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ON THE MATRIX WHICH IS THE SUM OF A TRIPOTENT AND A QUASINILPOTENT MATRICES

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Abstract. We investigate Hirano polar matrices over a local ring, and completely determine when a 2×2 matrix over a local ring is the sum of a tripotent and a quasinilpotent matrix.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a\in R$ is defined by $comm(a)=\{x\in R\mid xa=ax\}$. The double commutant of $a\in R$ is defined by $comm^2(a)=\{x\in R\mid xy=yx \text{ for all } y\in comm(a)\}$. An element $a\in R$ is quasinilpotent if $1-ax\in U(R)$ for any $x\in comm(a)$. We use R^{qnil} to denote the set of all quasinilpotents in R. That is, $R^{qnil}=\{a\in R\mid 1-ax\in U(R) \text{ for every } x\in comm(a)\}$. Clearly, every nilpotent and element in the Jacobson radical of a ring is quasinilpotent. Following Wang, an element a in a ring R has s-Drazin inverse if there exists $b\in comm^2(a)$ such that $b=b^2a, a-ab\in N(R)$ (see [13]). As is well known, an element $a\in R$ has s-Drazin inverse if and only if it is strongly nil-clean, i.e., it is the sum of an idempotent and a nilpotent that commute (see [1, 11]). Replace $N(R)$ of nilpotents by R^{qnil} the set of quasinilpotents, Gurgun introduced gs-Drazin inverse of an element in a ring. It was proved that an element $a\in R$ has gs-Drazin inverse if and only if there exists $e^2=e\in comm^2(a)$ such that $a-e\in R^{qnil}$ (see [8, Theorem 3.2]).

An element p in a ring R is a tripotent if $p^3=p$. It is readily seen that idempotents and negative of idempotents are tripotents and among units only the order 2 units (also called square roots of 1) are tripotents. In [2], the authors investigated the structure of rings in which every element is the sum of a tripotent and a nilpotent that commute. The motivation of this paper is to determine when a 2×2 matrix over a local ring is the sum of a tripotent and a quasinilpotent matrix that commutate. An element a in a ring R is quasipolar if there exists an idempotent $e\in comm^2(a)$ such that $a+e\in U(R)$ and $ae\in R^{qnil}$. Quasipolar elements in a ring were studied by many authors from different view of points, e.g., [4, 5] and [6]. We call an element $a\in R$ is Hirano polar if there exists a tripotent $p\in comm^2(a)$ such that $a-p\in R^{qnil}$. In Section 2 we investigate

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Hirano polar elements in a ring and prove that every Hirano polar element in a ring is quasipolar.

Let $a \in R$. $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. In Section 3, we are concerned on Hirano polar matrices over local rings. Let R be a local ring, and let $A \in M_2(R)$. We prove that A is Hirano polar if and only if $A \in M_2(R)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$, or A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in \pm 1 + J(R), \beta \in J(R)$.

A ring R is bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b$ and $l_b - r_a$ are both surjective. A ring R is cobleached provided that for any $a \in J(R), b \in U(R)$, $l_a - r_b$ and $r_b - r_a$ are both injective. For instance, every commutative local ring is cobleached. Finally, in the last section, we further characterize Hirano J-polar 2×2 matrices over a cobleached local ring in terms of solvability of their characteristic equations. Let R be a cobleached local ring, and let $A \in M_2(R)$. We prove that A is Hirano polar if and only if $A \in M_2(R)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$.

Throughout the paper, all rings are associative with an identity. We use $J(R), N(R)$ and $U(R)$ to denote the Jacobson radical and the set of nilpotents of R and units in R, respectively. $GL_2(R)$ denotes the sets of all 2×2 invertible matrices over R. $\mathbb{N}$ stands for the set of all natural numbers.

2. Hirano polar elements

Following Cui and Chen, an element $a \in R$ is J-quasipolar if there exists an idempotent $e \in comm^2(a)$ such that $a + e \in J(R)$ (see [5]). We begin with

EXAMPLE 2.1. Every J-quasipolar element in a ring is Hirano polar.

Proof. Let $a \in R$ be a J-quasipolar, then there exists some idempotent $e \in R$ such that $a + e \in J(R)$. It is obvious that $-e \in R$ is tripotent and $a - (-e) \in J(R) \subseteq R^{qnil}$. $\square$

EXAMPLE 2.2. Let $A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_3)$. Then A is Hirano polar, but it is not J-quasipolar.

Proof. Clearly, $A^3 = A$, and so A is Hirano polar. Since $A - A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \notin J(M_2(\mathbb{Z}_3))$. Therefore A is not J-quasipolar. $\square$

LEMMA 2.3. *Let $a \in R^{qnil}$ and $e^2 = e \in comm^2(a)$. Then $ae \in R^{qnil}$.*

Proof. Let $x \in comm(ae)$. Then $xae = aex$, and so $(exe)a = ex(ae) = eaex = aex = a(exe)$, i.e., $exe \in comm(a)$. Hence, $1 - a(exe) \in U(R)$, and so $1-(ae)x \in U(R)$ which implies that $ae \in R^{qnil}$. □

THEOREM 2.4. *Every Hirano polar element in a ring is quasipolar.*

Proof. Let $a \in R$, then there there exists $p^3 = p \in comm^2(a)$ such that $a - p \in R^{qnil}$. It is clear that $(1-p^2)^2 = 1-p^2$. Let $a-p=w$ so $a+1-p^2 = w+1-p^2+p$, as $(1+p^2-p)^2 = 1$, we can write $1-p^2+p+w = (1-p^2+p)(1+(1-p^2+p)w) \in U(R)$, since $w \in R^{qnil}$. Then $a+(1-p^2) \in U(R)$. As $a-p=w, a(1-p^2) = w(1-p^2)$ that is in R^{qnil} by applying Lemma 2.4, because $w \in R^{qnil}$ and $p \in comm^2(a)$ also $1-p^2$ is an idempotent. □

EXAMPLE 2.5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$. Then A is quasipolar, but it is not Hirano polar.

Proof. As $M_2(\mathbb{Z}_2)$ is a finite ring so it is strongly π-regular and then quasipolar. Now let A is a Hirano polar ring, then there exists a triptent E such that $A = E+W$ for some $W \in R^{qnil}$, clearly $W^2 = 0$ as $M_2(\mathbb{Z}_2)$ is of bounded index 2 and so $(A-A^3)^2 = 0$ that is a contradiction as $(A-A^3)^2 = A \neq 0$. □

THEOREM 2.6. *Let R be a ring, and let $a \in R$. If $\frac{1}{2} \in R$, then the following are equivalent:*

(1) a is Hirano polar.

(2) There exist two idempotents $e, f \in comm^2(a)$ and a $w \in R^{qnil}$ such that $a = e - f + w$.

Proof. $(1) \Rightarrow (2)$ Let $a \in R$, then there exists some tripotent $p \in R$ such that $a-p \in R^{qnil}$. Let $e = \frac{1}{2}(p^2-p)$ and $f = \frac{1}{2}(p^2+p)$. By computing e^2-e and f^2-f it is obvious that $e^2 = e$ and $f^2 = f$, also $p = f-e$. Then $a-e+f \in R^{qnil}$.

$(2) \Rightarrow (1)$ By hypothesis there exist two idempotents $e, f \in R$ and some $w \in R^{qnil}$ such that $a = e-f+w$. Let $p = e-f$, it is obvious that $p^3 = p$ and $a-p = w \in R^{qnil}$. □

COROLLARY 2.7. *Let A be a Banach algebra, and let $a \in A$. Then the following are equivalent:*

(1) a is Hirano polar.

(2) There exist two idempotents $e, f \in comm^2(a)$ such that

$$\lim_{n\to\infty} \| (a-(e-f))^n \|^{\frac{1}{n}} = 0.$$

Proof. $\Rightarrow$ Let $a \in A$ be a Hirano polar element, as $2 \in A$ is invertible, then by Theorem 2.6, there exist two idempotents e, f such that $a = f - e + w$ for some $w \in A^{qnil}$, which implies that $a - (e - f) \in A^{qnil}$, in view of [9, page 251] we deduce that

$$\lim_{n\to\infty} || (a-(e-f))^n ||^{\frac{1}{n}} = 0.$$

$\Leftarrow$ Let $a \in A$ and there exist two idempotents $e, f \in comm^2(a)$ such that

$$\lim_{n\to\infty} || (a-(e-f))^n ||^{\frac{1}{n}} = 0.$$

In view of [9, page 251] $a - (e - f) \in A^{qnil}$ and so a is Hirano polar. $\square$

3. Hirano polar matrices

The goal of this section is to characterize when a 2×2 matrix over local rings is Hirao polar in terms of diagonal reduction. The following lemma is crucial.

LEMMA 3.1. *Let R be a ring, and let $a \in R$ and $u \in U(R)$. Then $a \in R$ is Hirano polar if and only if $u^{-1}au \in R$ is Hirano polar.*

Proof. $\Longrightarrow$ By hypothesis, there exists $p^3 = p \in comm^2(a)$ such that $w := a + p \in R^{qnil}$. Then $u^{-1}pu = (u^{-1}pu)^3, u^{-1}au + u^{-1}pu = u^{-1}wu$. Let $x \in comm(u^{-1}au)$. Then $u^{-1}aux = xu^{-1}au$; hence, $auxu^{-1} = uxu^{-1}a$. Then $uxu^{-1} \in comm(a)$, and so $uxu^{-1}p = puxu^{-1}$. This shows $xu^{-1}pu = u^{-1}pux$, and therefore $u^{-1}pu \in comm^2(u^{-1}au)$. Let $y \in comm(u^{-1}wu)$. Then $uyu^{-1} \in comm(w)$; hence, $1 - w(uyu^{-1}) \in U(R)$. By using Jacobson's Lemma, $1 - (u^{-1}wu)y \in U(R)$. Therefore $u^{-1}wu \in R^{qnil}$, as needed.

$\Longleftarrow$ This is symmetric. $\square$

LEMMA 3.2. *(* [7, Lemma 3.3]*) Let R be a local ring, and let $A \in M_2(R)$. Then*

(1) $A \in GL_2(R)$; or

(2) $A^2 \in M_2(J(R))$; or

(3) A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$.

We come now to the demonstration for which this section has been developed.

THEOREM 3.3. *Let R be a local ring, and let $A \in M_2(R)$. Then A is Hirano polar if and only if*

(1) $A \in M_2(R)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$, or

(2) A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in \pm 1 + J(R), \beta \in J(R)$.

Proof. $\Longrightarrow$ In view of Theorem 2.4, A is pseudopolar. By virtue of [7, Theorem 3.5], we have three cases.

Case 1. $A \in GL_2(R)$. Since A is Hirano polar, there exists $V^3 = V \in comm^2(A)$ such that $Z := A + V$ with $Z \in M_2(R)^{qnil}$. Then $V = Z - A \in GL_2(R)$; hence, $V^2 = I_2$. Set $U = -V$. Then $A = U + Z, U \in comm^2(A)$ and $U^2 = I_2$.

Case 2. $A^2 \in M_2(J(R))$. For any $X \in comm(A)$, we see that $I_2 - A^2X^2 \in GL_2(R)$, and so $I_1 - AX \in GL_2(R)$. This shows that $A \in M_2(R)^{qnil}$.

Case 3. A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$. Since A is Hirano polar, we easily check that $B := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is Hirano polar. Then we can find some $E^3 = E \in comm^2(B)$ such that $W = B + E, W \in M_2(R)^{qnil}$. Set $E = (e_{ij})$. Then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix};$$

hence, we have

$$\alpha e_{12} - e_{12}\beta = 0, \beta e_{21} - e_{21}\alpha = 0.$$

This implies that $e_{12} = e_{21} = 0$. Hence, $E = \begin{pmatrix} e_{11} & 0 \\ 0 & e_{22} \end{pmatrix}$. Set $W = (w_{ij})$. Then $w_{12} = w_{21} = 0, w_{11}^2, w_{22}^2 \in J(R)$. Since R is local, $w_{11}, w_{22} \in J(R)$.

Clearly, $e_{11} \in U(R)$, we see that $e_{11}^2 = 1$, and so $(e_{11} - 1)(e_{11} + 1) = 0$. Since every element in R is invertible or in $J(R)$, we have $e_{11} \in \pm 1 + J(R)$. Hence, $\alpha \in \pm 1 + J(R)$. Also we see that $e_{22} \in J(R)$ and $e_{22}^3 = e_{22}$, and so $e_{22}(1 - e_{22}^2) = 0$. Hence $e_{22} = 0$; hence, $\beta \in J(R)$, as desired.

$\Longleftarrow$ Case 1. $A \in M_2(R)^{qnil}$. Then $A + 0 = A$ with $A \in M_2(R)^{qnil}$.

Case 2. $A = U + W, U \in comm^2(A), U^2 = I_2, N \in N(M_2(R)),\ W \in M_2(R)^{qnil}$. Set $V = -U$. Then $A + U = W$ where $U^3 = U, U \in comm^2(A), W \in M_2(R)^{qnil}$.

Case 3. It will suffice to check $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is Hirano polar, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in \pm 1 + J(R), \beta \in J(R)$. We observe that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \pm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha \pm 1 & 0 \\ 0 & \beta \end{pmatrix}.$$

Let $X = (x_{ij}) \in comm \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Then

$$\alpha x_{12} = x_{12}\beta, \beta x_{21} = x_{21}\alpha.$$

As $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective, we get $x_{12} = x_{21} = 0$. Hence $X \in comm(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$, and so

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2 \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Therefore $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is Hirano polar, as required. □

COROLLARY 3.4. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A is Hirano polar if and only if*

(1) $A \in M_2(R)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$, or

(2) There exists $E^2 = E \in comm(A)$ such that $A \pm E \in M_2(J(R))$.

Proof. $\Longrightarrow$ By Theorem 3.3, we may assume that A is isomorphic to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in \pm 1 + J(R), \beta \in J(R)$. As in the proof of Theorem 3.3, we see that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \pm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(J(R))$$

with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2 \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Clearly, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ is an idempotent, as required.

$\Longleftarrow$ We may assume that $A \pm E \in M_2(J(R))$ with $E^2 = E \in comm(A)$. By virtue of [6, Lemma 2.3], $E \cong 0, I_2$ or E is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Clearly, $0, I_2 \in comm^2(A)$. We may assume that

$$U^{-1}EU = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$U^{-1}AU \pm U^{-1}EU = U^{-1}WU \in M_2(J(R)).$$

Since $E \in comm(A)$, we see that

$$U^{-1}AU \in comm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Write $U^{-1}AU = \begin{pmatrix} x & y \\ s & t \end{pmatrix}$. Then

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ s & t \end{pmatrix},$$

and so $y = s = 0$.

Clearly,

$$\begin{pmatrix} 1+x & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} x & y \\ s & t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(J(R)).$$

Then $1+x, t \in J(R)$.

For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in comm \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix}$, we have

$$xb - bt = 0, tc - cx = 0.$$

Since R is cobleached, $b = c = 0$; hence,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in comm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2 \begin{pmatrix} x & y \\ s & t \end{pmatrix},$$

and so $U^{-1}EU \in comm^2(U^{-1}AU)$. Therefore $E \in comm^2(A)$, as desired. □

COROLLARY 3.5. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is Hirano polar if and only if*

(1) $A = N + W$, or $A = I_2 + N + W$ where $N^2 = 0, W \in M_2(J(R))$, or

(2) there exists $E^2 = E \in comm(A)$ such that $A \pm E \in M_2(J(R))$.

Proof. Since R is commutative, we obtain the result by Corollary 3.4 and [7, Lemma 3.2]. □

It is convenient at this stage to characterize Hirano polar matrices over a division ring.

THEOREM 3.6. *Let D be a division ring, and let $A \in M_2(D)$. Then the following are equivalent:*

(1) A is Hirano polar.

(2) $A = E - F + N$, where $E^2 = E, F^2 = F \in comm^2(A)$ and $N^2 = 0$.

(3) $A - A^3$ is nilpotent.

Proof. (1) $\Rightarrow$ (2) In light of Theorem 3.3, $A \in M_2(D)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(D)^{qnil}$, or A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in \pm 1 + J(D), \beta \in J(D)$.

Let $X \in M_2(D)^{qnil}$. Then $X \notin GL_2(R)$. Since there exists $V \in GL_2(D)$ such that $V^{-1}XV = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}$, we may assume that $x_{22} = 0$. If $x_{12} = 0$, then $x_{11} = 0$. If $x_{12} \neq 0$ and $x_{11} \neq 0$, then $\begin{pmatrix} x_{11}^{-1} & 0 \\ 0 & x_{12}^{-1}x_{11}^{-1}x_{12} \end{pmatrix} \in comm(V^{-1}XV)$. Hence,

$$I_2 - \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11}^{-1} & 0 \\ 0 & x_{12}^{-1}x_{11}^{-1}x_{12} \end{pmatrix} \in GL_2(D),$$

an absurd. Therefore $x_{11}=0$, and so $X^2=0$. This implies that $M_2(D)^{qnil}=\{X\in M_2(D)\mid X^2=0\}$, and so we have three cases.

Case 1. $A\in M_2(D)^{qnil}$. Then $A^2=0$.

Case 2. $A=U+W$, $U\in comm^2(A)$, $U^2=I_2$ and $W^2=0$. If $2\neq 0$, then $A=\frac{I_2+U}{2}-\frac{I_2-U}{2}+W$. One easily checks that

$$(\frac{I_2+U}{2})^2=\frac{I_2+U}{2},(\frac{I_2-U}{2})^2=\frac{I_2-U}{2}.$$

If $2=0$, then $A=I_2+(U-I_2)+W$, where $(U-I_2)^2=0$, and so $(U-I_2)+W\in M_2(R)$ is nilpotent.

Case 3. As $J(D)=0$, we see that $\alpha=\pm 1$ and $\beta=0$. Then A is similar to $\begin{pmatrix}\pm 1 & 0\\ 0 & 0\end{pmatrix}$. Therefore A or $-A$ is an idempotent, as desired.

$(2)\Rightarrow(3)$ Since $E,F\in comm^2(A)$, we see that $EF=FE$ and $(E-F)N=N(E-F)$, and so $(E-F)^3=E-F$. Moreover, $A-A^3=(E-F)+N-(E-F)^3-3(E-F)^2N=(I_2-3(E-F)^2)N\in M_2(R)$ is nilpotent, as desired.

$(3)\Rightarrow(1)$ Case 1. $2\neq 0$. Then $2\in U(D)$. Let $B=\frac{A^2+A}{2},C=\frac{A^2-A}{2}$. Then $A=B-C$. We easily check that

$$B^2-B=\frac{(A-A^3)(A+2I_2)}{4},C^2-C=\frac{(A-A^3)(A-2I_2)}{4}.$$

Hence $B^2-B,C^2-C\in N(M_2(R))$. In light of [15, Lemma 3.5], there exists idempotents $E,F\in\mathbb{Z}[A]$ such that $B-E,C-F\in N(M_2(D))$. Therefore $A=E-F+(B-E)-(C-F)$, where $(E-F)^3=E-F\in\mathbb{Z}[A]\subseteq comm^2(A),(B-E)-(C-F)\in N(M_2(D))$.

Case 2. $2=0$. Since $A^2-A^4\in M_2(D)$ is nilpotent, we can find an idempotent $E\in\mathbb{Z}[A^2]$ such that $W:=A^2-E\in M_2(D)$ is nilpotent. Hence, $A=E+(A-A^2)+W$. But $(A-A^2)^2=A^2-A^4$, and so $A-A^2$ is nilpotent. As $(A-A^2)W=W(A-A^2)$, we see that $(A-A^2)+W\in M_2(R)$ is nilpotent.

Therefore A is Hirano polar, as asserted. □

COROLLARY 3.7. *Let D be a division ring, and let $A\in M_2(D)$. Then the following are equivalent:*

(1) A is Hirano polar.

(2) A is the sum of a tripotent and a nilpotent that commute.

(3) $A^2+A^6=2A^4$.

Proof. $(1)\Rightarrow(2)$ This is obvious, as $M_2(D)^{qnil}=\{X\in M_2(D)\mid X^2=0\}$.

$(2)\Rightarrow(3)$ Write $A=E+W,E^3=E\in comm(A)$ and $W\in N(M_2(D))$. Then $A-A^3\in M_2(D)$ is nilpotent. As $M_2(D)$ is of bounded index 2, we have $(A-A^3)^2=0$. Therefore $A^2+A^6=2A^4$, as desired.

$(3)\Rightarrow(1)$ Clearly, $(A-A^3)^2=0$. This completes the proof, by Theorem 3.6. □

4. Solvability of quadratic equations

We now investigate Hirano polar matrices over a cobleached local ring by means of the solvability of quadratic equations.

THEOREM 4.1. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A is Hirano polar if and only if*

(1) $A \in M_2(R)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$, or

(2) A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$.

Proof. $\Longrightarrow$ As in the proof of Theorem 3.3, we may assume

$$U^{-1}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for some $U \in GL_2(R)$. Write $U^{-1} = \begin{pmatrix} x & y \\ s & t \end{pmatrix}$. Then we have

$$\begin{aligned} &y = \alpha x; \\ &x\lambda + y\mu = \alpha y; \\ &t = \beta s; \\ &s\lambda + \mu = \beta t. \end{aligned}$$

Thus we see that $t \in J(R), y, s, x \in U(R)$.

Let $\delta = y^{-1}\alpha y$ and $\gamma = t^{-1}\beta t^{-1}t$. Then $\delta \in \pm 1 + J(R), \gamma \in J(R)$. We easily check that $\delta^2 - \delta\mu = \lambda$; whence, $\delta^2 - \delta\mu - \lambda = 0$. Similarly, we have $\gamma^2 - \gamma\mu = \lambda$. Therefore the equation $x^2 - \mu x - \lambda = 0$ has a root $\delta \in \pm 1 + J(R)$ and a root $\gamma \in J(R)$, as desired.

$\Longleftarrow$ Suppose that the equation $x^2 - x\mu - \lambda = 0$ has a root $\alpha \in \pm 1 + J(R)$ and a root $\beta \in J(R)$. Then $\alpha^2 = \alpha\mu + \lambda; \beta^2 = \beta\mu + \lambda$. Hence,

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \beta - \alpha \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in GL_2(R).$$

Therefore $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in \pm 1 + J(R)$ and $\beta \in J(R)$. By virtue of Theorem 3.3, we complete the proof. $\square$

COROLLARY 4.2. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is Hirano polar if and only if*

(1) $A = N+W$, *or* $A = U+N+W, U \in comm^2(A), U^2 = I_2, N^2 = 0$ *and* $W \in M_2(J(R))$, *or*

(2) $x^2 - tr(A)x + det(A)$ *has a root* $\alpha \in \pm 1 + J(R)$ *and a root* $\beta \in J(R)$.

Proof. $\Longrightarrow$ By virtue of Theorem 4.1, we may assume that A is isomorphic to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$ and the equation $x^2 - \mu x - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$. Hence $\lambda = -det(A)$ and $\mu = tr(A)$, as desired.

$\Longleftarrow$ Case 1. A is Hirano polar.

Case 2. Since $det(A) = \alpha\beta \in J(R)$, we see that $A \notin GL_2(R)$. As $tr(A) = \alpha + \beta \in 1 + J(R)$, we have $det(I_2 - A) = 1 - tr(A) + det(A) \in J(R)$; hence, $I_2 - A \notin GL_2(R)$. In view of [12, Lemma 2.4], A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. Thus $\lambda = -det(A)$ and $tr(A) = u$, and so the equation $x^2 - \mu x - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$. Therefore A is Hirano polar by Theorem 4.1. $\square$

We note that $\pm 1 + J(R)$ can not be replaced by $U(R)$ in the preceding corollary, as the following shows.

EXAMPLE 4.3. Let $R = \{\frac{f}{g} \mid f, g \in \mathbb{Z}_2[t], g \neq 0\}$. Then R is a field with $J(R) = 0$. Let $A = \begin{pmatrix} 1 & 1+t \\ 1 & 1+t \end{pmatrix} \in M_2(R)$. Then $det(A) = 0$ and $tr(A) = t \in U(R)$. Hence, $x^2 - tr(A)x + det(A)$ has a root $tr(A) \in U(R)$ and a root $0 \in J(R)$. But $tr(A) \notin \pm 1 + J(R)$. If $A^2 \in M_2(J(R))$, then A is nilpotent, an absurd. If $A = U + W, U \in comm^2(A), U^2 = I_2, W^2 \in M_2(J(R))$, then $(I_2 - A^2)^2 = 0$. But $I_2 - A^2 = \begin{pmatrix} 1+t & t+t^2 \\ t & 1+t+t^2 \end{pmatrix}$, an absurd. Therefore A is not Hirano polar, by Corollary 4.2.

Let R be a commutative local ring, and let $A \in M_2(R)$. If A is Hirano polar, it follows from [6, Lemma 4.1] that $(A - A^3)^2 \in M_2(J(R))$. But we have

EXAMPLE 4.4. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. Then A is not Hirano polar, but $(A - A^3)^2 \in M_2(J(R))$.

Proof. Clearly, $J(\mathbb{Z}_{(2)}) = 2\mathbb{Z}_{(2)}, A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}, (I_2 - A^2)^2 = \begin{pmatrix} -6 & -10 \\ -15 & -21 \end{pmatrix}$. Thus the condition (1) in Corollary 4.2 does not satisfied. Moreover, $tr(A) = 5$ and $det(A) = -2$. Since $p(x) = x^2 - 5x - 2$ is irreducible in $\mathbb{Q}[x]$, we see that $x^2 - tr(A)x + det(A) = 0$ is no solvable in $\mathbb{Z}_{(2)}$, and so the condition (2) is Corollary 4.2 does not satisfied. Therefore A is not Hirano polar. But $A - A^3 = \begin{pmatrix} -36 & -52 \\ -78 & -114 \end{pmatrix} \in M_2(J(R))$, as required. $\square$

Evidently, Hirano polar matrices over a cobleached local ring R can be characterized by left roots of a polynomial over R. But a left root of polynomials in a ring need not be a right root. We now derive

THEOREM 4.5. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A is Hirano polar if and only if*

(1) $A \in M_2(R)^{qnil}$, or $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$, or

(2) A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - \mu x - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$.

Proof. $\Longrightarrow$ In view of Lemma 3.2, we have three cases. Case 1. $A = U + W$ where $U^2 = I_2, U \in comm^2(A), W^2 \in M_2(J(R))$. Case 2. $A^2 \in M_2(J(R))$. Case 3, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. It suffices to consider Case 3. In view of Theorem 3.3, there exists $U \in GL_2(R)$ such that

$$U^{-1} \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where $\alpha \in \pm 1 + J(R), \beta \in J(R)$. Let $\delta = s\alpha s^{-1}$ and $\gamma = t\beta t^{-1}$ Then $\delta \in \pm 1 + J(R), \gamma \in J(R)$. We easily check that $\delta^2 - \mu\delta = \lambda$ hence, $\delta^2 - \mu\delta - \lambda = 0$. Likewise, $\gamma^2 - \mu\gamma - \lambda = 0$. Therefore the equation $x^2 - \mu x - \lambda = 0$ has a root $\delta \in \pm 1 + J(R)$ and a root $\gamma \in J(R)$, as desired.

$\Longleftarrow$ Suppose that the equation $x^2 - \mu x - \lambda = 0$ has a root $\alpha \in \pm 1 + J(R)$ and a root $\beta \in J(R)$. As in the proof of Theorem 4.1, we prove that $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in \pm 1 + J(R)$ and a root $\beta \in J(R)$. In light of Theorem 3.3, we complete the proof. $\square$

With this information we can now extend the main results in [5] to a general local ring which may be not commutative (see [5, Theorem 4.9]).

COROLLARY 4.6. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A is J-quasipolar if and only if*

(1) $A \in M_2(J(R))$, or $I_2 + A \in M_2(J(R))$, or

(2) A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $-1 + J(R)$ and a root in $J(R)$.

Proof. $\Longrightarrow$ By hypothesis, there exists $E^2 = E \in comm^2(A)$ such that $A + E \in M_2(J(R))$. In view of Example 2.1, A is Hirano polar. By virtue of Theorem 4.5, we have three cases.

Case I. $A \in M_2(R)^{qnil}$. Then $(A + I_2) - (I_2 - E) \in M_2(J(R))$, and so $I_2 - E = I_2$. Hence $E = 0$, and so $A \in M_2(J(R))$.

Case II. $A = U + W, U \in comm^2(A), U^2 = I_2, W \in M_2(R)^{qnil}$. Then $A \in GL_2(R)$, and so $E = I_2$. This shows that $I_2 + A \in M_2(J(R))$.

Case III. A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$. If $x^2 - x\mu - \lambda = 0$ has a root in $\alpha \in 1 + J(R)$ and a root in $\beta \in J(R)$. As in the proof of Theorem 3.3, we see that $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is J-quasipolar. Hence, $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(J(R))$, and so $2 \in J(R)$. This implies that $\alpha \in -1 + J(R)$, as desired.

$\Longleftarrow$ If $A \in M_2(J(R))$, or $I_2 + A \in M_2(J(R))$, then A is J-quasipolar. Suppose that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $-1 + J(R)$ and a root in $J(R)$ Analogously to Theorem 3.3, we check that A is J-quasipolar, as asserted. $\square$

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