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A NOTE ON MORE INEQUALITIES FOR SECTOR MATRICES

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Abstract. In this note, we correct an inequality and a proof of another result due to Liu and Wang [Bull. Iranian Math. Soc. 2018; 44: 1059-1066].

1. Introduction

Let $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$, whose components have been rearranged in decreasing order. If

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i, \quad k=1,2,\dots,n,$$

then we say that x is weakly majorized by y and denote $x\prec_w y$. If $x\prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then we say that x is majorized by y and denote $x\prec y$ (see [12, p. 56]). We denote by $\mathbb{M}_n$ the set of $n\times n$ complex matrices. For $A\in\mathbb{M}_n$, the conjugate transpose of A is denoted by A^*, and the matrices $\Re A=\frac{1}{2}(A+A^*)$ and $\Im A=\frac{1}{2i}(A-A^*)$ are called the real part and imaginary part of A, respectively (e.g., [2, p. 6]). Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\|=\|A\|$ for any $A\in\mathbb{M}_n$ and unitarily matrices $U,V\in\mathbb{M}_n$. If the eigenvalues of a square matrix $A\in\mathbb{M}_n$ are all real, then we denote $\lambda_j(A)$ the jth largest eigenvalue of A. The singular values of a complex matrix $A\in\mathbb{M}_n$ are the eigenvalues of $|A|:=(A^*A)^{\frac{1}{2}}$, and we denote $\sigma_j(A):=\lambda_j(|A|)$ the jth largest singular value of A. A positive semidefinite matrix A will be expressed as $A\geqslant 0$. Likewise, we write $A>0$ to refer that A is a positive definite matrix.

The numerical range of $A\in\mathbb{M}_n$ is defined by

$$W(A)=\{x^*Ax | x\in\mathbb{C}^n, x^*x=1\}.$$

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For $\alpha \in [0, \pi/2)$, let

$$S_\alpha = \{z \in \mathbb{C} | \Re z \geqslant 0, |\Im z| \leqslant (\Re z)\tan(\alpha)\}$$

be a sector region on the complex plane. A matrix whose numerical range is contained in a sector region S_α is called a sector matrix [5, 6, 7, 10, 13].

If $A, B \in \mathbb{M}_n$ are both positive definite matrices, then the geometric mean of A and B is defined by

$$A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

For more information about matrix geometric mean, we refer to [2, p. 105].

Generalizing this, Drury [3] defined the geometric mean for two sector matrices $A, B \in \mathbb{M}_n$ via the formula

$$A \sharp B = \left(\frac{2}{\pi}\int_0^\infty (tA + t^{-1}B)^{-1}\frac{dt}{t}\right)^{-1}, \tag{1}$$

where we continue to use the standard notation $A \sharp B$ for the geometric mean. This new geometric mean possesses a lot of similar properties compared to the geometric mean for positive definite matrices. For example, $A \sharp B = B \sharp A$, $(A \sharp B)^{-1} = A^{-1} \sharp B^{-1}$. Moreover, if $W(A) \subset S_\alpha$ and $W(B) \subset S_\alpha$, then $W(A \sharp B) \subset S_\alpha$. We refer to [3] for more details.

Let $A, B \in \mathbb{M}_n$ be positive definite matrices. The following noncommutative AM-GM-HM inequalities are known (e.g. [2, p. 107]):

$$\left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1} \leqslant A \sharp B \leqslant \frac{A+B}{2}. \tag{2}$$

Lin [8, Theorem 3] extended the AM-GM inequality to sector matrices as follows.

THEOREM 1.1. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\Re(A \sharp B) \leqslant \sec^2(\alpha)\Re\left(\frac{A+B}{2}\right).$$

Similarly, Liu and Wang [11, Theorem 1.2] generalized the GM-HM inequality to sector matrices and gave an assertion as follows.

ASSERTION 1.2. Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then

$$\Re\left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1} \leqslant \sec^2(\alpha)\Re(A \sharp B). \tag{3}$$

The result (3) is true, but the authors' proof of it is invalid because they used the statement

$$\sec^2(\alpha)((\Re(A^{-1}))^{-1}\sharp(\Re(B^{-1}))^{-1}) \leqslant \sec^2(\alpha)(\Re(A)\sharp\Re(B)). \tag{4}$$

This inequality (4) is refuted by the example

$$\alpha = \frac{\pi}{4}, \ A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \ B = \begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}.$$

Then, we will prove the above assertion 1.2 by other means in next section.

Let $A \in \mathbb{M}_n$ be partitioned as a 2×2 block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are square.} \tag{5}$$

Zhang [14, Theorem 1] proved the following majorization inequality.

THEOREM 1.3. *Let $A \in \mathbb{M}_n$ be partitioned as given in (5) and let A be positive definite. Then for any complex number z with $|z| = 1$,*

$$\lambda(A) \prec \tfrac{1}{2}\lambda([A_{11}+A_{22}+i(zA_{21}-\bar{z}A_{12})]\oplus O)+\tfrac{1}{2}\lambda([A_{11}+A_{22}+i(\bar{z}A_{12}-zA_{21})]\oplus O).$$

Using Theorem 1.3 and the triangle inequality, Liu and Wang [11, Corollary 4.2] got the unitarily invariant norm inequality for partitioned positive definite matrices

$$\|A\| \leqslant \|A_{11}+A_{22}\| + \|A_{12}-z^2A_{21}\|. \tag{6}$$

Moreover, the authors [11, Theorem 4.3] tried to extend the result (6) to sector matrices as follows.

ASSERTION 1.4. Let $A \in \mathbb{M}_n$ be partitioned as given in (5) and let $W(A) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any complex number z with $|z| = 1$, we have

$$\|A\| \leqslant \sec(\alpha)(\|A_{11}+A_{22}\| + \|A_{12}-z^2A_{21}\|). \tag{7}$$

Nevertheless, there is a gap in their proof. The authors of [11] get

$$\Re\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \Re A_{11} & \Re A_{12} \\ \Re A_{21} & \Re A_{22} \end{pmatrix},$$

and

$$\Re A_{12} - z^2\Re A_{21} = \Re(A_{12}-z^2A_{21}),$$

for any complex number z with $|z| = 1$ based on that $A \in \mathbb{M}_n$ is a partitioned sector matrix as given in (5). However, such two equalities do not hold. Therefore, the proof of assertion 1.4 is invalid.

In this paper, we will first provide two different proofs of assertion 1.2 and then present our result in place of (7) by the correct equalities.

2. Main results

We begin this section with some lemmas which are useful to establish and prove our main results. The first one is due to Fan and Hoffman.

LEMMA 2.1. *(see [1, p. 73]) If $A \in \mathbb{M}_n$. Then*

$$\lambda_j(\Re A) \leqslant \sigma_j(A), \quad j = 1, \dots, n.$$

Consequently,

$$\|\Re A\| \leqslant \|A\|. \tag{8}$$

LEMMA 2.2. *(see [4, Theorem 3.1]) If $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$, then*

$$(\Re A)^{-1} \leqslant \sec^2(\alpha)\Re(A^{-1}). \tag{9}$$

LEMMA 2.3. *(see [8, Lemma 2]) If $A \in \mathbb{M}_n$ has a positive definite real part, then*

$$\Re(A^{-1}) \leqslant (\Re A)^{-1}. \tag{10}$$

LEMMA 2.4. *(see [9, Theorem 1.1]) If $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$, then*

$$\Re(A)\sharp\Re(B) \leqslant \Re(A\sharp B). \tag{11}$$

LEMMA 2.5. *(see [13, Theorem 3.1]) If $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then*

$$\|A\| \leqslant \sec(\alpha)\|\Re A\|. \tag{12}$$

For the above preparation, we first show two kinds of new proofs of assertion 1.2 (i.e., [11, Theorem 1.2]).

Proof of assertion 1.2.

$$\begin{aligned}
\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} &\leqslant 2\left(\Re\left(A^{-1}+B^{-1}\right)\right)^{-1} && \text{(by (10))} \\
&= 2(\Re(A^{-1})+\Re(B^{-1}))^{-1} \\
&\leqslant (\Re(A^{-1})\sharp\Re(B^{-1}))^{-1} && \text{(by (2))} \\
&= (\Re(A^{-1}))^{-1}\sharp(\Re(B^{-1}))^{-1} \\
&\leqslant (\sec^2(\alpha)\Re A)\sharp(\sec^2(\alpha)\Re B) && \text{(by (9) and Monotonicity)} \\
&= \sec^2(\alpha)(\Re A\sharp\Re B) \\
&\leqslant \sec^2(\alpha)\Re(A\sharp B) && \text{(by (11))}. \quad \square
\end{aligned}$$

Also, we give an alternative proof by the formula (1) defined by Drury [3].

Proof.

$$\begin{aligned}
\Re(A\sharp B) &= \Re((A^{-1}\sharp B^{-1})^{-1}) \\
&= \Re\left(\frac{2}{\pi}\int_0^\infty (tA^{-1}+t^{-1}B^{-1})^{-1}\frac{dt}{t}\right) \\
&= \frac{2}{\pi}\int_0^\infty \Re(tA^{-1}+t^{-1}B^{-1})^{-1}\frac{dt}{t} \\
&\geqslant \frac{2}{\pi}\int_0^\infty \cos^2(\alpha)(\Re(tA^{-1}+t^{-1}B^{-1}))^{-1}\frac{dt}{t} \qquad \text{(by (9))} \\
&= \frac{2}{\pi}\int_0^\infty \cos^2(\alpha)(\Re(tA^{-1})+\Re(t^{-1}B^{-1}))^{-1}\frac{dt}{t} \\
&= \cos^2(\alpha)(\Re A^{-1}\sharp\Re B^{-1})^{-1} \\
&\geqslant \cos^2(\alpha)\left(\frac{\Re A^{-1}+\Re B^{-1})}{2}\right)^{-1} \qquad \text{(by (2))} \\
&= \cos^2(\alpha)\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)^{-1} \\
&\geqslant \cos^2(\alpha)\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \qquad \text{(by (10))}. \quad \square
\end{aligned}$$

Now we correct the result (7) of assertion 1.4 by the following theorem.

THEOREM 2.6. *Let $A \in \mathbb{M}_n$ be partitioned as given in (5) and let $W(A) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any complex number z with $|z| = 1$, we have*

$$\|A\| \leqslant \sec(\alpha)\left(\|A_{11}+A_{22}\| + \left\|\frac{A_{12}+A_{21}^*}{2} - z^2\frac{A_{21}+A_{12}^*}{2}\right\|\right).$$

Proof. Compute

$$\begin{aligned}
\|A\| &\leqslant \sec(\alpha)\|\Re A\| \quad \text{(by (12))} \\
&= \sec(\alpha)\left\|\begin{pmatrix} \Re A_{11} & \dfrac{A_{12}+A_{21}^*}{2} \\ \dfrac{A_{21}+A_{12}^*}{2} & \Re A_{22}\end{pmatrix}\right\| \\
&\leqslant \sec(\alpha)\left(\|\Re A_{11}+\Re A_{22}\| + \left\|\frac{A_{12}+A_{21}^*}{2} - z^2\frac{A_{21}+A_{12}^*}{2}\right\|\right) \quad \text{(by (6))} \\
&\leqslant \sec(\alpha)\left(\|A_{11}+A_{22}\| + \left\|\frac{A_{12}+A_{21}^*}{2} - z^2\frac{A_{21}+A_{12}^*}{2}\right\|\right) \quad \text{(by (8))}. \quad \square
\end{aligned}$$

This immediately yields by $z^2 = 1$:

COROLLARY 2.7. *Let $A \in \mathbb{M}_n$ be partitioned as given in (5) and let $W(A) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. If, in addition, $A_{12} = A_{21}$, then we have*

$$\|A\| \leqslant \sec(\alpha)\|A_{11}+A_{22}\|. \tag{13}$$

REFERENCES

[1] R. BHATIA, *Matrix analysis*, GTM 169, Springer, New York, 1997.

[2] R. BHATIA, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[3] S. W. DRURY, *Principal powers of matrices with positive definite real part*, Linear Multilinear Algebra. 63 (2015), 296–301.

[4] S. DRURY, M. LIN, *Singular value inequalities for matrices with numerical ranges in a sector*, Oper. Matrices, 8 (2014), 1143–1148.

[5] X. FU, Y. LIU, *Rotfel'd inequality for partitioned matrices with numerical ranges in a sector*, Linear Multilinear Algebra. 64 (2016), 105–109.

[6] C. K. LI, N. SZE, *Determinantal and eigenvalue inequalities for matrices with numerical ranges in a sector*, J. Math. Anal. Appl. 410 (2014), 487–491.

[7] M. LIN, *Extension of a result of Haynsworth and Hartfiel*, Arch. Math. 104 (2015), 93–100.

[8] M. LIN, *Some inequalities for sector matrices*, Oper Matrices. 10 (2016), 915–921.

[9] M. LIN, F. SUN, *A property of geometric mean of accretive operators*, Linear Multilinear Algebra. 65 (2017), 433–437.

[10] J. LIU, *Generalizations of the Brunn-Minkowski inequality*, Linear Algebra Appl. 508 (2016), 206–213.

[11] J. LIU, Q. WANG, *More inequalities for sector matrices*, Bull. Iranian Math. Soc. 44 (2018), 1059–1066.

[12] X. ZHAN, *Matrix Theory*, American Mathematical Society, Providence, 2013.

[13] F. ZHANG, *A matrix decomposition and its applications*, Linear Multilinear Algebra. 63 (2015), 2033–2042.

[14] Y. ZHANG, *Eigenvalue majorization inequalities for positive semidefinite block matrices and their blocks*, Linear Algebra Appl. 446 (2014), 216–223.

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