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WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE

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Abstract. In this paper, we study weighted composition operators on the Fock space. We show that a weighted composition operator is cohyponorma if and only if it is normal. Moreover, we give a complete characterization of closed range weighted composition operators. Finally, we find norms of some weighted composition operators.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. For an analytic map φ, let φ_0 be the identity function, $\varphi_1 = \varphi$ and $\varphi_{n+1} = \varphi \circ \varphi_n$ for $n = 1, 2, \ldots$. We call them the iterates of φ. It is well-known that if φ, neither the identity nor an elliptic automorphism of $\mathbb{D}$ (i.e., φ is an automorphism of $\mathbb{D}$ with a fixed point in $\mathbb{D}$), is an analytic map on the unit disk into itself, then there exists a point w in $\overline{\mathbb{D}}$ such that φ_n converges to w uniformly on compact subsets of $\overline{\mathbb{D}}$. The point w is called the Denjoy-Wolff point of φ. The Denjoy-Wolff point w is the unique fixed point of φ in $\overline{\mathbb{D}}$ so that $|\varphi'(w)| \leqslant 1$ (see [7]).

Recall that the Fock space $\mathscr{F}^2$ is a Hilbert space of all entire functions on $\mathbb{C}$ that are square integrable with respect to the Gaussian measures $d\mu(z) = \pi^{-1}e^{-|z|^2}dA(z)$, where dA is the usual Lebesgue measure on $\mathbb{C}$. The Fock space $\mathscr{F}^2$ is a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}d\mu(z)$$

and reproducing kernel function $K_w(z) = e^{\overline{w}z}$ for any $w \in \mathbb{C}$. Note that for any $w \in \mathbb{C}$, $\|K_w\| = e^{|w|^2/2}$. For each $w \in \mathbb{C}$, we define the normalized reproducing kernel as $k_w(z) = \frac{K_w(z)}{\|K_w\|} = e^{\overline{w}z - |w|^2/2}$. For each nonnegative integer n, let $e_n(z) = z^n/\sqrt{n!}$. The set $\{e_n\}$ is an orthonormal basis for $\mathscr{F}^2$ (see [16]).

For entire function φ on $\mathbb{C}$, the composition operator C_φ on $\mathscr{F}^2$ is defined as $C_\varphi(f) = f \circ \varphi$ for any $f \in \mathscr{F}^2$; moreover, for $\psi \in \mathscr{F}^2$, the weighted composition operator $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}f = \psi \cdot (f \circ \varphi)$. There exists some literature on composition operators acting on the Hardy and Bergman spaces. The books [7] and [11] are the important references.

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Carswell et al. [4] characterized the bounded and compact composition operators on the Fock space over $\mathbb{C}^n$. Specified to the one-dimensional case, they stated that C_φ is bounded if and only if $\varphi(z) = az$, where $|a| = 1$ or $\varphi(z) = az+b$ with $|a| < 1$. In [12], Ueki found the criteria to characterize the boundedness and compactness of weighted composition operators on the Fock space. After that in [10], Le obtained much easier criteria for the boundedness and compactness of weighted composition operators on the Fock space. On the Hardy space, normal weighted composition operators were studied; moreover, unitary weighted composition operators were characterized (see [3]). Also in [6], cohyponormal weighted composition operators were obtained. After that in [8], hyponormal weighted composition operators were investigated on the Hardy and weighted Bergman spaces. Unitary weighted composition operators $C_{\psi,\varphi}$ on the Fock space were obtained in [13]. Also invertible weighted composition operators on the Fock space were characterized in [14]. In the second section, we find normaloid, hyponormal and cohyponormal composition operators. After that we obtain all hyponormal weighted composition operators $C_{\psi,\varphi}$, where $\psi = K_c$ for each $c \in \mathbb{C}$. Moreover, we study a class of normaloid weighted composition operators. Next, we show that for $\varphi(z) = az+b$, $C_{\psi,\varphi}$ is cohyponormal if and only if $\psi = \psi(0)K_{b\frac{\bar{a}-1}{a-1}}$. Closed range composition operators were studied on the Hardy and weighted Bergman spaces in [1], [9] and [17]. In the third section, we characterize closed range weighted composition operators on the Fock space. In the fourth section, we find norm of $C_{\psi,\varphi}$ on $\mathscr{F}^2$, when $\psi = K_c$ for any $c \in \mathbb{C}$.

2. Normaloid weighted composition operators

Suppose that T is a bounded operator on a Hilbert space. Throughout this paper, the spectrum of T, the essential spectrum of T, and the point spectrum of T are denoted by $\sigma(T)$, $\sigma_e(T)$, $\sigma_p(T)$ respectively. Also the spectral radius of T is denoted by $r(T)$.
Le [10] studied the boundedness of weighted composition operator on the Fock space. His result shows that if $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$, then $\varphi(z) = az+b$, where $|a| \leqslant 1$; furthermore, he proved that if $|a| = 1$ and $C_{\psi,az+b}$ is bounded on $\mathscr{F}^2$, then $\psi = \psi(0)K_{-\bar{a}b}$. We use these facts frequently in this paper and so throughout this paper, we assume that $\varphi(z) = az+b$, where $|a| \leqslant 1$. [4], [13] and [15] were written on another Fock space (see [16, p. 33]), but their results hold for $\mathscr{F}^2$ which is considered in this paper, with identical arguments.

In the following proposition, we investigate $\sigma_p(C_{\psi,\varphi})$, when $\varphi(z) = az+b$ and $|a| < 1$. Note that in the case that $|a| = 1$, as we saw in the preceding paragraph, $\psi = \psi(0)K_{-\bar{a}b}$. Then $C_{\psi,\varphi}$ is a constant multiple of a unitary operator from [13, Corollary 1.2]. Moreover, in [13, Corollary 1.4], the spectrum of unitary weighted composition operators were characterized.

PROPOSITION 2.1. *Let ψ and φ be entire functions on $\mathbb{C}$ and $C_{\psi,\varphi}$ be a bounded weighted composition operator on $\mathscr{F}^2$. Suppose that $\varphi(z) = az+b$, where $|a| < 1$ and $b \in \mathbb{C}$. If $\lambda \in \sigma_p(C_{\psi,\varphi})$, then $|\lambda| \leqslant \psi(\frac{b}{1-a})$. Moreover, if $\psi(\frac{b}{1-a}) = 0$ and φ and ψ are nonconstant, then $C_{\psi,\varphi}$ has no eigenvalues.*

Proof. Suppose that $|a|<1$. First we find a weighted composition operator $C_{\widetilde{\psi},\widetilde{\varphi}}$ which is unitary equivalent to $C_{\psi,\varphi}$ such that the fixed point of $\widetilde{\varphi}$ lies in $\mathbb{D}$ and $\widetilde{\varphi}$ is a self-map of $\mathbb{D}$. There exists a positive integer N such that $|a|+\frac{1}{N}|1-a|<1$. It is not hard to see that there is a complex number $u\in\mathbb{C}$ such that $|u+\frac{b}{1-a}|<\frac{1}{N}$. By [13, Corollary 1.2], we know that $C_{k_u,z-u}$ is unitary (the operator $C_{k_u,z-u}$ is known as the Weyl unitary). By [10, Proposition 3.1], $C^*_{k_u,z-u}=C_{k_{-u},z+u}$, so

$$\begin{aligned}C_{k_u,z-u}C_{\psi,\varphi}C_{k_{-u},z+u} &= \frac{1}{\|K_u\|^2}C_{e^{\overline{u}z},z-u}C_{\psi,\varphi}C_{e^{-\overline{u}z},z+u}\\ &= \frac{1}{\|K_u\|^2}e^{\overline{u}z}\cdot\psi(z-u)\cdot e^{(-\overline{u}(az+b))\circ(z-u)}C_{(z+u)\circ(az+b)\circ(z-u)}\\ &= \frac{1}{\|K_u\|^2}e^{\overline{u}z}\cdot\psi(z-u)\cdot e^{-\overline{u}(az-au+b)}C_{az+u(1-a)+b}. \qquad (1)\end{aligned}$$

Let $\widetilde{\varphi}(z)=az+u(1-a)+b$ and

$$\widetilde{\psi}(z)=\frac{1}{\|K_u\|^2}e^{\overline{u}z}\cdot\psi(z-u)\cdot e^{-\overline{u}(az-au+b)}. \qquad (2)$$

It is easy to see that the fixed point of $\widetilde{\varphi}$ is $u+\frac{b}{1-a}$ which belongs to $\mathbb{D}$. Because $|a|+|u(1-a)+b|<1$, $\widetilde{\varphi}$ is a self-map of $\mathbb{D}$. Since $C_{\psi,\varphi}$ is unitary equivalent to $C_{\widetilde{\psi},\widetilde{\varphi}}$, $\sigma_p(C_{\psi,\varphi})=\sigma_p(C_{\widetilde{\psi},\widetilde{\varphi}})$. Assume that λ is a nonzero eigenvalue for $C_{\widetilde{\psi},\widetilde{\varphi}}$ with corresponding eigenvector h. We obtain

$$\lambda^n h(z)=\Pi_{j=0}^{n-1}\widetilde{\psi}(\widetilde{\varphi}_j(z))h(\widetilde{\varphi}_n(z)) \qquad (3)$$

for each $z\in\mathbb{C}$ and positive integer n. For any fixed point $z\in\mathbb{C}$, we obtain

$$\begin{aligned}|h(\widetilde{\varphi}_n(z))| &= |\langle h\circ\widetilde{\varphi}_n,K_z\rangle|\leqslant\|h\circ\widetilde{\varphi}_n\|\|K_z\|=\|h\circ\widetilde{\varphi}_n\|e^{\frac{|z|^2}{2}}=\|C_{\widetilde{\varphi}_n}(h)\|e^{\frac{|z|^2}{2}}\\ &\leqslant\|C_{\widetilde{\varphi}_n}\|\|h\|e^{\frac{|z|^2}{2}}. \qquad (4)\end{aligned}$$

Since h is not the zero function, we can choose $z\in\mathbb{D}$ such that $h(z)\neq0$. Since $u+\frac{b}{1-a}$ is the Denjoy-Wolff point of $\widetilde{\varphi}$, $\widetilde{\varphi}_j(z)\to u+\frac{b}{1-a}$ and $\widetilde{\psi}(\widetilde{\varphi}_j(z))\to\widetilde{\psi}(u+\frac{b}{1-a})$ as $j\to\infty$. Take n-th roots of the absolute value each side of Equation (3), use Equation (4) and let $n\to\infty$, we get $|\lambda|\leqslant|\widetilde{\psi}(u+\frac{b}{1-a})|r(C_\varphi)$. From Equation (2) and [15, Theorem 1.1], we see that $|\lambda|\leqslant|\widetilde{\psi}(u+\frac{b}{1-a})|r(C_\varphi)=|\psi(\frac{b}{1-a})|$. Then

$$|\lambda|\leqslant|\psi(\frac{b}{1-a})|. \qquad (5)$$

Now assume that $\widetilde{\psi}(u+\frac{b}{1-a})=\psi(\frac{b}{1-a})=0$ and φ and ψ are nonconstant. Thus, by Equation (5), $\lambda=0$ is the only possible eigenvalue for $C_{\psi,\varphi}$. Since ψ is not the zero function and φ is not constant, the Open Mapping Theorem implies that 0 cannot be an eigenvalue for $C_{\psi,\varphi}$ (see idea of the proof of [2, Lemma 4.1]). $\square$

Suppose that T is a bounded operator. The operator T is hyponormal (cohyponormal) if $T^*T \geqslant TT^*$ ($T^*T \leqslant TT^*$). Also T is normaloid if $\|T\| = r(T)$. It is well known that hyponormal (cohyponormal) operators are normaloid. In the following proposition, we characterize hyponormal, cohyponormal and normaloid composition operators on $\mathscr{F}^2$.

PROPOSITION 2.2. *Let $\varphi(z) = az+b$, where $|a| \leqslant 1$ and $b \in \mathbb{C}$. Then C_φ is a bounded normaloid (hyponormal or cohyponormal) operator if and only if $b = 0$.*

Proof. Let C_φ be normaloid (hyponormal or cohyponormal). Suppose that $b \neq 0$. Then [4, Theorem 1] states that $|a| < 1$. By [4, Theorem 4], $\|C_\varphi\| = e^{\frac{1}{2}\frac{|b|^2}{1-|a|^2}}$ (note that the inner product for $\mathscr{F}^2$ in this paper is different from [4], so the norm of a composition operator is not exactly the same as [4, Theorem 4]; furthermore, in Remark 4.1, $\|C_\varphi\|$ will be described). Also [15, Theorem 1.1] implies that $r(C_\varphi) = 1$. Since C_φ is normaloid (hyponormal or cohyponormal), $e^{\frac{1}{2}\frac{|b|^2}{1-|a|^2}} = 1$. Hence $b = 0$ which is a contradiction.
Conversely, suppose that $\varphi(z) = az$, where $|a| \leqslant 1$. Invoking [4, Lemma 2], $C_\varphi^* = C_{\overline{a}z}$. Then C_{az} is normal and the result follows. □

Suppose that $C_{\psi,\varphi}$ is a bounded weighted composition operator and $\varphi(z) = az+b$. Note that if $a = 1$, then from [13, Corollary 1.2] and as we saw in the second paragraph of this section, $C_{\psi,\varphi}$ is a constant multiplie of a unitary operator. Thus, $C_{\psi,\varphi}$ is normal, normaloid, hyponormal and cohyponormal. Hence we state the following proposition for $a \neq 1$.

PROPOSITION 2.3. *Let $\psi = K_c$ and $\varphi(z) = az+b$, where $|a| \leqslant 1$, $a \neq 1$ and $b, c \in \mathbb{C}$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$. Then the following are equivalent.*
(a) $C_{\psi,\varphi}$ is hyponormal.
(b) $C_{\psi,\varphi}$ is cohyponormal.
(c) $C_{\psi,\varphi}$ is normaloid.
(d) $c = b\frac{\overline{a}-1}{a-1}$.

Proof. There is $u \in \mathbb{C}$ such that $c = u(\overline{a}-1)$. By Equation (1) and some calculation, $C_{\psi,\varphi}$ is unitarily equivalent to $C_{\widetilde{\psi},\widetilde{\varphi}}$, where $\widetilde{\psi}(z) = e^{-|u|^2}e^{\overline{u}z}\cdot\psi(z-u)\cdot e^{-\overline{u}(az-au+b)}$ $= \psi(\frac{b}{1-a})$ and $\widetilde{\varphi}(z) = az+u(1-a)+b$. We can see that $C_{\psi,\varphi}$ is unitarily equivalent to $\psi(\frac{b}{1-a})C_{az+u(1-a)+b}$.

(a) ⇒ (d). Suppose that $C_{\psi,\varphi}$ is hyponormal. Then $C_{az+u(1-a)+b}$ is hyponormal. Proposition 2.2 implies that $u(1-a)+b = 0$. Since $u = \frac{c}{\overline{a}-1}$, we conclude that $c = b\frac{\overline{a}-1}{a-1}$.

(d) ⇒ (a). Assume that $c = b\frac{\overline{a}-1}{a-1}$. Let $u = \frac{b}{a-1}$. By Equation (1) and some calculation, $C_{\psi,\varphi}$ is unitarily equivalent to $e^{\frac{|b|^2}{1-\overline{a}}}C_{\widetilde{\varphi}}$, where $\widetilde{\varphi}(z) = az$. We infer from Proposition 2.2 that $C_{\widetilde{\varphi}}$ is hyponormal and so $C_{\psi,\varphi}$ is hyponormal.

(b) ⇔ (d). The idea of the proof is similar to (a) ⇔ (d).

(c) ⇔ (d). The idea of the proof is similar to (a) ⇔ (d). □

Suppose that $\varphi(z) = az+b$, where $|a| = 1$ and ψ is an entire function. If $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$, then as we saw in the second paragraph of this section, $\psi(z) = \psi(0)K_{-\beta}$, where $\beta = \overline{a}b$. Hence $C_{\psi,\varphi}$ is a constant multiple of the unitary operator (see [13, Corollary 1.2]). It shows that in this case $C_{\psi,\varphi}$ is normaloid and so in the next theorem, we assume that $|a| < 1$.

THEOREM 2.4. *Suppose that ψ is an entire function and is not identically zero. Assume that $\varphi(z) = az+b$, where $|a| < 1$ and $b \in \mathbb{C}$. Let for each $\lambda \in \sigma_e(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(\frac{b}{1-a})|$. Then $C_{\psi,\varphi}$ is normaloid if and only if $\psi = \psi(0)K_{b\frac{\overline{a}-1}{a-1}}$.*

Proof. Let $p = \frac{b}{1-a}$ (note that it is obvious that p is the fixed point of φ). By [13, Corollary 1.2], $C_{k_p, z-p}$ is unitary. Furthermore, [10, Proposition 3.1] states that $C^*_{k_p,z-p} = C_{k_{-p},z+p}$. We obtain

$$H := C^*_{k_p,z-p} C_{\psi,\varphi} C_{k_p,z-p} = C_{k_{-p},z+p} C_{\psi,\varphi} C_{k_p,z-p} = C_{q,\widetilde{\varphi}}, \tag{6}$$

where

$$\widetilde{\varphi}(z) = \varphi(z+p) - p = a(z+p) + b - p = az \tag{7}$$

and

$$\begin{aligned} q(z) &= k_{-p}(z) k_p(\varphi(z+p)) \psi(z+p) = e^{\overline{p}b + |p|^2(a-1)} e^{\overline{p}(a-1)z} \psi(z+p) \\ &= e^{\overline{p}(a-1)z} \psi(z+p). \end{aligned} \tag{8}$$

Since $C_{\psi,\varphi}$ is unitary equivalent to $C_{q,\widetilde{\varphi}}$, $\sigma_e(C_{\psi,\varphi}) = \sigma_e(C_{q,\widetilde{\varphi}})$. It is not hard to see that $q(0) = \psi(p)$. Now suppose that $C_{\psi,\varphi}$ is normaloid. Then $C_{q,\widetilde{\varphi}}$ is normaloid. Since $C_{\psi,\varphi}$ and $C_{q,\widetilde{\varphi}}$ are unitary equivalent, for each $\lambda \in \sigma_p(C_{q,\widetilde{\varphi}})$, $|\lambda| \leqslant |q(0)|$. We infer from [5, Proposition 6.7, p. 210] and [5, Proposition 4.4, p. 359] that $r(C_{q,\widetilde{\varphi}}) \leqslant |q(0)|$. Since $C_{q,\widetilde{\varphi}}$ is normaloid, $|q(0)| \geqslant \|C_{q,\widetilde{\varphi}}\| \geqslant \|C_{q,\widetilde{\varphi}}(1)\| = \|q\|$. We know that $\{\frac{z^m}{\sqrt{m!}} : m \geqslant 0\}$ is an orthonormal basis for $\mathscr{F}^2$. Then $\|q\| \geqslant |q(0)|$. It shows that q must be constant. Thus, Equation (8) shows that $\psi(z) \cdot e^{\overline{p}(a-1)(z-p)}$ is constant. Then $\psi(z) = \psi(0) e^{-\overline{p}(a-1)z} = \psi(0) e^{\overline{b}\frac{a-1}{\overline{a}-1}z} = \psi(0) K_{b\frac{\overline{a}-1}{a-1}}(z)$.
Conversely, suppose that $\psi = \psi(0)K_{b\frac{\overline{a}-1}{a-1}}$. By Equation (8), q is constant. Equations (6) and (7) state that $C_{\psi,\varphi}$ is unitarily equivalent to a constant multiple of C_{az}. Since by Proposition 2.2, C_{az} is normaloid, $C_{\psi,\varphi}$ is also normaloid. □

We know that for each $c \in \mathbb{C}$, $C_{K_c,\varphi}$ is compact, where $\varphi(z) = az+b$ and $|a| < 1$ (see [15, Corollary 2.4]). Hence $\sigma_e(C_{K_c,\varphi}) = \{0\}$. Thus, $C_{K_c,\varphi}$ satisfies the conditions of Theorem 2.4 and so if $C_{K_c,\varphi}$ is normaloid, then c must be $b\frac{\overline{a}-1}{a-1}$ (see also Proposition 2.3).

COROLLARY 2.5. *Suppose that ψ is an entire function and is not identically zero. Assume that $\varphi(z) = az+b$, where $|a| < 1$ and $b \in \mathbb{C}$. Then $C_{\psi,\varphi}$ is compact and normaloid if and only if $\psi = \psi(0)K_{b\frac{\overline{a}-1}{a-1}}$.*

Suppose that ψ is an entire function and $\varphi(z) = az+b$, where $|a| \leqslant 1$. In the next theorem, we show that $C_{\psi,\varphi}$ is cohyponormal if and only if $C_{\psi,\varphi}$ is normal (see [10, Theorem 3.3] and note that if $|a| = 1$ and $a \neq 1$, then $\frac{\overline{a}-1}{a-1} = -\overline{a}$).

THEOREM 2.6. *Suppose that* ψ *is an entire function and* $\varphi(z) = az+b$, *where* $|a| \leqslant 1$. *Then* $C_{\psi,\varphi}$ *is cohyponormal if and only if* $\psi = \psi(0)K_{b\frac{\overline{a}-1}{a-1}}$ *for* $a \neq 1$ *and* $\psi = \psi(0)K_{-b}$ *for* $a = 1$.

Proof. We break the proof into two cases. First assume that $a = 1$. If $\psi = \psi(0)K_{-b}$, then by [13, Corollary 1.2], $C_{\psi,\varphi}$ is a constant multiple of a unitary operator. Thus, $C_{\psi,\varphi}$ is cohyponormal. Now let $C_{\psi,\varphi}$ be cohyponormal. As we saw in the second paragraph of this section $\psi = \psi(0)K_{-b}$.

Now assume that $a \neq 1$. Suppose that $C_{\psi,\varphi}$ is cohyponormal. Let $C_{q,\widetilde{\varphi}}$ be as in Equation (6), where q and $\widetilde{\varphi}$ were obtained in Equations (7) and (8). It is obvious that $C_{q,\widetilde{\varphi}}$ is also cohyponormal. Then $\|C^*_{q,\widetilde{\varphi}}K_0\| \geqslant \|C_{q,\widetilde{\varphi}}K_0\|$. We obtain $|q(0)| \geqslant \|q\|$. As we saw in the proof of Theorem 2.4, q must be constant, so Equation (8) stats that $\psi(z) = \psi(0)e^{\overline{b}\frac{a-1}{\overline{a}-1}z}$.
The other direction follows easily from Proposition 2.3. $\square$

3. Closed range weighted composition operator

In this section, we prove that $C_{\psi,\varphi}$ has closed range if and only if $C_{\psi,\varphi}$ is a constant multiple of a unitary operator (see [13, Corollary 1.2]).

THEOREM 3.1. *Let* φ *and* ψ *be entire functions on* $\mathbb{C}$ *such that* ψ *is not identically zero. Suppose that* $C_{\psi,\varphi}$ *is bounded on* $\mathscr{F}^2$. *Then* $C_{\psi,\varphi}$ *has closed range if and only if and* $\varphi(z) = az+b$, *with* $|a| = 1$, $b \in \mathbb{C}$ *and* $\psi = \psi(0)K_{-\overline{a}b}$.

Proof. First suppose that $|a| = 1$ and $\psi = (0)K_{-\overline{a}b}$. By [13, Corollary 1.2], we have $C_{\psi,\varphi}$ is a constant multiple of a unitary operator. Therefore, $C_{\psi,\varphi}$ has closed range.
Conversely, let $C_{\psi,\varphi}$ have closed range on $\mathscr{F}^2$. As we stated in the second paragraph of Section 2, $\varphi(z) = az+b$, with $|a| \leqslant 1$. Suppose that $|a| < 1$. By Equations (6) and (7), $C_{\psi,\varphi}$ is unitarily equivalent to $C_{q,\widetilde{\varphi}}$, where $\widetilde{\varphi}(z) = az$, so without loss of generality, we assume that $\varphi(z) = az$ (note that $C_{\psi,\varphi}$ has closed range if and only if $C_{q,\widetilde{\varphi}}$ has closed range). Since $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$, $C_{\psi,az}(e^z) = \psi(z)e^{az}$ belongs to $\mathscr{F}^2$. Now we define a bounded linear functional $F_{\psi(z)e^{az}}$ by $F_{\psi(z)e^{az}}(f) = \langle f(z), \psi(z)e^{az}\rangle$ for each $f \in \mathscr{F}^2$. We know that $\frac{K_w}{\|K_w\|}$ converges to zero weakly as $|w| \to \infty$. Then

$$\lim_{|w|\to\infty} \langle \frac{K_w}{\|K_w\|}, \psi(z)e^{az}\rangle = 0.$$

It shows that

$$\lim_{|w|\to\infty} \frac{|\psi(w)||e^{aw}|}{e^{|w|^2/2}} = 0. \tag{9}$$

Now if $a = |a|e^{i\theta}$, we take $w = re^{-i\theta}$, where r is a positive real number. Then $|e^{aw}| = e^{|aw|}$. Equation (9) shows that

$$\lim_{r\to\infty} |\psi(re^{-i\theta})|^2 e^{|ar|^2 - r^2} = 0. \tag{10}$$

From Equation (10), we obtain

$$\lim_{r\to\infty}\|C^*_{\psi,\varphi}\frac{K_{re^{-i\theta}}}{\|K_{re^{-i\theta}}\|}\|^2 = \lim_{r\to\infty}|\psi(re^{-i\theta})|^2\frac{\|e^{re^{i\theta}\overline{a}z}\|^2}{e^{r^2}}$$
$$= \lim_{r\to\infty}|\psi(re^{-i\theta})|^2 e^{r^2(|a|^2-1)}$$
$$= 0. \tag{11}$$

Since ψ is not identically zero, it is easy to see that there exits a sequence $\{r_n\}$ such that for any n, r_n is a positive real number, $r_n\to\infty$ as $n\to\infty$ and $\psi(r_ne^{-i\theta})\neq 0$ for each n. We have $C^*_{\psi,\varphi}\frac{K_{r_ne^{-i\theta}}}{\|K_{r_ne^{-i\theta}}\|}=\overline{\psi(r_ne^{-i\theta})}\frac{K_{ar_ne^{-i\theta}}}{\|K_{r_ne^{-i\theta}}\|}\neq 0$ and so $\frac{K_{r_ne^{-i\theta}}}{\|K_{r_ne^{-i\theta}}\|}\notin \mathrm{Ker}(C^*_{\psi,\varphi})$. Equation (11) and [5, Proposition 6.1, p. 363] show that $C^*_{\psi,\varphi}$ does not have closed range. Thus, $C_{\psi,\varphi}$ does not have closed range (see [5, Proposition 6.2, p. 364]). Hence $|a|=1$ and the result follows from the second paragraph of Section 2. □

4. Norm of weighted composition operator

Suppose that φ and ψ are entire functions on $\mathbb{C}$ and $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$. It is well-known that for any $w\in\mathbb{C}$,

$$C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}.$$

We use this formula in the following remark. Furthermore, in this section for each $c\in\mathbb{C}$, M_{K_c} is multiplication by the kernel function K_c.

REMARK 4.1. For $\varphi(z)=az+b$, where $|a|\leqslant 1$ and $b,c\in\mathbb{C}$, $\|C_\varphi\|$ was found in [4, Theorem 4]. In [4], the inner product of the Fock space is different from ours. An analogue of [4, Theorem 4] holds for $\mathscr{F}^2$ with our definition. One can follow the outline of the proof of [4, Theorem 4] to find $\|C_\varphi\|$ on $\mathscr{F}^2$. Moreover, we state another proof for finding $\|C_\varphi\|$. We break it into two cases.
First assume that $\varphi(z)=az+b$, where $|a|<1$. By [4, Theorem 2], C_φ is compact. Then [4, Lemma 2] implies that $C^*_\varphi C_\varphi = M_{K_b}C_{\overline{a}z}C_{az+b}=M_{K_b}C_{|a|^2z+b}$ is compact (note that by the similar proof which was stated in [4, Lemma 2], we can see that $C^*_{az+b}=C_{K_b,\overline{a}z}$). We know that $\|C_\varphi\|^2=\|C^*_\varphi C_\varphi\|=r(M_{K_b}C_{|a|^2z+b})$. Since $M_{K_b}C_{|a|^2z+b}$ is compact, $r(M_{K_b}C_{|a|^2z+b})=\sup\{|\lambda|:\lambda\in\sigma_p(M_{K_b}C_{|a|^2z+b})\}$ (see [5, Theorem 7.1, p. 214]). By Proposition 2.1, for each $\lambda\in\sigma_p(M_{K_b}C_{|a|^2z+b})$, $|\lambda|\leqslant |K_b(\frac{b}{1-|a|^2})|=e^{\frac{|b|^2}{1-|a|^2}}$. We have $(M_{K_b}C_{|a|^2z+b})^*K_{\frac{b}{1-|a|^2}}=e^{\frac{|b|^2}{1-|a|^2}}K_{\frac{b}{1-|a|^2}}$. Then $e^{\frac{|b|^2}{1-|a|^2}}\in\sigma_p((M_{K_b}C_{|a|^2z+b})^*)$ and so by [5, Theorem 7.1, p. 214], $e^{\frac{|b|^2}{1-|a|^2}}\in\sigma_p(M_{K_b}C_{|a|^2z+b})$. Thus, $\|C_{az+b}\|=e^{\frac{1}{2}\frac{|b|^2}{1-|a|^2}}$.

Now assume that $\varphi(z)=az+b$ with $|a|=1$. By [4, Theorem 1], $b=0$. From [4, Lemma 2], one can easily see that $\|C_\varphi\|^2=\|C^*_\varphi C_\varphi\|=\|C_{\overline{a}z}C_{az}\|=\|C_{|a|^2z}\|=\|C_z\|=1$. Then in this case $\|C_\varphi\|=1$.

In the preceding sections, we saw that among weighted composition operators, $C_{K_c,\varphi}$ is much important, when $c \in \mathbb{C}$. Hence in the following theorem, we try to find $\|C_{K_c,\varphi}\|$.

THEOREM 4.2. *Let $\psi = K_c$ and $\varphi(z) = az+b$, where $|a| \leqslant 1$ and $b,c \in \mathbb{C}$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$.*

(a) If $|a|<1$, then $\|C_{\psi,\varphi}\| = |e^{\overline{c}\frac{b}{1-a}}|e^{\frac{1}{2}\frac{|\frac{c(1-a)}{\overline{a}-1}+b|^2}{1-|a|^2}}$.

(b) If $|a|=1$ and $a \neq 1$, then $\|C_{\psi,\varphi}\| = |e^{\frac{|b|^2}{1-\overline{a}}}|$.

(c) If $a=1$, then $\|C_{\psi,\varphi}\| = e^{\frac{|b|^2}{2}}$.

Proof. (a) Assume that $|a|<1$. By the proof of Proposition 2.3, $C_{\psi,\varphi}$ is unitarily equivalent to $\psi(\frac{b}{1-a})C_{az+u(1-a)+b}$, where $u = \frac{c}{\overline{a}-1}$. Since $|a|<1$, [4, Theorem 4] implies that $\|C_{\psi,\varphi}\| = |\psi(\frac{b}{1-a})|\|C_{az+u(1-a)+b}\| = |\psi(\frac{b}{1-a})|e^{\frac{1}{2}\frac{|\frac{c(1-a)}{\overline{a}-1}+b|^2}{1-|a|^2}}$ (see also Remark 4.1).
(b) Assume that $|a|=1$ and $a \neq 1$. Again by the proof of Proposition 2.3, $\|C_{\psi,\varphi}\| = |\psi(\frac{b}{1-a})|\|C_{az+\frac{c(1-a)}{\overline{a}-1}+b}\| = |\psi(\frac{b}{1-a})|\|C_{az+ca+b}\|$. Since $|a|=1$ and $C_{K_c,\varphi}$ is bounded, from the second paragraph of Section 2, $c = -\overline{a}b$. Then $ca+b=0$. Therefore, $\|C_{\psi,\varphi}\| = |\psi(\frac{b}{1-a})|\|C_{az}\| = |\psi(\frac{b}{1-a})|$ (see [4, Theorem 4] and Remark 4.1). Thus, $\|C_{\psi,\varphi}\| = |e^{\frac{-a|b|^2}{1-a}}| = |e^{\frac{|b|^2}{1-\overline{a}}}|$.
(c) Assume that $a=1$. As we saw in the second paragraph of Section 2, $c = -\overline{a}b = -b$. Then $\psi(z) = e^{-\overline{b}z}$. Now we must find $\|C_{e^{-\overline{b}z},z+b}\|$. By [13, Corollary 1.2], $e^{\frac{-|b|^2}{2}}C_{e^{-\overline{b}z},z+b}$ is unitary. Hence $\|C_{\psi,\varphi}\| = e^{\frac{|b|^2}{2}}$. □

REFERENCES

[1] J. R. AKEROYD AND S. R. FULMER, *Closed-range composition operators on weighted Bergman spaces*, Integr. Equ. Oper. Theory **72** (2012), 103–114.
[2] P. S. BOURDON, *Spectra of some composition operators and associated weighted composition operators*, J. Oper. Theory **67 (2)** (2012), 537–560.
[3] P. S. BOURDON AND S. K. NARAYAN, *Normal weighted composition operators on the Hardy space $H^2(\mathbb{D})$*, J. Math. Anal. Appl. **367** (2010), 278–286.
[4] B. J. CARSWELL, B. D. MACCLUER AND A. SCHUSTER, *Composition operators on the Fock space*, Acta Sci. Math. (Szeged) **69** (2003), 871–887.
[5] J. B. CONWAY, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.
[6] C. C. COWEN, S. JUNG, AND E. KO, *Normal and cohyponormal weighted composition operators on H^2*, Operator Theory: Advances and Applications **240** (2014), 69–85.
[7] C. C. COWEN AND B. D. MACCLUER, *Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics*, CRC Press, Boca Raton, FL, 1995.
[8] M. FATEHI AND M. HAJI SHAABANI, *Norms of hyponormal weighted composition operators on the Hardy and weighted Bergman spaces*, Operators and Matrices **12**(4) (2018), 997–1007.
[9] P. GHATAGE AND M. TJANI, *Closed range composition operators on Hilbert function spaces*, J. Math. Anal. Appl. **431** (2) (2015), 841–866.
[10] T. LE, *Normal and isometric weighted composition operators on the Fock space*, Bull. London Math. Soc. **46** (2014), 847–856.

[11] J. H. SHAPIRO, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.
[12] S. UEKI, *Weighted composition operator on the Fock space*, Proc. Amer. Math. Soc. **135** (2007), 1405–1410.
[13] L. ZHAO, *Unitary weighted composition operators on the Fock space of* $\mathbb{C}^n$, Complex Anal. Oper. Theory **8** (2014), 581–590.
[14] L. ZHAO, *Invertible weighted composition operators on the Fock space of* $\mathbb{C}^n$, J. Funct Spaces **2015**. Art. ID 250358.
[15] L. ZHAO AND C. PANG, *A class of weighted composition operators on the Fock space*, Journal of Mathematical Research with Applications, **35** (3) (2015), 303–310.
[16] K. ZHU, *Analysis on Fock Spaces, Graduate Texts in Mathematics* **263**, Springer, New York, 2012.
[17] N. ZORBOSKA, *Compositon operators with closed range*, Trans. Amer. Math. Soc. **344** (1994), 791–801.

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