Introduction

Let X be a complex Banach space. By $B(X)$ and X^* we denote the algebra of all bounded linear operators on X and the topological dual of X, respectively. For $A \in B(X)$, A^* is its adjoint. Two operators A,B in $B(X)$ are called similar, denoted by $A \sim B$, if there exists an invertible operator S in $B(X)$ such that $A = SBS^{-1}$.

Our main result reads as follows. Recall that a map $T : X \to X$ is called semilinear if it is additive and there is an automorphism $h : \mathbb{C} \to \mathbb{C}$ such that $T(\lambda x) = h(\lambda)x$ for all $x \in X$ and $\lambda \in \mathbb{C}$. Given two operators A,B, their Jordan product is defined by $A \circ B = AB + BA$.

THEOREM 1.1. *Let X be a complex Banach space of dimension $\geqslant 3$ and $\phi : B(X) \to B(X)$ a bijective map satisfying*

$$\phi(A \circ B) \sim \phi(A) \circ \phi(B) \tag{1.1}$$

for all $A,B \in B(X)$. Then one of the following holds.

(1) There is a semilinear bijection $T : X \to X$ such that

$$\phi(A) = TAT^{-1}, \ \ A \in B(X).$$

Moreover, if X is infinite-dimensional, then T is bounded and linear or conjugate-linear.

(2) The space X is reflexive and there is a semilinear bijection $T : X^* \to X$ such that

$$\phi(A) = TA^*T^{-1}, \ \ A \in B(X).$$

Moreover, if X is infinite-dimensional, then T is bounded and linear or conjugate-linear.

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There are two distinct motivations. First is the works on Jordan multiplicative map. A map $\phi : B(X) \to B(X)$ is called Jordan multiplicative if $\phi(A \circ B) = \phi(A) \circ \phi(B)$ for all $A, B \in B(X)$. In [14], the second author showed that a bijective Jordan mutiplicative map of $B(X)$ is additive. Various generalizations are available. For example, papers [11, 24, 25] weakened the bijectivity assumption; papers [1, 2, 9, 10, 16] altered the underlying algebra. In the present paper, we weaken the equality into the "approximate" equality.

The second motivation for our study is the works on simility-preserving maps. A map $\phi : B(X) \to B(X)$ is said to be similarity-preserving if $\phi(A) \sim \phi(B)$ whenever $A \sim B$. Hiai [6] and Lim [13] characterized similarity-preserving linear map on the matrix algebra. Various generalizations are available. For example, papers [8, 15, 23, 17] studied infinite-dimensional space case; papers [4, 7] weakened the linearity; papers [18, 19, 20] considered other type of similarity. In the present paper, we consider non-linear similarity-preserving maps concerning the Jordan product.

2. Proofs

This section is due to proving Theorem 1.1. Throughout this section, X is a complex Banach space with dimension at least 3, ϕ is a surjection of $B(X)$ satisfying Eq.(1.1). An operator A is called nilpotent if there is a positive integer $n \in \mathbb{N}$ such that $A^n = 0$. By $\mathscr{N}(X)$ we denote the set of all nilpotent operators in $B(X)$. For non-zero vectors $x \in X$ and $f \in X^*$, the rank-one operator $x \otimes f$ is defined as the map: $y \mapsto f(y)x$, $y \in X$. Then the symbol $\mathscr{N}_1(X)$ stands for the set of all rank-one operators in $\mathscr{N}(X)$.

We begin with an easy an useful observation.

LEMMA 2.1. *Let A and $x \otimes f$ be in $B(X)$. Then the following are equivalent:*

(1) $A \circ x \otimes f \in \mathscr{N}(X)$.

(2) $f(Ax) = 0$ *and* $f(x)f(A^2x) = 0$.

(3) $(A \circ x \otimes f)^3 = 0$.

Proof. That (2) $\Rightarrow$ (3) is an easy computation and that (3) $\Rightarrow$ (1) is obvious. To show that (1) $\Rightarrow$ (2), we suppose that $A \circ x \otimes f \in \mathscr{N}(X)$. Then its trace is zero and therefore $f(Ax) = 0$. Thus $(A \circ x \otimes f)^2 = f(x)Ax \otimes fA + f(A^2x)x \otimes f$ and hence $f(x)f(A^2x) = 0$ since it is nilpotent. $\square$

LEMMA 2.2. *Let $A \in B(X)$. Then $A \in \mathbb{C}I$ if and only if $A \circ N \in \mathscr{N}(X)$ for all $N \in \mathscr{N}(X)$.*

Proof. The necessity is obvious. To verify the sufficiency, let $x \otimes f \in \mathscr{N}_1(X)$. Then $A \circ x \otimes f \in \mathscr{N}(X)$ and hence $f(Ax) = 0$ for all $x \otimes f \in \mathscr{N}_1(X)$ by Lemma 2.1. This implies $A \in \mathbb{C}I$. $\square$

LEMMA 2.3. *Suppose that X has dimension at least five. Let $A \in B(X)$ be such that $A^2 = 0$. Then the following are equivalent:*

(1) rank$(A) \geqslant 2$.

(2) There exists an operator $S \in B(X)$ such that $A \circ S \in \mathcal{N}(X)$ but $(A \circ S)^4 \neq 0$.

Proof. (2) $\Rightarrow$ (1). Suppose on the contrary that $\text{rank}(A) \leqslant 1$. Then for any $S \in B(X)$, if $A \circ S$ is nilpotent, then $(A \circ S)^3 = 0$ by Lemma 2.1, a contradiction.

(1) $\Rightarrow$ (2). We distinguish two cases.

Case 1: $\text{rank}(A) = 2$. Write $A = x_1 \otimes f_3 + x_2 \otimes f_4$, where $x_1, x_2 \in X$ are linearly independent, $f_3, f_4 \in X^*$ are linearly independent, $f_j(x_i) = 0$ for all $i = 1, 2$ and $j = 3, 4$. Since X is has dimension at least 5, we can find $x_3, x_4, x_5 \in X$ and f_1, f_2, f_5 such that $f_i(x_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant 5$. Now set $S = x_3 \otimes f_5 + x_4 \otimes f_1 + x_5 \otimes f_2$. Then $(A \circ S)^4 = x_2 \otimes f_3 \neq 0$ and $(A \circ S)^5 = 0$.

Case 2: $\text{rank}(A) \geqslant 3$. Then there exist vectors x_1, x_2, x_3 in X such that Ax_1, Ax_2, Ax_3 are linearly independent. Hence since $A^2 = 0$, the vectors $x_1, x_2, x_3, Ax_1, Ax_2, Ax_3$ are linearly independent. Take $f_1, f_2 \in X^*$ such that

$$\begin{aligned} &f_1(Ax_2) = 1, && f_1(x_1) = f_1(x_2) = f_1(Ax_1) = 0; \\ &f_2(Ax_3) = f_2(x_1) = 1, && f_2(x_2) = f_2(Ax_1) = f_2(Ax_2) = 0. \end{aligned}$$

Set $S = x_1 \otimes f_1 + x_2 \otimes f_2$. Then $(A \circ S)^4 = Ax_1 \otimes f_2 A \neq 0$ and $(A \circ S)^5 = 0$. $\square$

DEFINITION 2.4. We say that an operator is s-idempotent if it is a scalar multiple of an idempotent operator.

Observe that an operator is s-idempotent if and only if it is a scalar multiple of its square.

LEMMA 2.5. *Suppose that X has dimension at least five. Let A be s-idempotent. Then the following are equivalent.*

(1) rank$(A) \geqslant 2$.

(2) There exists an operator $S \in B(X)$ such that $A \circ S \in \mathcal{N}(X)$ and the rank of $(A \circ S)^2$ is greater than one.

Proof. Without loss of generality, we may assume that A is idempotent.

(2) $\Rightarrow$ (1). Suppose $\text{rank}(A) = 1$. Write $A = x \otimes f$ with $f(x) = 1$. Suppose $S \circ x \otimes f \in \mathcal{N}(X)$ for an operator $S \in B(X)$. Then $f(Sx) = 0$ and $f(S^2 x) = 0$ by Lemma 2.1. Consequently, $(S \circ x \otimes f)^2 = Sx \otimes fS$ has rank at most one.

(1) $\Rightarrow$ (2). We distinguish two cases.

Case 1: $2 \leqslant \text{rank}(A) \leqslant 3$. Then we can take x_1, x_2 from the image of A and x_3, x_4 from the image of $I - A$ such that x_1, x_2, x_3, x_4 are linearly independent. Take f_1, f_2, f_3, f_4 from X^* such that $f_i(x_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant 4$. Set $S = x_2 \otimes f_4(I - A) +$

$x_4 \otimes f_1 A + x_1 \otimes f_3(I-A)$. Then $(A \circ S)^2 = x_4 \otimes f_3(I-A) + x_2 \otimes f_1 A$ has rank two and $(A \circ S)^4 = 0$.

Case 2: $\operatorname{rank}(A) \geqslant 4$. Since A is idempotent, we can take vectors x_1, x_2, x_3, x_4 from the range of A and vectors f_1, f_2, f_3, f_4 from the range of A^* such that $f_i(x_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant 4$. Set $S = x_1 \otimes f_2 + x_2 \otimes f_3 + x_3 \otimes f_4$. Then $(A \circ S)^2 = 4(x_1 \otimes f_3 + x_2 \otimes f_4)$ has rank two and $(A \circ S)^4 = 0$. $\square$

Proof of Theorem 1.1. For clarity of exposition, we proceed in steps.

STEP 1. We have $\phi(0) = 0$.

By the surjectivity of ϕ, we can take $A \in B(X)$ such that $\phi(A) = 0$. Then by Eq.(1.1),

$$\phi(0) = \phi(A \circ 0) \sim \phi(A) \circ \phi(0) = 0.$$

So $\phi(0) = 0$.

STEP 2. Let $A, B \in B(X)$.

(1) $\phi(2^{2^n-1} A^{2^n}) \sim 2^{2^n-1} \phi(A)^{2^n}$ for all $n \in \mathbb{N}$.

(2) $\phi(2^{2^n-1} (A \circ B)^{2^n}) \sim 2^{2^n-1} (\phi(A) \circ \phi(B))^{2^n}$ for all $n \in \mathbb{N}$.

(3) $A \in \mathscr{N}(X)$ if and only if $\phi(A) \in \mathscr{N}(X)$.

(4) $A \circ B \in \mathscr{N}(X)$ if and only if $\phi(A) \circ \phi(B) \in \mathscr{N}(X)$.

It is easy to see that (3) follows from (1), Step 1 and the injectivity of ϕ, and that (4) follows from (2), Step 1 and the injectivity of ϕ.

By Eq.(1.1), we have $\phi(2A^2) \sim 2\phi(A)^2$. Suppose that

$$\phi(2^{2^n-1} A^{2^n}) \sim 2^{2^n-1} \phi(A)^{2^n}.$$

Note that if two operators are similar then their squares are similar. Thus

$$(2^{2^n-1} \phi(A)^{2^n})^2 \sim (\phi(2^{2^n-1} A^{2^n}))^2 \sim \frac{1}{2} \phi(2 \cdot (2^{2^n-1} A^{2^n})^2) = \frac{1}{2} \phi(2^{2^{n+1}-1} A^{2^{n+1}}).$$

So

$$\phi(2^{2^{n+1}-1} A^{2^{n+1}}) \sim 2^{2^{n+1}-1} \phi(A)^{2^{n+1}}.$$

Therefore, by the induction, we prove (1).

Now by (1) and Eq.(1.1), we have

$$\phi(2^{2^n-1} (A \circ B)^{2^n}) \sim 2^{2^n-1} \phi(A \circ B)^{2^n} \sim 2^{2^n-1} (\phi(A) \circ \phi(B))^{2^n}$$

for all $n \in \mathbb{N}$. This proves (2).

STEP 3. $\phi(A) \in \mathbb{C}I$ if and only if $A \in \mathbb{C}I$.

It is a consequence of Step 2 and Lemma 2.2.

STEP 4. There is a bijective map θ of $\mathbb{C}$ onto itself such that $\phi(zI) = \theta(z)I$ and $\phi(2zA) \sim 2\theta(z)\phi(A)$ for all $z \in \mathbb{C}$ and $A \in B(X)$.

This is an easy consequence of Step 3 and Eq.(1.1).

STEP 5. $\phi(\mathcal{N}_1(X)) = \mathcal{N}_1(X)$.

Apply [5, Theorem 2.1] when X is finite-dimensional and apply Step 2(3) and Lemma 2.3 when X is infinite-dimensional.

STEP 6. Let $A \in B(X)$ be non-nilpotent. Then $\phi(A)$ is s-idempotent if and only if A is s-idempotent. In particular, if A is idempotent, then $\frac{1}{\theta(1)}\phi(A)$ is idempotent.

First suppose that A is s-idempotent. Then $A = \lambda P$ for some nonzero scalar λ and some idempotent operator P. We can suppose that $P \neq I$; for otherwise, we are done by Step 3. Let $x \otimes f \in \mathcal{N}_1(X)$ be such that $f(\phi(A)x) = 0$. By Step 5, we can take $y \otimes g \in \mathcal{N}_1(X)$ such that $\phi(y \otimes g) = x \otimes f$. Compute

$$(\phi(A) \circ x \otimes f)^2 = f(\phi(A)^2 x) x \otimes f, \tag{2.1}$$

$$(\phi(A) \circ x \otimes f)^4 = 0. \tag{2.2}$$

By Eq.(2.2) and Step 2(3), $A \circ y \otimes g \in \mathcal{N}(X)$. Then $g(Ay) = 0$ by Lemma 2.1 and hence $(A \circ y \otimes g)^2 = 0$. This together with Step 2 and Eq.(2.1) leads to $f(\phi(A)^2 x) = 0$ for all $x \otimes f \in \mathcal{N}_1(X)$ with $f(\phi(A)x) = 0$. By [12, Lemma 2.4], there exist scalars $\alpha, \beta \in \mathbb{C}$ such that

$$\phi(A)^2 + \alpha\phi(A) + \beta I = 0.$$

By the surjectivity of ϕ, we can take $S \in B(X)$ such that $\phi(S) = \phi(A) + \alpha I$. Then

$$\phi(A \circ S) \sim \phi(A) \circ \phi(S) = -2\beta I.$$

By Step 3, $A \circ S = \gamma I$ for some scalar γ. Then $\gamma(I-P) = (I-P)(A \circ S)(I-P) = 0$. So $\gamma = 0$ and hence $\beta = 0$. Thus $\phi(A)^2 + \alpha\phi(A) = 0$. This implies that A is s-idempotent.

In a similar way, we can show that if $\phi(A)$ is s-idempotent then A is s-idempotent.

Now suppose that A is idempotent. Then $\phi(A) = \mu Q$, where $\mu \in \mathbb{C}$ and Q is an idempotent operator. Then $\phi(2A) = \phi(2A^2) \sim 2\phi(A)^2 = 2\mu^2 Q$. On the other hand, by Step 4, $\phi(2A) \sim 2\theta(1)\phi(A) = 2\mu\theta(1)Q$. We have $2\mu^2 Q \sim 2\mu\theta(1)Q$. This implies that $\mu = \theta(1)$, completing the proof.

STEP 7. Let $A \notin \mathcal{N}(X)$. Then $\phi(A)$ is of rank-one if and only so is A.

If X is finite-dimensional, the result can be concluded from [5, Theorem 2.1]. In the following, we assume that X is infinite-dimensional.

First suppose that A is of rank-one. Then A is s-idempotent and hence so is $\phi(A)$ by Step 6. Suppose $\phi(S) \circ \phi(A) \in \mathcal{N}(X)$ for some $S \in B(X)$. Then $S \circ A \in \mathcal{N}(X)$. By Lemma 2.5, $(S \circ A)^2$ has rank at most one. Then by Step 5, $\phi(2(A \circ S)^2)$ has rank at most one. Since

$$(\phi(S) \circ \phi(A))^2 \sim \frac{1}{2}\phi(2(A \circ S)^2),$$

it follows that $(\phi(S) \circ \phi(A))^2$ has rank at most one. By Lemma 2.5, $\phi(A)$ is of rank-one.

In a similar way, we can show that if $\phi(A)$ is of rank-one then so is A.

STEP 8. Let P be an idempotent and λ be a nonzero scalar. Then $\phi(\lambda P)$ and $\phi(P)$ are linearly dependent.

Let $x\otimes f$ be any rank-one operator. Then by Steps 5 and 7, there is a rank-one operator $y\otimes g$ such that $\phi(y\otimes g)=x\otimes f$. Note that $\phi(\lambda P)$ and $\phi(P)$ are both s-idempotent by Step 6. Then

$$f(\phi(P)x)=0 \Leftrightarrow \phi(P)\circ x\otimes f\in\mathscr{N}(X) \Leftrightarrow P\circ y\otimes g\in\mathscr{N}(X) \Leftrightarrow (\lambda P)\circ y\otimes g\in\mathscr{N}(X)$$
$$\Leftrightarrow \phi(\lambda P)\circ x\otimes f\in\mathscr{N}(X) \Leftrightarrow f(\phi(\lambda P)x)=0.$$

So $f(\phi(P)x)=0$ if and only if $f(\phi(\lambda P)x)=0$ for all $x\in X$ and $f\in X^*$. This implies that $\phi(\lambda P)$ and $\phi(P)$ are linearly dependent.

Recall that $\mathscr{P}(X)$ denotes the set of all idempotent operators in $B(X)$. In the following, we let $\psi=\frac{1}{\theta(1)}\phi|_{\mathscr{P}(X)}$, the restriction of $\frac{1}{\theta(1)}\phi$ to $\mathscr{P}(X)$.

STEP 9. The following are true.

(1) The map ψ is a bijection from $\mathscr{P}(X)$ onto $\mathscr{P}(X)$.

(2) For $P,Q\in\mathscr{P}(X)$, $PQ=QP=0$ if and only if $\psi(P)\psi(Q)=\psi(Q)\psi(P)=0$.

We only show (1) since (2) is a direct verification.

First we know that the image of ψ is contained in $\mathscr{P}(X)$ by Step 6.

Next we show the surjectivity. Suppose that $\phi(A)$ is idempotent for some $A\in B(X)$. Then A is s-idempotent by Step 6. Write $A=\lambda P$ for some scalar $\lambda\in\mathbb{C}$ and some idempotent $P\in\mathscr{P}(X)$. Then by Step 8, there is a scalar $\mu\in\mathbb{C}$ such that

$$\phi(A)=\mu\phi(P)=\mu\theta(1)\psi(P).$$

Since both $\phi(A)$ and $\psi(P)$ are idempotent, we conclude $\mu\theta(1)=1$ and then $\psi(P)=\phi(A)$.

Finally, we show the injectivity. Let P_1 and P_2 be idempotents and suppose that $\psi(P_1)=\psi(P_2)$. Then $\phi(P_1)=\phi(P_2)$ and hence $P_1=P_2$ by the injectivity of ϕ.

Now by Step 9, ψ is a bijective map on $\mathscr{P}(X)$ preserving orthogonality in both directions. We can apply [21, Corollary 4.13] when X is finite-dimensional and [22, Corollary 1.4 and Corollary 1.5] when X is infinite-dimensional. Then one of the following holds:

(1) There is a semilinear bijection $T:X\to X$ such that

$$\phi(P)=\theta(1)TPT^{-1},\ P\in\mathscr{P}(X).$$

Moreover, if X is infinite-dimensional, then T is bounded and linear or conjugate-linear.

(2) The space X is reflexive and there is a semilinear bijection $T:X^*\to X$ such that

$$\phi(P)=\theta(1)TP^*T^{-1},\ P\in\mathscr{P}(X).$$

Moreover, if X is infinite-dimensional, then T is bounded and linear or conjugate-linear.

Without loss of generality, in the rest of proof we assume the first case above holds and show that case 1 in Theorem 1.1 holds. Suppose that h is an automorphism of $\mathbb{C}$ such that $T(\lambda x) = h(\lambda)Tx$ for all $\lambda \in \mathbb{C}$ and $x \in X$. Then it is easy to verify that $T^{-1}(h(\lambda)x) = \lambda T^{-1}x$ for all $\lambda \in \mathbb{C}$ and $x \in X$. Therefore $T^{-1}AT \in B(X)$ for all $A \in B(X)$ and hence $T^{-1}\phi(\cdot)T$ is a bijection of $B(X)$ satisfying Eq.(1.1). So we may replace ϕ by $T^{-1}\phi T$ and then we have that

$$\phi(P) = \theta(1)P,\ P \in \mathscr{P}(X). \tag{2.3}$$

Our aim is to show that $\phi(A) = A$ for all $A \in B(X)$.

STEP 10. There exists a function $\tau : B(X) \to \mathbb{C} \setminus \{0\}$ such that $\phi(A) = \tau(A)A$ for all $A \in B(X)$.

First we suppose that $A^2 = 0$. Then $\phi(A)^2 = 0$ by Step 2. Therefore, for any $x \otimes f \in \mathscr{P}(X)$, by Eq.(2.3) we have

$$\begin{aligned} f(\phi(A)x) = 0 &\Leftrightarrow \theta(1)(\phi(A) \circ x \otimes f) \in \mathscr{N}(X) \Leftrightarrow \phi(A) \circ \phi(x \otimes f) \in \mathscr{N}(X) \\ &\Leftrightarrow A \circ x \otimes f \in \mathscr{N}(X) \Leftrightarrow f(Ax) = 0, \end{aligned}$$

and so $\phi(A) = \tau(A)A$ for some nonzero scalar $\tau(A)$ by [3, Lemma 2.16].

We now turn to the general case. Let $A \in B(X)$. By Step 3, we can assume $A \notin \mathbb{C}I$. For $x \in X, f \in X^*$ with $f(x) = 0$ and $f(Ax) = 0$, since $A \circ x \otimes f \in \mathscr{N}(X)$, by the preceding result we have $\tau(x \otimes f)(\phi(A) \circ x \otimes f) \in \mathscr{N}(X)$, which implies $f(\phi(A)x) = 0$. It follows from [12, Lemma 2.4] that

$$\phi(A) = \tau(A)A + \mu(A)I \tag{2.4}$$

for some scalars $\tau(A)$ and $\mu(A)$.

It now suffices to show that $\mu(A) = 0$. For this, first we suppose that A is of rank-one. Then $\phi(A)$ is of rank-one. This together with Eq.(2.4) leads to $\mu(A) = 0$. Next, we suppose that A has rank at most two. Then there exists an operator B of rank-one such that $A \circ B = 0$. Thus we have $\phi(A) \circ \phi(B) = 0$. By the preceding case, we can write $\phi(B) = \alpha B$ for some scalar α. Then from $(\tau(A)A + \mu(A)I) \circ (\alpha B) = 0$, we get $\mu(A) = 0$.

Finally, we consider the general case. Choose $y \in X$ such that y and Ay are linearly independent. Take $g \in X$ such that $g(y) = 1$ and $g(Ay) = 0$. Since $A \circ y \otimes g$ has rank at most two, it follows that $\phi(A \circ y \otimes g) = \beta(A \circ y \otimes g)$ for some scalar β. From

$$\phi(A \circ y \otimes g) \sim \phi(A) \circ \phi(y \otimes g),$$

it follows

$$\theta(1)(\tau(A)A + \mu(A)I)y \otimes g + y \otimes g(\tau(A)A + \mu(A)I)) \sim \beta(A \circ y \otimes g).$$

Then by comparing the trace of the left and the right, we get $\mu(A) = 0$, completing the proof.

STEP 11. There is a scalar $\lambda_0 \in \mathbb{C}$ such that $\tau(x\otimes f) = \lambda_0$ for all $x\otimes f \in \mathscr{N}_1(X)$.

It suffices to show $\tau(x_1\otimes f_1) = \tau(x_2\otimes f_2)$ for any $x_1\otimes f_1, x_2\otimes f_2 \in \mathscr{N}_1(X)$. We distinguish some cases.

Case 1: x_1 and x_2 are linearly independent, and $f_1 = f_2$. Then we can choose $x_0 \in X$ such that $f_1(x_0) = f_2(x_0) = 1$. Notice that x_0, x_1, x_2 are linearly independent. Then we can take f_0 in X^* such that $f_0(x_0) = 0$ and $f_0(x_1) = f_0(x_2) = 1$. It is easy to see that $x_0\otimes f_0 \in \mathscr{N}_1(X)$ and $x_0\otimes f_i + x_i\otimes f_0 \in \mathscr{P}(X)$, $i = 1,2$. Since

$$x_0\otimes f_i + x_i\otimes f_0 = (x_0\otimes f_0)\circ(x_i\otimes f_i),\ i = 1,2,$$

it follows that

$$\phi(x_0\otimes f_i + x_i\otimes f_0) \sim \phi(x_0\otimes f_0)\circ\phi(x_i\otimes f_i),\ i = 1,2.$$

Hence we have

$$\theta(1)(x_0\otimes f_i + x_i\otimes f_0) \sim \tau(x_i\otimes f_i)\tau(x_0\otimes f_0)(x_0\otimes f_i + x_i\otimes f_0), i = 1,2.$$

From this, we get $\tau(x_i\otimes f_i)\tau(x_0\otimes f_0) = \theta(1)$, $i = 1,2$. So $\tau(x_1\otimes f_1) = \tau(x_2\otimes f_2)$.

Case 2: f_1 and f_2 are linearly independent, and $x_1 = x_2$.

By an argument similar to that in Case 1, we have $\tau(x_1\otimes f_1) = \tau(x_2\otimes f_2)$.

Case 3: $x_1\otimes f_1$ and $x_2\otimes f_2$ are linearly dependent, say $x_2\otimes f_2 = \alpha x_1\otimes f_1$ for some scalar α. Take $y \in \ker(f_1)$ such that y and x_1 are linearly independent. Then by Case 1, we have $\tau(x_1\otimes f_1) = \tau(y\otimes f_1) = \tau(\alpha x_1\otimes f_1) = \tau(x_2\otimes f_2)$.

Case 4: $f_1(x_2) = 0$. Then

$$\tau(x_1\otimes f_1) = \tau(x_2\otimes f_1) = \tau(x_2\otimes f_2),$$

where the first equality is due to Cases 1 and 3, the second equality is due to Cases 2 and 3.

Finally, we consider the general case. Take $x_3 \in \ker f_1 \cap \ker f_2$. Then by Case 4, we have $\tau(x_1\otimes f_1) = \tau(x_3\otimes f_2) = \tau(x_2\otimes f_2)$, completing the proof.

STEP 12. We have $\lambda_0^2 = \theta(1)$.

Let $x_1, x_2 \in X$ and $f_1, f_2 \in X^*$ be such that $f_i(x_j) = \delta_{ij}$ for all $1 \leqslant i,j \leqslant 2$. Then $x_1\otimes f_2, x_2\otimes f_1 \in \mathscr{N}_1(X)$ and $x_1\otimes f_1 + x_2\otimes f_2 \in \mathscr{P}(X)$. Since

$$(x_1\otimes f_2)\circ(x_2\otimes f_1) = x_1\otimes f_1 + x_2\otimes f_2,$$

it follows that

$$\lambda_0^2((x_1\otimes f_2)\circ(x_2\otimes f_1)) \sim \theta(1)(x_1\otimes f_1 + x_2\otimes f_2).$$

So $\lambda_0^2 = \theta(1)$.

STEP 13. If A is non-zero, non-invertible and non-s-idempotent, then $\tau(A) = \lambda_0$.

First suppose that there is $x_0 \in X$ such that Ax_0 doesn't lie in the linear span of x_0 and A^2x_0. Then we can find $f_0 \in X^*$ such that $f_0(x_0) = f_0(A^2x_0) = 0$ while $f(Ax_0) = 1$. Then $x_0 \otimes f_0 \in \mathscr{N}_1(X)$ and $A \circ x_0 \otimes f_0 \in \mathscr{P}(X)$. Therefore, we have

$$\theta(1)(A \circ x_0 \otimes f_0) \sim \tau(A)\lambda_0(A \circ x_0 \otimes f_0).$$

So $\tau(A)\lambda_0 = \theta(1)$ and hence $\tau(A) = \lambda_0$ since $\lambda_0^2 = \theta(1)$.

Suppose now that $Ax \in \operatorname{span}\{x, A^2x\}$ for all $x \in X$. Then by [12, Lemma 2.4], there are scalars α and β such that $\alpha A^2 + A + \beta I = 0$. Since A is non-invertible, we get $\beta = 0$. Thus we have $\alpha A^2 + A = 0$. This implies that $A = 0$ or A is s-idempotent, a contradiction.

STEP 14. $\lambda_0 = \theta(1) = 1$.

Take linearly independent vectors $x_1, x_2 \in X$ and $f_1, f_2 \in X^*$ such that $f_1(x_1) = 0$ and $f_1(x_2) = f_2(x_1) = f_2(x_2) = 1$. It is not difficult to verify that $x_1 \otimes f_2 + x_2 \otimes f_1$ is non-invertible and non-s-idempotent. Note that $x_1 \otimes f_1 \in \mathscr{N}_1(X)$, $x_2 \otimes f_2 \in \mathscr{P}(X)$ and

$$x_1 \otimes f_2 + x_2 \otimes f_1 = (x_1 \otimes f_1) \circ (x_2 \otimes f_2).$$

It follows from Step 13 that

$$\lambda_0(x_1 \otimes f_2 + x_2 \otimes f_1) \sim \lambda_0\theta(1)(x_1 \otimes f_2 + x_2 \otimes f_1).$$

Comparing the trace, we get $\theta(1) = 1$.

Take linearly independent vectors $y_1, y_2 \in X$ and $g_1, g_2 \in X^*$ such that $g_i(y_j) = 1$, $i, j = 1, 2$. It is not difficult to verify that $y_1 \otimes g_2 + y_2 \otimes g_1$ is non-invertible and non-s-idempotent. Note that $y_1 \otimes g_1, y_2 \otimes g_2 \in \mathscr{P}(X)$ and

$$y_1 \otimes g_2 + y_2 \otimes g_1 = (y_1 \otimes g_1) \circ (y_2 \otimes g_2).$$

It follows

$$\lambda_0(y_1 \otimes g_2 + y_2 \otimes g_1) \sim \theta(1)^2(y_1 \otimes g_2 + y_2 \otimes g_1) = y_1 \otimes g_2 + y_2 \otimes g_1.$$

Comparing the trace, we get $\lambda_0 = 1$.

STEP 15. $\phi(A) = A$ for all $A \in B(X)$.

First we suppose $A \notin \mathbb{C}I$. Then there are $x \in X$ and $f \in X^*$ such that $f(x) = 0$ and $f(Ax) = 1$. It is not difficult to verify that $A \circ x \otimes f$ is non-invertible and either idempotent or non-s-idempotent. Then from Step 13 and the relation

$$\phi(A) \circ \phi(x \otimes f) \sim \phi(A \circ x \otimes f),$$

we have

$$\tau(A)(A \circ x \otimes f) \sim A \circ x \otimes f.$$

Comparing the trace, we get $\tau(A) = 1$.

Finally, let $\mu \in \mathbb{C}$, and let P be an idempotent of rank-one. Then from the proceeding result, Step 14 and the relation

$$\phi(2\mu P) \sim \phi(\mu I) \circ \phi(P),$$

we get that $2\mu P \sim 2\theta(\mu)P$. So $\theta(\mu) = \mu$ and then $\phi(\mu I) = \mu I$ for all $\mu \in \mathbb{C}$. □

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