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ORTHOMAPS ON FORMALLY REAL SIMPLE JORDAN ALGEBRAS

GREGOR DOLINAR, BOJAN KUZMA AND NIK STOPAR

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Abstract. We characterize maps on finite-dimensional formally real simple Jordan algebras with the property $\phi(A\circ B)=\phi(A)\circ\phi(B)$ for all A,B. Although we do not assume additivity it turns out that every such map is either a real linear automorphism or a constant function. The main technique is a reduction to orthomaps, that is, maps which preserve zeros of Jordan product.

1. Introduction and statement of the results

A formally real Jordan algebra (FRJA for short) is an abelian (i.e., a commutative) algebra over the set of real numbers $\mathbb{R}$ with a Jordan product $\circ$ which satisfies a weak form of associativity

$$X^2\circ(Y\circ X)=(X^2\circ Y)\circ X \tag{1}$$

and a formally real identity

$$\sum_{i=1}^{k} X_i^2=0 \quad \text{if and only if} \quad X_1=X_2=\ldots=X_k=0. \tag{2}$$

FRJAs (originally called r-number algebras) were introduced by Pascuale Jordan in 1933 (see [11]) in an attempt to find the algebraic background for quantum mechanics. In this theory quantum observables play a fundamental role and in a mathematical formulation of quantum mechanic they are represented by (possibly unbounded) self-adjoint operators acting on a complex separable Hilbert space. The set $\mathscr{S}(H)$ of self-adjoint operators (i.e., Hermitian matrices in a finite-dimensional case) is a real vector space which is closed under squaring operation $A\mapsto A^2$. Jordan's insight was to linearize this operation and obtain a binary operation

$$A\circ B:=\tfrac{1}{2}(AB+BA) \tag{3}$$

which is now known as the Jordan product. Jordan then noticed that this product, which is clearly commutative, satisfies also the axioms (1)–(2) and as such makes the space of self-adjoint operators into a FRJA.

A FRJA is simple if it has no proper nontrivial ideals. It was shown by Wigner, Jordan, and von Neumann [10] that every finite-dimensional FRJA is a direct sum of simple ones, and every simple finite-dimensional FRJA belongs to one among the following families:

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(i) $H_n(\mathbb{R})$, the algebra of symmetric n-by-n matrices over reals,

(ii) $H_n(\mathbb{C})$, the algebra of hermitian n-by-n matrices over complex numbers,

(iii) $H_n(\mathbb{H})$, the algebra of hermitian n-by-n matrices over quaternions

(iv) $H_3(\mathbb{O})$, the (exceptional) algebra of 3-by-3 hermitian matrices over octonions,

(v) Spin_n $(n \geqslant 2)$, the algebra of spin factors,

where in all but the last family $n \geqslant 1$ and the product is given by (3). As for the spin factors, recall that Spin_n is obtained by formally adjoining the identity $(1,0)$ to the usual scalar product on $\mathbb{R}^n$. More precisely, Spin_n is an algebra $\mathbb{R}\times\mathbb{R}^n$ with a Jordan product

$$(\lambda,x)\circ(\mu,y) = (\lambda\mu + x^t y, \lambda y + \mu x).$$

Spin factor (λ,x) will be abbreviated as $\lambda + x$, further $(\lambda,0)$ will be written as λ and called scalar spin factor, and $(0,x)$ will be written as x. Note that the algebra $\mathrm{Spin}_1 = \mathbb{R}\times\mathbb{R}$ is not simple because it has an ideal generated by $(1,1)\in\mathrm{Spin}_1$. Let us remark that Spin_n can be embedded into $H_{2^n}(\mathbb{R})$ and also that 2-by-2 hermitian matrices are nothing but spin factors, i.e.

$$H_2(\mathbb{R}) = \mathrm{Spin}_2, \quad H_2(\mathbb{C}) = \mathrm{Spin}_3, \quad H_2(\mathbb{H}) = \mathrm{Spin}_5, \quad \text{and} \quad H_2(\mathbb{O}) = \mathrm{Spin}_9,$$

see, e.g., [5] and a book by McCrimmon [3, p. 47], where an interested reader can find more information about the historical background and developments in FRJAs.

Two elements A and B from a FRJA $\mathscr{A}$ are called Jordan-orthogonal if $A\circ B = 0$. This relation induces a simple graph $\Gamma(\mathscr{A})$ whose vertex set consists of nonzero elements in $\mathscr{A}$ and where two vertices are connected if and only if they are Jordan-orthogonal. Note that $\Gamma(\mathscr{A})$ has no loops since by (2), $A\circ A = 0$ implies $A = 0$. Recall that in a given graph a subset of vertices which are all connected to each other is called a clique. A clique of maximal possible cardinality is called a maximum clique. In the paper [4] we classified such cliques and we have also shown that maximum cliques in finite-dimensional simple FRJA have only finitely many elements. We refer the reader to [1] for a similar concept of an orthograph with respect to the usual associative product on matrix algebras.

Recall that an element P in a FRJA with $P^2 = P$ is called a projection. A nonzero projection P is minimal if it cannot be decomposed into a sum of two nonzero Jordan-orthogonal projections, that is, $P = P_1 + P_2$ with $P_i^2 = P_i$ and $P_1\circ P_2 = 0$ implies that $P_i \in \{0,P\}$. The rank of a FRJA is the cardinality of a maximal set consisting of pairwise Jordan-orthogonal minimal projections (such set is also known as a Jordan frame).

The main result of this paper is the classification of Jordan maps on a finite-dimensional simple FRJA, which are defined in the following way.

DEFINITION 1.1. Let $\mathscr{A}$ be a FRJA. A map $\phi:\mathscr{A}\to\mathscr{A}$ is called a Jordan map if

$$\phi(A\circ B) = \phi(A)\circ\phi(B); \qquad A,B\in\mathscr{A}.$$

Let us remark that unlike Jordan homomorphisms we assume no additional condition, like additivity, on ϕ. If in addition ϕ is bijective and linear over $\mathbb{R}$, then we call it a real linear (Jordan) automorphism.

For an element $A \in \mathscr{A}$, where $\mathscr{A} \in \{H_n(\mathbb{R}), H_n(\mathbb{C}), H_n(\mathbb{H}), H_3(\mathbb{O})\}$ for some $n \geqslant 1$, we denote by $\operatorname{tr}(A)$ the trace of A, i.e., the sum of its diagonal elements. Note that in all our cases $\operatorname{tr}(A)$ is a real number. We remark that trace is preserved by any real linear automorphism of $\mathscr{A}$. For $\mathscr{A} \in \{H_n(\mathbb{R}), H_n(\mathbb{C}), H_n(\mathbb{H})\}$ this follows easily from remark 1.4 below and for $\mathscr{A} = H_3(\mathbb{O})$ it is proved in [9, lemma 14.96].

Let us state our main result.

THEOREM 1.2. *Let $\mathscr{A}$ be a finite-dimensional formally real simple Jordan algebra with* $\dim \mathscr{A} \geqslant 3$. *Then every nonconstant Jordan map $\phi\colon \mathscr{A} \to \mathscr{A}$ is a real linear automorphism.*

REMARK 1.3. The assumption that $\dim \mathscr{A} \geqslant 4$ rules out the possibility $\mathscr{A} \simeq \mathrm{Spin}_2 \simeq H_2(\mathbb{R})$.

REMARK 1.4. The structure of real linear automorphisms in theorem 1.2 is as follows:

(i) If $\mathscr{A} = H_n(\mathbb{F})$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then $\phi(A) = UA^{\sigma}U^*$ for some unitary $U \in M_n(\mathbb{F})$ and some continuous automorphism σ of $\mathbb{F}$. Here σ is identity if $\mathbb{F} = \mathbb{R}$, σ is identity or conjugation if $\mathbb{F} = \mathbb{C}$ and σ is an inner automorphism if $\mathbb{F} = \mathbb{H}$ (see proof of proposition 1.5, Step 4).

(ii) If $\mathscr{A} = \mathrm{Spin}_n$, then $\phi(\lambda + x) = \lambda + Wx$ for some orthogonal matrix $W \in M_n(\mathbb{R})$ (see proof of theorem 1.2).

(iii) If $\mathscr{A} = H_3(\mathbb{O})$, then its automorphism group is isomorphic to exceptional Lie group F_4 of dimension 52 (see [9, definition 14.92]).

The proof of theorem 1.2 is a consequence of the following proposition.

PROPOSITION 1.5. *Let $\mathscr{A}$ be a finite-dimensional formally real simple Jordan algebra with* $\dim \mathscr{A} \geqslant 4$. *Assume $\phi\colon \mathscr{A} \to \mathscr{A}$ is a map which preserves zero products. Assume in addition that $\phi(X) = 0$ only if $X = 0$. Then there exists a real linear automorphism Φ of $\mathscr{A}$ and a real-valued function $\gamma\colon \mathscr{A} \to \mathbb{R} \setminus \{0\}$ such that*

$$\phi(R) = \gamma(R)\Phi(R) \qquad \textit{for all} \quad R \in \Omega,$$

where $\Omega \subseteq \mathscr{A}$ is the set of scalar multiples of minimal projections if $\operatorname{rank} \mathscr{A} \geqslant 3$, *and $\Omega \subseteq \mathscr{A}$ is the set of scalar multiples of nonscalar involutions if* $\operatorname{rank} \mathscr{A} = 2$.

REMARK 1.6. Unlike theorem 1.2, proposition 1.5 is wrong if $\dim \mathscr{A} = 3$, that is, if $\mathscr{A} = H_2(\mathbb{R}) = \mathrm{Spin}_2$. This is because each nonzero element from Spin_2 distinct from $\mathbb{R}(\|x\| \pm x) \cup \mathbb{R}(0 + x)$ is orthogonal only to 0, while the only nonzero elements from Spin_2 which are orthogonal to $(\|x\| \pm x)$ belong to $\mathbb{R}(\|x\| \mp x)$ and the only nonzero elements orthogonal to involution $\left(0+(\cos\theta,\sin\theta)^t\right)$ belong to $\mathbb{R}\big(0+(\cos(\theta+\pi/2),\sin(\theta+\pi/2))^t\big)$. So, for example, the map $\phi\colon\ \mathrm{Spin}_2\to\mathrm{Spin}_2$ which is identity outside the set of all scalar multiples of involutions and maps scalar multiples of involutions into scalar multiples of idempotents according to the rule

$$\begin{cases}\lambda\mapsto\lambda; & \lambda\in\mathbb{R}\\ \lambda\left(\cos\vartheta,\sin\vartheta\right)^t\mapsto\frac{1}{2}\left(1+(1,0)^t\right); & \lambda\in\mathbb{R}\backslash\{0\},\ \vartheta\in[0,\pi/2)\\ \lambda\left(\cos\vartheta,\sin\vartheta\right)^t\mapsto\frac{1}{2}\left(1-(1,0)^t\right); & \lambda\in\mathbb{R}\backslash\{0\},\ \vartheta\in[\pi/2,\pi)\end{cases}$$

preserves orthogonality, but it is easily seen that it is not of the form given in proposition 1.5.

REMARK 1.7. The proof of proposition 1.5 shows even more. In the case $\mathscr{A}=\mathrm{Spin}_n$, $n\geqslant 3$ every map ϕ which satisfies the assumptions of proposition 1.5 takes the form

$$\phi(x)=\gamma(x)Vx,\qquad x\in\mathbb{R}^n\subseteq\mathrm{Spin}_n, \tag{4}$$

for some orthogonal matrix $V\in M_n(\mathbb{R})$ and some real-valued function $\gamma\colon\ \mathbb{R}^n\subseteq\mathrm{Spin}_n\to\mathbb{R}\setminus\{0\}$.

REMARK 1.8. Maps satisfying assumptions from proposition 1.5 in general do not have a nice form outside the set Ω, unless some additional regularity is imposed. As an example consider, on $\mathscr{A}=\mathrm{Spin}_n$, the map $\phi\colon\ \mathscr{A}\to\mathscr{A}$ which maps elements from the set $\Xi:=\{\lambda+x;\ \ x\in\mathbb{R}^n,\lambda\in\mathbb{R}\setminus\{-\|x\|,0,\|x\|\}\}$ into 1 and is identity on $\mathscr{A}\setminus\Xi$. Such ϕ satisfies all the assumptions from proposition 1.5 because if $\alpha\notin\{-\|x\|,0,\|x\|\}$, then $\alpha+x\in\mathrm{Spin}_n$ is Jordan-orthogonal only to 0.

2. Proofs

Proposition 1.2 together with table 1, and lemmas 2.1, 3.2, and 3.5 from our paper [4] are indispensable to prove our main results. We will restate them here for convenience, but before doing that let us introduce some additional terminology.

Two elements A,B in a maximum clique are *co-cellular* if $\mathbb{R}A^2=\mathbb{R}B^2$. Notice that if A,B belong to one of the maximum cliques listed in table 1, then it is easy to see that they are co-cellular if and only if they are trace-zero and belong to the same 2-by-2 diagonal block. Given a subset $\Xi\subseteq\mathscr{A}$ we define $\mathbb{R}\Xi:=\{\lambda X;\ \lambda\in\mathbb{R},X\in\Xi\}$.

PROPOSITION 2.1. (see proposition 1.2 in [4]) *Let $\mathscr{A}$ be a finite-dimensional simple FRJA. If $\Xi\subseteq\mathscr{A}\setminus\{0\}$ consists of pairwise Jordan-orthogonal elements, then $|\Xi|\leqslant k_{\max}$, where integers $k_{\max}$ are given in table 1. The equality $|\Xi|=k_{\max}$ holds if and only if there exists an automorphism Φ of $\mathscr{A}$ such that $\mathbb{R}\Xi=\mathbb{R}\Phi(\mathscr{F})$, where the set $\mathscr{F}$ is a standard maximum clique defined and given in table 1.*

$\mathcal{A}$	$k_{\max}$	$\mathcal{F}$	Remark
$H_n(\mathbb{R})$	n	$\mathcal{F}_{n,k}(\mathbb{R}) = \bigcup_{i=1}^{k} \{D_i, F_i\} \cup \bigcup_{j=2k+1}^{n} \{E_{jj}\}$	$0 \le 2k \le n$
$H_n(\mathbb{C})$	$\frac{3n-1}{2}$	$\mathcal{F}_n(\mathbb{C}) = \bigcup_{i=1}^{(n-1)/2} \{D_i, F_i, G_i\} \cup \{E_{nn}\}$	n odd
	$\frac{3n}{2}$	$\mathcal{F}_n(\mathbb{C}) = \bigcup_{i=1}^{n/2} \{D_i, F_i, G_i\}$	n even
$H_n(\mathbb{H})$	$\frac{5n-3}{2}$	$\mathcal{F}_n(\mathbb{H}) = \bigcup_{i=1}^{(n-1)/2} \{D_i, F_i, G_i, J_i, K_i\} \cup \{E_{nn}\}$	n odd
	$\frac{5n}{2}$	$\mathcal{F}_n(\mathbb{H}) = \bigcup_{i=1}^{n/2} \{D_i, F_i, G_i, J_i, K_i\}$	n even
Spin_n	n	$\mathcal{F}_n(\mathrm{Spin}) = \{e_1, \ldots, e_n\}$	$n \ge 3$
	2	$\mathcal{F}_2(\mathrm{Spin}) = \{e_1, e_2\}$ or $\mathcal{F}'_2(\mathrm{Spin}) = \{(1 - e_1), (1 + e_1)\}$	$n = 2$
$H_3(\mathbb{O})$	10	$\mathcal{F}_3(\mathbb{O}) = \left\{ \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&\mathbf{e}_1&0\\\overline{\mathbf{e}_1}&0&0\\0&0&0 \end{pmatrix}, \ldots, \begin{pmatrix} 0&\mathbf{e}_7&0\\\overline{\mathbf{e}_7}&0&0\\0&0&0 \end{pmatrix} \right\}$	$\mathbf{e}_1, \ldots, \mathbf{e}_7 \in \mathbb{O}$ standard unit octonions

LEGEND:

$D_i = E_{(2i-1)(2i-1)} - E_{(2i)(2i)}$ $\quad$ $F_i = E_{(2i-1)(2i)} + E_{(2i)(2i-1)}$

$G_i = \mathbf{i}(E_{(2i-1)(2i)} - E_{(2i)(2i-1)})$ $\quad$ $J_i = \mathbf{j}(E_{(2i-1)(2i)} - E_{(2i)(2i-1)})$

$K_i = \mathbf{k}(E_{(2i-1)(2i)} - E_{(2i)(2i-1)})$ $\quad$ $e_i \in \mathbb{R}^n$ standard basis

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ standard quaternion imaginary units

Table 1

LEMMA 2.2. (see lemma 2.1 in [4]) *Let $n \geqslant 1$ and let $\Omega \subseteq \mathrm{Spin}_n$ be a maximal subset which consists of nonzero pairwise Jordan-orthogonal elements. Then one of the following occurs:*

(i) $\Omega = \{\lambda\}$, $\lambda \in \mathbb{R} \setminus \{0\}$.

(ii) $\Omega = \{\lambda(\|x\| + x), \mu(-\|x\| + x)\}$, $x \in \mathbb{R}^n$ *and* $\lambda, \mu \in \mathbb{R} \setminus \{0\}$.

(iii) $\Omega = \{x_1, \ldots, x_n\}$ *for pairwise orthogonal nonzero vectors* $x_i \in \mathbb{R}^n$.

Hence $|\Omega| \leqslant n$, and for $n \geqslant 3$ the equality holds if and only if $\Omega \subseteq \mathbb{R}^n$.

LEMMA 2.3. (see lemma 3.2 in [4]) *Let $\Gamma = \Gamma(\mathscr{A})$ be an orthograph of a finite-dimensional simple FRJA $\mathscr{A}$ with $\omega(\Gamma) \geqslant 3$. Then, the following statements are equivalent:*

(i) $\mathscr{A} = H_{k_0}(\mathbb{R})$ *or* $\mathscr{A} = \mathrm{Spin}_{k_0}$ *for some* $k_0 \geqslant 3$.

(ii) There exists a vertex $A \in \Gamma$ together with a maximum clique $\mathscr{F} = \{A, V_2, \ldots, V_{k_0}\} \subseteq \Gamma$ for which the following holds: For every permutation σ on $\{2, \ldots, k_0\}$ one can find four vertices V_{13}, V_{23}, V'_{13}, V'_{23} such that the sets $\mathscr{F}' = \{V_{13}, V'_{13}, V_{\sigma(2)}, V_{\sigma(4)} \ldots, V_{\sigma(k_0)}\}$ and $\mathscr{F}'' = \{A, V_{23}, V'_{23}, V_{\sigma(4)} \ldots, V_{\sigma(k_0)}\}$ are maximum cliques and such that for every choice of $\{U, U'\} = \{V_{13}, V'_{13}\}$, $\{T, T'\} = \{V_{23}, V'_{23}\}$ one can find three additional vertices W, W', Z with the following two properties:

(a) we have paths $Z - V_{\sigma(3)}$, $U - W - T$, $U' - W' - T'$, *and* $W - Z - W'$.
(b) $\{X, V_{\sigma(4)}, \ldots, V_{\sigma(k_0)}\}$ *is a clique for every* $X \in \{W, W', Z\}$.

Moreover, if $\mathscr{A} = H_{k_0}(\mathbb{R})$, *then a vertex* $A \in \Gamma$ *satisfies* $\operatorname{rank} A = 1$ *if and only if* A *is a member of a maximum clique* $\mathscr{F}$ *with the properties stated in (ii).*

LEMMA 2.4. (see lemma 3.5 in [4]) *Let* Γ *be an orthograph of a finite-dimensional simple FRJA* $\mathscr{A}$ *with* $n = \omega(\Gamma) \geqslant 3$ *such that no vertex in* Γ *satisfies the assumption (ii) of lemma 2.3. Assume two vertices* A, B *belong to the same maximum clique. Then the following statements are equivalent:*

(i) A *and* B *are co-cellular.*

(ii) There exists a vertex C *together with vertices* $V_4, \ldots, V_n$ *such that* $\mathscr{F} = \{A, B, C, V_4, \ldots, V_n\}$ *is a maximum clique with the following property: there exist four vertices* V_{13}, V'_{13} *and* V_{23}, V'_{23}, *such that* $\mathscr{F}' = \{V_{13}, V'_{13}, C, V_4, \ldots, V_n\}$ *and* $\mathscr{F}'' = \{V_{23}, V'_{23}, B, V_4, \ldots, V_n\}$ *are maximum cliques and for every choice of* $\{U, U'\} = \{V_{13}, V'_{13}\}$ *and* $\{T, T'\} = \{V_{23}, V'_{23}\}$ *one can find three additional vertices* W, W' *and* Z *which form paths*

$$U - W - T, \quad U' - W' - T', \quad W - Z - W', \quad Z - A$$

and for which $\{X, V_4, \ldots, V_n\}$ *is a clique for every* $X \in \{W, W', Z\}$.

Let us continue with some additional notation for the next lemma. The Jordan annihilator of an element A in a formally real Jordan algebra $\mathscr{A}$ is a real vector space defined by

$$A^{\#} = \{X \in \mathscr{A};\ A \circ X = 0\}.$$

Given two hermitian matrices $A \in H_n(\mathbb{F})$ and $B \in H_m(\mathbb{F})$ ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$) we denote the block-diagonal matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in H_{n+m}(\mathbb{F})$ shortly as $A \oplus B$. This notation clearly extends to more than two factors and to sets in place of matrices. As an example,

$$0_2 \oplus H_2(\mathbb{F}) \oplus 0_{n-4} = \left\{ \begin{pmatrix} 0_2 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0_{n-4} \end{pmatrix};\ A \in H_2(\mathbb{F}) \right\} \subseteq H_n(\mathbb{F}),$$

where 0_k denotes the zero matrix of size k. Similarly, I_k will denote the identity matrix of size k.

LEMMA 2.5. *Let* $\mathscr{A} \in \{H_n(\mathbb{F});\ n = 2k \geqslant 4,\ \mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}\}$ *and let* $\mathscr{F} = \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{n/2} \subseteq \Gamma(\mathscr{A})$ *be a partition of a standard maximum clique listed in table 1 into its co-cellular elements. Then, for* $i = 1, \ldots, n/2$,

$$\bigcap_{S \in \mathscr{F} \setminus \mathscr{F}_i} S^{\#} = 0_{2(i-1)} \oplus H_2(\mathbb{F}) \oplus 0_{n-2i} \quad \text{and} \quad \bigcap_{S \in \mathscr{F} \setminus (\mathscr{F}_1 \cup \mathscr{F}_2)} S^{\#} = H_4(\mathbb{F}) \oplus 0_{n-4}.$$

Proof. The proof is similar to the proof of [4, lemma 3.1]. Borrowing notation from table 1 we have that $\mathscr{F}_j = \{D_j, F_j, \ldots\}$ contains all co-cellular elements of trace-zero. Now, in the algebra $\mathscr{A} = H_n(\mathbb{F}) = H_{2k}(\mathbb{F})$, (with $\mathbb{F} = \{\mathbb{C}, \mathbb{H}\}$), we easily compute that

$$D_j^{\#} = \{H_{2(j-1)}(\mathbb{F}) \oplus \left(\begin{smallmatrix} 0 & \alpha \\ \alpha & 0 \end{smallmatrix}\right) \oplus H_{n-2j}(\mathbb{F});\ \alpha \in \mathbb{F}\}. \tag{5}$$

From table 1 we infer that, besides D_j, the set $\mathscr{F}_j$ contains exactly $\dim_{\mathbb{R}} \mathbb{F} \in \{2,4\}$ elements. Since they are pairwise Jordan-orthogonal, they are linearly independent over $\mathbb{R}$ and hence the set $\mathscr{F}_j$ forms a basis for the subspace $0_{2(j-1)} \oplus H_2(\mathbb{F}) \oplus 0_{n-2j} \subseteq H_{2k}(\mathbb{F})$. Together with (5) this implies that

$$\bigcap_{S \in \mathscr{F}_j} S^{\#} = H_{2(j-1)}(\mathbb{F}) \oplus 0_2 \oplus H_{n-2j}(\mathbb{F}) \qquad j \in \{1,\ldots,n/2\} \setminus \{i\}.$$

Then, $\bigcap_{S \in \mathscr{F} \setminus \mathscr{F}_i} S^{\#} = \bigcap_{j \neq i} (\bigcap_{S \in \mathscr{F}_j} S^{\#})$ is the intersection of the above sets and the first identity of lemma follows. The second identity also follows with a trivial modification $j \in \{1,\ldots,n/2\} \setminus \{1,2\}$ instead of $j \neq i$. $\square$

Proof of proposition 1.5. We divide the proof into several steps.

Step 1. ϕ induces a graph homomorphism on orthograph $\Gamma = \Gamma(\mathscr{A})$. Namely, by the assumption that $\phi(X) = 0$ implies $X = 0$ we see that ϕ leaves the vertex set of Γ, that is, the set $V(\Gamma) = \mathscr{A} \setminus \{0\}$ invariant. If $A, B \in \Gamma$ are connected, then $A \circ B = 0$ so by the assumptions also $\phi(A) \circ \phi(B) = 0$. Since $\phi(A), \phi(B) \neq 0$ we see from the axiom (2) of FRJA that $\phi(B)$ and $\phi(A)$ must be different but connected vertices in Γ, as claimed.

Step 2. Assume $\mathscr{A} \neq \mathrm{Spin}_n$. Then, ϕ maps minimal projections into scalar multiples of minimal projections.

To see this let $A \in \mathscr{A}$ be a minimal projection. If $\mathscr{A} = H_n(\mathbb{R})$ then, by (ii) of lemma 2.3, $\{A\}$ can be enlarged to a maximum clique $\mathscr{F}$ with the properties stated in (ii) of lemma 2.3. Note that, by Step 1 ϕ is a homomorphism of the orthograph so it maps vertices from a clique injectively into a clique. As such, it maps maximum cliques onto maximum cliques. It is now easily seen that the properties (ii) of lemma 2.3 are inherited by graph homomorphism ϕ, so by lemma 2.3, $\phi(A)$ is a scalar multiple of a minimal projection.

Suppose next $\mathscr{A} \in \{H_n(\mathbb{C}), H_n(\mathbb{H}), H_3(\mathbb{O})\}$ with n odd. By applying a suitable automorphism of Jordan algebra we may assume $A = E_{nn}$ (in the case of complex or quaternionic Hermitian matrices the existence of automorphism follows from Brenner's paper [2, theorem 2] and the fact that an upper-triangular Hermitian matrix is diagonal, while in the case of octonions this is Freudenthal's theorem, see, e.g., [6, theorem V.2.5, p. 90]). Hence, A is a part of a standard maximum clique $\mathscr{F}$ listed in table 1 and is the only vertex which is not co-cellular to some other vertex from $\mathscr{F}$. Recall that ϕ is a homomorphism of orthograph, so it is injective on maximum cliques. It easily follows that item (ii) of lemma 2.4 is inherited by graph homomorphisms, and so co-cellular vertices from $\mathscr{F}$ are mapped into co-cellular vertices in a maximum clique $\phi(\mathscr{F})$. Since, by table 1 maximum clique $\phi(\mathscr{F})$ has finitely many vertices and $\phi|_{\mathscr{F}}$ is injective, hence bijective, it follows that $X, Y \in \mathscr{F}$ are co-cellular if and only if $\phi(X), \phi(Y) \in \phi(\mathscr{F})$ are. As $A = E_{nn} \in \mathscr{F}$ is not co-cellular to any vertex from $\mathscr{F} \setminus \{A\}$ the same must hold also for $\phi(A)$. From proposition 2.1 and the fact that each maximum clique equals the standard clique modulo automorphism of $\mathscr{A}$ and modulo scalar multiples, we deduce that $\phi(A)$ is a scalar multiple of a minimal projection.

Suppose lastly $\mathscr{A} \in \{H_n(\mathbb{C}), H_n(\mathbb{H})\}$ with even $n \geqslant 4$. Let $B \in \mathscr{A}$ be a minimal projection, orthogonal to A and let $Z = A - B \in \mathscr{A}$. Then there exists a unitary matrix U such that $(UAU^*, UBU^*) = (E_{11}, E_{22})$ (for existence of U in quaternionic case we again refer to Brenner [2], but see also [12, theorem 5.3.6]). By replacing ϕ with $\phi(U\,.\,U^*)$ we may assume from the start that $Z = E_{11} - E_{22}$. It follows that Z belongs to a standard maximum clique $\mathscr{F}$ and, conversely, since n is even, every member of a standard maximum clique is a difference of two orthogonal minimal projections. By Step 1, ϕ maps maximum cliques onto maximum cliques, and we infer that, modulo a scalar multiple, $\phi(Z)$ is also a difference of two Jordan-orthogonal minimal projections.

Let $\mathscr{F} = \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{n/2}$ be a partition of a standard clique into co-cellular elements. Since ϕ induces a graph homomorphism, which by lemma 2.4 preserves co-cellularity, we may assume, after composing ϕ with a suitable unitary matrix, that $\mathbb{R}\phi(\mathscr{F}_i) = \mathbb{R}\mathscr{F}_i$. Now, $A = E_{11}$ is Jordan-orthogonal to every member from $\mathscr{F}_2 \cup \ldots \cup \mathscr{F}_{n/2} = \mathscr{F} \setminus \mathscr{F}_1$ and the same is true for $\phi(E_{11})$. So $\phi(E_{11}) \in \cap_{S \in \mathscr{F} \setminus \mathscr{F}_1} S^\#$. By lemma 2.5 we easily deduce that

$$\phi(A) = \phi(E_{11}) \in (H_2(\mathbb{F}) \oplus 0_2) \oplus 0_{n-4}, \qquad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{H}, \text{ as required.} \tag{6}$$

Similarly,

$$\phi(E_{44}) \in (0_2 \oplus H_2(\mathbb{F})) \oplus 0_{n-4}. \tag{7}$$

Moreover, let $\{X_1, \ldots, X_k\} \subseteq (0 \oplus H_2(\mathbb{F}) \oplus 0) \oplus 0_{n-4}$, $k \in \{3, 5\}$ be the maximal set of co-cellular elements, each a difference of two Jordan-orthogonal minimal projections. The same holds for their ϕ-images, modulo scalar multiplication, and since each X_i is Jordan-orthogonal to $\mathscr{F} \setminus (\mathscr{F}_1 \cup \mathscr{F}_2)$ we easily deduce that $\phi(X_i) \in H_4(\mathbb{F}) \oplus 0_{n-4}$. Being co-cellular, there exists a unitary similarity $V = V_1 \oplus I_{n-4} \in H_4(\mathbb{F}) \oplus I_{n-4}$ such that $V^*\phi(X_i)V \in H_2(\mathbb{F}) \oplus 0_2 \oplus 0_{n-4}$. Then, by (6)–(7) and as E_{11}, E_{44} are Jordan-orthogonal to every X_i,

$$V^*\phi(E_{11})V, \quad V^*\phi(E_{44})V \in 0_2 \oplus H_2(\mathbb{F}) \oplus 0_{n-4}.$$

Moreover, $V^*\phi(E_{11})V$ and $V^*\phi(E_{44})V$ are Jordan-orthogonal but not co-cellular, because by (6)–(7) $\mathbb{R}\phi(E_{11})^2 \neq \mathbb{R}\phi(E_{44})^2$. Since $V^*\phi(E_{11})V$ can be diagonalized inside $0_2 \oplus H_2(\mathbb{F}) \oplus 0_{n-4}$, it easily follows that both $\phi(A) = \phi(E_{11})$ and $\phi(E_{44})$ are scalar multiples of minimal projections, as claimed.

Step 3. $\phi(\mathbb{R}P) \subseteq \mathbb{R}\phi(P)$ for every minimal projection P.

Again we may assume $P = E_{11} = \frac{1}{\lambda_1}\phi(P)$ for suitable nonzero $\lambda_1 \in \mathbb{R}$. By applying unitary similarities in succession we can achieve that $\phi(E_{ii}) = \lambda_i E_{ii}$ for every $i = 1, \ldots, n$ (in case $\mathscr{A} = H_3(\mathbb{O})$ we use [5, lemma 3.1]). Then, $\phi(\alpha P) = \phi(\alpha E_{11})$ is Jordan-orthogonal to $\phi(E_{ii}) = \lambda_i E_{ii}$ for every $i = 2, \ldots, n$. Note that $\lambda_i \in \mathbb{R} \setminus \{0\}$ so, it commutes with every quaternion and every octonion. Therefore, writing $\phi(\alpha E_{11}) = \sum_{kl} \gamma_{kl} E_{kl}$ one gets (see equation (3)) that $\phi(\alpha E_{11}) \circ (\lambda_i E_{ii}) = 0$ implies $\gamma_{ik} = \gamma_{ki} = 0$ for $k = 1, \ldots, n$. Since this holds with $i = 2, \ldots, n$ the claim follows.

Step 4. By Step 2, ϕ induces a well-defined map φ on the set of minimal projections $\mathscr{P}$, by

$$\varphi(P) = \frac{1}{\operatorname{tr}(\phi(P))}\phi(P).$$

We identify $\mathscr{P}$ with a projective space (in case $\mathscr{A} \in \{H_n(\mathbb{R}), H_n(\mathbb{C}), H_n(\mathbb{H})\}$ this is done by $P \mapsto \operatorname{Im}(P) \in \mathbb{P}(\mathbb{F}^n)$ with the inverse $[x] = \mathbb{F}x \mapsto \frac{1}{\|x\|^2}xx^*$, in case $\mathscr{A} = H_3(\mathbb{O})$ the set $\mathscr{P}$ *is a projective space*). By [5, lemma 1.1] we see that in case $\mathscr{A} \neq H_3(\mathbb{O})$ we have $(xx^*)\circ(yy^*) = 0 \in H_n(\mathbb{F})$ if and only if $x^*y = 0 \in \mathbb{F}$, i.e. $x \perp y$, while in case $\mathscr{A} = H_3(\mathbb{O})$ the orthogonality is defined by $P \perp Q$ if $\operatorname{tr}(P\circ Q) = 0$ which, by [5, lemma 3.2] is again equivalent to $P\circ Q = 0$. Since ϕ preserves zeros of Jordan product, the induced map φ is hence an orthomap (i.e. $P\perp Q$ implies $\varphi(P)\perp\varphi(Q)$). Such maps were classified in [5, theorems 1.3–1.4] where it was shown that they are induced by a real linear automorphism of Jordan algebra $\mathscr{A}$, whose structure is described in remark 1.4 (i). More precisely, there exists a real linear Jordan automorphism $\Phi\colon \mathscr{A} \to \mathscr{A}$ such that $\varphi(P) = \Phi(P)$. By Step 3 we then have

$$\phi(\alpha P) = \gamma(\alpha P)\Phi(\alpha P); \qquad \alpha \in \mathbb{R}\setminus\{0\} \tag{8}$$

for some real-valued function $\gamma\colon (\mathbb{R}\setminus\{0\})\mathscr{P} \to \mathbb{R}\setminus\{0\}$. The equation (8) is valid also for $\alpha = 0$ because $0\circ 0 = 0$ implies $\phi(0)\circ\phi(0) = 0$ and thus $\phi(0) = 0$ by the axiom (2). This proves the proposition in case $\mathscr{A} \neq \mathrm{Spin}_n$.

Step 5. Lastly, let $\mathscr{A} = \mathrm{Spin}_n$. Choose any nonzero $x \in \mathbb{R}^n \subseteq \mathrm{Spin}_n$. We can find a set of $n-1$ pairwise Jordan-orthogonal nonzero spin factors $\{x_2,\dots,x_n\}$ which are also Jordan-orthogonal to x, see item (iii) in lemma 2.2. By the assumptions $\phi(x)$ is nonzero and also Jordan-orthogonal to $n-1$ nonzero spin factors $\{\phi(x_2),\dots,\phi(x_n)\}$, which are moreover pairwise Jordan-orthogonal to each other. Since $n \geqslant 3$, lemma 2.2 implies $\phi(x) \in \mathbb{R}^n$.

If $\alpha \in \mathbb{R}$ is nonzero, then $\alpha x \in \mathrm{Spin}_n$ is again Jordan-orthogonal to $\{x_2,\dots,x_n\}$. Hence $\phi(\alpha x)$ is Jordan-orthogonal to $\{\phi(x_2),\dots,\phi(x_n)\}$. Recall that Jordan--orthogonality in $\mathbb{R}^n \subseteq \mathrm{Spin}_n$ coincides with orthogonality with respect to the usual scalar product in $\mathbb{R}^n$. This implies that $\phi(\alpha x) \in \mathbb{R}\phi(x)$ and therefore ϕ induces a map on $\mathbb{P}(\mathbb{R}^n)$ which clearly preserves orthogonality of elements in $\mathbb{P}(\mathbb{R}^n)$. By [5, theorem 1.3.] (see also [7, theorem 4.1]) we obtain that

$$\phi(x) = \gamma_x V x \tag{9}$$

for some orthogonal matrix V and some nonzero scalar $\gamma_x \in \mathbb{R}$ which depends on x. As in the concluding arguments in Step 4 we see that $\phi(0) = 0$ so (9) holds also for $x = 0$. Hence, $\phi(x) = \gamma_x\Phi(x)$ for all $x \in \mathbb{R}^n$, where $\Phi(\lambda + x) = \lambda + Vx$ is a real linear automorphism of Spin_n. $\square$

2.1. Proof of theorem 1.2

For $\mathscr{A} = H_n(\mathbb{C})$, $n \geqslant 3$, a version of theorem 1.2 for the product $A\circ B = AB + BA$ was proved in [8, proposition 5.2]. The same arguments, with obvious modifications like $I_k \oplus (2B)$ in place of $\frac{1}{2}I_k \oplus B$, can be used also for the product $A\circ B = \frac{1}{2}(AB+BA)$

and in the case of $\mathscr{A} \in \{H_n(\mathbb{R}), H_n(\mathbb{C}), H_n(\mathbb{H}), H_3(\mathbb{O}); n \geqslant 3\}$ with the following additional modifications for $H_3(\mathbb{O})$: "rank" should be replaced by "a number of nonzero eigenvalues counted with their multiplicities" and unitary similarity by Freudenthal's theorem (possibly composed with a similarity by permutation matrix; this similarity is an automorphism of $H_3(\mathbb{O})$ and permutes diagonal entries). Note that by Freudenthal's theorem, for each $A \in H_3(\mathbb{O})$ there exists a Jordan automorphism Φ such that $\Phi(A)$ is diagonal and so a sum of nonzero scalar multiples of $r \in \{0,1,2,3\}$ pairwise orthogonal minimal projections. Thus, "rank" can also be replaced by this integer r.

It remains to prove the theorem for spin factors. Recall that $\mathrm{Spin}_0 = \mathbb{R}$. Throughout the rest the paper it is meant that $n \geqslant 2$, unless explicitly stated otherwise.

LEMMA 2.6. *Suppose* $\phi : \mathrm{Spin}_n \to \mathrm{Spin}_k$, $0 \leqslant k \leqslant n$, $n \geqslant 2$, *is a nonzero Jordan map. Then* $\phi(X) = 0$ *implies* $X = 0$.

Proof. Suppose $\phi(\alpha + x) = 0$. If $\alpha = 0$ and $x \neq 0$, then $\phi(1) = \phi((0+x) \circ (0 + \|x\|^{-2}x)) = 0$, which implies that ϕ is zero. If $\alpha \neq 0$ and $x \neq 0$, then there exists nonzero $y \in \mathbb{R}^n \subseteq \mathrm{Spin}_n$ such that $y^t x = 0$, giving $\phi(0+\alpha y) = \phi((\alpha + x) \circ (0+y)) = 0$. By the previous step ϕ is zero. If $\alpha \neq 0$ and $x = 0$, then $\phi(1) = \phi((\alpha + 0) \circ (\alpha^{-1} + 0)) = 0$ and ϕ is again zero. $\square$

LEMMA 2.7. *Suppose a Jordan map* $\phi\colon \mathrm{Spin}_n \to \mathrm{Spin}_n$, $n \geqslant 2$, *satisfies* $\phi(0) \neq 0$. *Then* ϕ *is a constant map.*

Proof. Since 0 is a projection the same holds for $P := \phi(0)$. Decompose $P = \lambda + e$ for appropriate $\lambda \in \mathbb{R}$ and $e \in \mathbb{R}^n$; then $(\lambda + e) = P = P^2 = \lambda^2 + \|e\|^2 + 2\lambda e$. Hence,

$$P = 1 \qquad \text{or} \qquad P = \tfrac{1}{2}(1+e) \quad \text{with} \quad \|e\| = 1.$$

Take any $X \in \mathrm{Spin}_n$. Then,

$$P = \phi(0) = \phi(0 \circ X) = \phi(0) \circ \phi(X) = P \circ \phi(X).$$

If $P = 1$ we immediately get $\phi(X) = 1 = const$. Otherwise write $\phi(X) = (\lambda + y)$ and compute $\frac{1}{2}(1+e) = P = P \circ \phi(X) = \frac{1}{2}(1+e) \circ (\lambda + y) = \frac{1}{2}(\lambda + e^t y + \lambda e + y)$. This is equivalent to

$$\lambda + e^t y = 1 \quad \text{and} \quad \lambda e + y = e,$$

that is, $y = (1-\lambda)e$ and so $\phi(X) = (\lambda + y) \in 1 + \mathbb{R}(1-P)$. Thus, we may write

$$\phi(X) = 1 + \psi(X)(1-P); \qquad \psi\colon \mathrm{Spin}_n \to \mathbb{R}.$$

An easy computation reveals that

$$\big(1 + \psi(X \circ Y)\big) = \big(1 + \psi(X)\big)\big(1 + \psi(Y)\big),$$

and as such, $\psi' := (1+\psi)\colon \mathrm{Spin}_n \to \mathbb{R}$ is a Jordan morphism. Clearly, $\psi'(0) = 1 + \psi(0) = 1 - 1 = 0$. By lemma 2.6 with $k = 0$ we obtain that $\psi'(X) = 0$ for some

$X \neq 0$ implies $\psi' = 0$. Therefore, if $\psi' \neq 0$, then it induces a homomorphism between orthographs of Spin_n and $\mathrm{Spin}_0 = \mathbb{R}$. Hence $\psi' \neq 0$ would map a maximum clique in Spin_n bijectively to a maximum clique in $\mathbb{R}$, which consists of only a single element. This contradicts the assumption $n \geqslant 2$. So, $\psi' = 0$ and therefore $\operatorname{Im}\phi = \{P\}$. $\square$

Proof of theorem 1.2 for Spin_2. By lemmas 2.6–2.7 we have $\phi(X) = 0$ if and only if $X = 0$. Let $x \in \mathbb{R}^2$ with $\|x\| = 1$ and assume that $\phi(x) \notin \mathbb{R}^2$. Chose a nonzero $y \in \mathbb{R}^2$ orthogonal to x. Then $\phi(x) \circ \phi(y) = \phi(0) = 0$, so by lemma 2.2 there exist nonzero $\alpha, \beta \in \mathbb{R}$ and a nonzero $v \in \mathbb{R}^2$, such that $\phi(x) = \alpha(\|v\| + v)$ and $\phi(y) = \beta(-\|v\| + v)$. This implies

$$\gamma = \beta\phi(x) - \alpha\phi(y) \in \mathbb{R}. \tag{10}$$

Note that $(0+x)\circ(1+x) = (1+x)$ and $(0+y)\circ(1+x) = (0+y)$. Applying ϕ we obtain

$$\phi(x)\circ\phi(1+x) = \phi(1+x) \qquad \text{and} \qquad \phi(y)\circ\phi(1+x) = \phi(y). \tag{11}$$

In view of (10), multiplying the first equation by β, the second by α, and subtracting the two gives $\gamma\phi(1+x) = \beta\phi(1+x) - \alpha\phi(y)$ or equivalently

$$\alpha\phi(y) = (\beta - \gamma)\phi(1+x).$$

Jordan multiplying this equation by $\phi(x)$, the left side becomes 0 while the right remains the same by (11). We conclude that $\alpha\phi(y) = 0$, a contradiction. This shows that $\phi(x) \in \mathbb{R}^2$. Now let $x \in \mathbb{R}^2$ be an arbitrary nonzero vector. By the above $\phi(\frac{x}{\|x\|}) \in \mathbb{R}^2$. Choose a unit $y \in \mathbb{R}^2$ orthogonal to x. Then $\phi(x)\circ\phi(y) = \phi(\frac{x}{\|x\|})\circ\phi(y) = 0$, which means $\phi(x) \in \mathbb{R}\phi(\frac{x}{\|x\|}) \subseteq \mathbb{R}^2$ by lemma 2.2.

By replacing ϕ with $\tilde{V}\phi$ for a suitable Jordan automorphism $\tilde{V}\colon \lambda + x \mapsto \lambda + Vx$ induced by the orthogonal matrix V we can assume that

$$\phi(e_i) = \lambda_i e_i \quad \text{ for } i = 1, 2.$$

Next, $\phi(1) = \phi(e_1^2) = \phi(e_1)^2 = (\lambda_1 e_1)^2 \in \mathbb{R}$ gives $\phi(1) = 1$. Thus, $1 = \phi(1) = \phi(e_i \circ e_i) = \lambda_i^2 e_i \circ e_i$ so that $\lambda_1^2 = \lambda_2^2 = 1$, and thus $\lambda_1, \lambda_2 = \pm 1$. By further composing ϕ with a Jordan automorphism induced by the orthogonal matrix $\operatorname{diag}(\lambda_1, \lambda_2)$ we can achieve that

$$\phi(e_i) = e_i; \qquad i = 1, 2.$$

Next, $1 = \phi(1) = \phi(e_1 \circ (e_1 + \alpha e_2)) = e_1 \circ \phi(e_1 + \alpha e_2)$ gives

$$\phi(e_1 + \alpha e_2) = (e_1 + \mu_\alpha e_2). \tag{12}$$

In particular,

$$\phi(\alpha) = \phi(e_2)\circ\phi(e_1 + \alpha e_2) = \mu_\alpha \in \mathbb{R}; \qquad \alpha \in \mathbb{R}. \tag{13}$$

Clearly then, the restriction $\phi|_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is multiplicative. Due to $(e_1 + \alpha e_2)\circ(e_1 + \beta e_2) = 1 + \alpha\beta$ we now get

$$\phi(1 + \alpha\beta) = (e_1 + \mu_\alpha e_2)\circ(e_1 + \mu_\beta e_2) = 1 + \mu_\alpha\mu_\beta = 1 + \phi(\alpha)\phi(\beta) = 1 + \phi(\alpha\beta).$$

Hence, by multiplicativity, $\phi(\alpha)\phi(\alpha^{-1}+\beta)=\phi(1+\alpha\beta)=1+\phi(\alpha)\phi(\beta)$ $=\phi(\alpha)\big(\phi(\alpha)^{-1}+\phi(\beta)\big)$ so that, with $\gamma=\alpha^{-1}$ and $\delta=\beta$,

$$\phi(\gamma+\delta)=\phi(\gamma)+\phi(\delta);\qquad \gamma,\delta\in\mathbb{R}.$$

Clearly, this holds also if $\gamma=0$, so a multiplicative $\phi|_{\mathbb{R}}$ is also additive. It is well-known that this implies $\phi|_{\mathbb{R}}=\mathrm{id}$. Going back to (12)–(13) we now find $\phi(e_1+\alpha e_2)=e_1+\alpha e_2$. Then

$$\phi(\beta e_1+\alpha e_2)=\phi(\beta)\circ\phi(e_1+\beta^{-1}\alpha e_2)=\beta(e_1+\beta^{-1}\alpha e_2)=\beta e_1+\alpha e_2$$

which clearly holds even if $\beta=0$. That is, $\phi(x)=x$ for every $x=(0+x)\in\mathbb{R}^2$.

It only remains to show $\phi(\lambda+x)=\lambda+x$ for each nonzero $\lambda\in\mathbb{R}$ and each nonzero $x\in\mathbb{R}^2$. To this end pick $y\in\mathbb{R}^2$ orthogonal to x. Then, applying ϕ on $(\lambda+x)\circ y=\lambda y$ gives $\phi(\lambda+x)\circ y=\lambda y$. This is possible only if

$$\phi(\lambda+x)=\lambda+\zeta x$$

for some $\zeta\in\mathbb{R}$. From here, applying ϕ on $(\lambda+x)\circ x=\|x\|^2+\lambda x$ gives

$$\begin{aligned}\zeta\|x\|^2+\lambda x=(\lambda+\zeta x)\circ x&=\phi(\lambda+x)\circ\phi(x)=\phi((\lambda+x)\circ x)\\&=\phi(\|x\|^2+\lambda x)=\|x\|^2+\xi(\lambda x).\end{aligned}$$

$\xi\in\mathbb{R}$ and so $\zeta=1$ (and $\lambda x=\xi(\lambda x)$, i.e., $\xi=1$). $\square$

Proof of theorem 1.2 *for* Spin_n, $n\geqslant 3$. Suppose ϕ is nonconstant. Then, by lemmas 2.6–2.7 it satisfies all the assumptions of proposition 1.5. By remark 1.7 then $\phi(x)=\gamma(x)Vx$ for every $x\in\mathbb{R}^n\subseteq\mathrm{Spin}_n$, where $\gamma\colon\ \mathrm{Spin}_n\to\mathbb{R}$ and V is an orthogonal matrix.

Let $x,y\in\mathbb{R}^n$ be arbitrary linearly independent nonorthogonal vectors. Let

$$\lambda=(0+x)\circ(0+y)=y^tx\neq 0.$$

Then $\phi(\lambda)=\phi(x)\circ\phi(y)=\gamma(x)\gamma(y)(Vy)^tVx=\gamma(x)\gamma(y)y^tx=\gamma(x)\gamma(y)\lambda$. Since linear functionals on $\mathbb{R}^n$ are induced by row vectors w^t (with $w\in\mathbb{R}^n$) via $x\mapsto w^tx$, and since x,y are linearly independent, then for $\lambda=y^tx\in\mathbb{R}$ there exists $z\in\mathbb{R}^n$, such that

$$z^tx=z^ty=\lambda.$$

Hence $\gamma(x)\gamma(y)\lambda=\gamma(z)\gamma(x)\lambda=\gamma(z)\gamma(y)\lambda$ and therefore $\gamma(x)=\gamma(y)$. If $x,y\in\mathbb{R}^n\setminus\{0\}$ are linearly dependent or orthogonal we can take a third vector $z\in\mathbb{R}^n\setminus(\mathbb{R}x\cup\mathbb{R}y)$ such that $z^tx\neq 0$, $z^ty\neq 0$ and by the previous we deduce that $\gamma(x)=\gamma(y)=\gamma(z)$. Hence $\gamma(x)=\gamma_0$ is a nonzero constant function when restricted to $\mathbb{R}^n\setminus\{0\}\subseteq\mathrm{Spin}_n$. Without loss of generality we may further assume that $\gamma(0)=\gamma_0$.

Let $\lambda+x\in\mathrm{Spin}_n$ and let $y\in\mathbb{R}^n$ be such that $y^tx=0$. Then $(\lambda+x)\circ y=\lambda y$, therefore $\phi(\lambda+x)\circ\phi(y)=\phi(\lambda y)$ and hence $(\mu+z)\circ(\gamma_0Vy)=\gamma_0\lambda Vy$, where $\phi(\lambda+$

$x) = \mu + z$. Comparing the vector parts we obtain that $\mu = \lambda$ and comparing the scalar parts we obtain

$$z^t(Vy) = 0, \qquad \text{for every } y \text{ orthogonal to } x. \tag{14}$$

Since V is an orthogonal matrix it follows from (14) that $z \in \mathbb{R}Vx$. Hence

$$\phi(\lambda + x) = \lambda + \alpha_x Vx, \qquad \text{for some } \alpha_x \in \mathbb{R}. \tag{15}$$

If $x = 0$ then, by (14), $z = 0$ which gives $\phi(\lambda + 0) = \lambda$.

Assume $x \neq 0$. Now, let $y \in \mathbb{R}^n$ be such that $y^t x \neq 0$. Applying ϕ on $(\lambda + x) \circ y = y^t x + \lambda y$ it follows by (15) that

$$(\lambda + \alpha_x Vx) \circ (\gamma_0 Vy) = y^t x + \alpha_{\lambda y} \lambda Vy.$$

Since $(Vy)^t(Vx) = y^t x$ this simplifies into

$$\gamma_0 \alpha_x y^t x + \gamma_0 \lambda Vy = y^t x + \alpha_{\lambda y} \lambda Vy.$$

Comparing the scalar parts we obtain $\alpha_x = 1/\gamma_0$ and since this holds for an arbitrary $x \in \mathbb{R}^n$ we also obtain $\alpha_{\lambda y} = 1/\gamma_0$. By comparing also the vector parts we deduce that $\gamma_0^2 = 1$. Hence, $\phi(\lambda + x) = \lambda + Wx$ where $W = \gamma_0 V$ is again an orthogonal matrix. Clearly, such ϕ is a real linear automorphism of Spin_n. $\square$

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Gregor Dolinar
University of Ljubljana
Faculty of Electrical Engineering
Tržaška cesta 25, SI-1000 Ljubljana, Slovenia
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, SI-1000 Ljubljana, Slovenia
e-mail: gregor.dolinar@fe.uni-lj.si

Bojan Kuzma
University of Primorska
FAMNIT
Glagoljaška 8, SI-6000 Koper, Slovenia
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, SI-1000 Ljubljana, Slovenia
e-mail: bojan.kuzma@upr.si

Nik Stopar
University of Ljubljana
Faculty of Electrical Engineering
Tržaška cesta 25, SI-1000 Ljubljana, Slovenia
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, SI-1000 Ljubljana, Slovenia
e-mail: nik.stopar@fe.uni-lj.si