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WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS FROM THE $\alpha-$BLOCH SPACE TO THE $\alpha-$BLOCH–ORLICZ SPACE

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Abstract. The boundedness and the compactness of the weighted differentiation composition operators from the $\alpha-$Bloch space $\mathscr{B}_\alpha$ to the $\alpha-$Bloch-Orlicz space $\mathscr{B}_\alpha^\varphi$ with $\alpha > 0$ are investigated respectively in this paper.

1. Introduction

Let $S(\mathbb{D})$ be the collection of all analytic self-maps of the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. The composition operator C_ϕ induced by $\phi \in S(\mathbb{D})$ is defined as $C_\phi f = f \circ \phi$ for each $f \in H(\mathbb{D})$, where $H(\mathbb{D})$ is the collection of all holomorphic functions on the unit disk. The n-th iterates of an analytic self-map $\phi \in S(\mathbb{D})$ are denoted by ϕ_n, where $n = 1,2,\cdots$. Specially, ϕ_0 stands for the identity self-map. For a given $\psi \in H(\mathbb{D})$, the pointwise multiplication operator can be defined by $M_\psi(f) = \psi \cdot f$, where $f \in H(\mathbb{D})$. By combining the composition operator C_ϕ and the multiplication operator M_ψ, the weighted composition operator ψC_ϕ is defined by $\psi C_\phi f(z) = \psi(z) f(\phi(z))$, where $f \in H(\mathbb{D})$. An extensive study on the theory of composition operators and the weighted composition operators has been established during the past several decades on various settings. We refer to some excellent papers [14][15][17][19][22] and the famous book [3] for properties on different classical spaces of holomorphic functions.

Let $n \in \mathbb{N}$ be a positive integer. The n-th differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^n f(z) = f^{(n)}(z), z \in \mathbb{D}.$$

It deduces into the well-known differentiation operator $Df(z) = f'(z), z \in \mathbb{D}$ when $n = 1$. As a product of the multiplication operator, the composition operator and the n-th differentiation operator, the weighted differentiation composition operator was introduced by Zhu in [21], which is defined by

$$D^n_{\phi,\psi} f = \psi f^{(n)} \circ \phi, f \in H(\mathbb{D}).$$

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The one-to-one analytic self-maps that map $\mathbb{D}$ onto itself, are called the *Möbius* transformation with the form $\lambda\varphi_a$, where $a\in\mathbb{D}$, $|\lambda|=1$ and $\varphi_a(z)=\frac{a-z}{1-\bar{a}z}, z\in\mathbb{D}$.

We next recall that the Bloch space is a Banach space of analytic functions on the unit disk, which is defined as

$$\mathscr{B}=\{f\in H(\mathbb{D}):\|f\|_{\mathscr{B}}=\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty\}.$$

The Bloch space $\mathscr{B}$ is maximal among all *Möbius-invariant* Banach spaces of analytic functions on $\mathbb{D}$, which means that $\|f\circ\varphi\|_{\mathscr{B}}=\|f\|_{\mathscr{B}}$ holds for all $f\in\mathscr{B}$ and $\varphi\in Aut(\mathbb{D})$ with the seminorm $\|\ \|_{\mathscr{B}}$. It is well-known that $\mathscr{B}$ is a Banach space endowed with the norm $\|f\|_1=|f(0)|+\|f\|_{\mathscr{B}}$.

For $0<\alpha<\infty$, the $\alpha-$Bloch space is defined by

$$\mathscr{B}_\alpha=\{f\in H(\mathbb{D}):\|f\|_{\mathscr{B}_\alpha}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty\}.$$

It is a Banach space endowed with the norm $\|f\|_\alpha=|f(0)|+\|f\|_{\mathscr{B}_\alpha}$.

The $\mu-$Bloch space $\mathscr{B}_\mu$ is defined by

$$\mathscr{B}_\mu=\{f\in H(\mathbb{D}):\|f\|_{\mathscr{B}_\mu}=\sup_{z\in\mathbb{D}}\mu(z)|f'(z)|<\infty\}.$$

Also it is well-known that $\mathscr{B}_\mu$ is a Banach space endowed with the norm $\|f\|_\mu=|f(0)|+\|f\|_{\mathscr{B}_\mu}$.

Specifically, the $\alpha-$Bloch space and the $\mu-$Bloch space generalize the Bloch space in a natural way. In the past decades, basic questions including the boundedness and compactness of the composition operators on various spaces of holomorphic functions were studied by many authors (see, e.g., [5], [13], [16] and the references therein).

A function $\varphi:[0,\infty)\to[0,\infty)$ is called the Young's function if φ is a strictly increasing convex function satisfying $\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. Using the Young's function, the study of the Bloch-Orlicz space $\mathscr{B}^\varphi$ in the recent years is motivated by the development of the Hardy-Orlicz space and the Bergman-Orlicz space (see, e.g., [2, 10, 12] and [6, 9, 11], respectively). The Bloch-Orlicz space is a generalization of the classical Bloch space on the unit disk, which was firstly defined by Julio C. Ramos Fernández in [4] as

$$\mathscr{B}^\varphi=\{f\in H(\mathbb{D}):\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\lambda|f'(z)|)<\infty\},$$

where λ is a positive number depending of f and φ is the Young's function. On the one hand, we can further assume without loss of generality that φ^{-1} is differentiable. If φ^{-1} is not differentiable, by considering the function $\Psi(t)=\int_0^t\frac{\varphi(x)}{x}dx$ for $t\geqslant 0$, then we can obtain that Ψ and Ψ^{-1} are both differentiable on $[0,\infty)$. Since $\frac{\varphi(t)}{t}$ increases on $[0,\infty)$, a direct calculation $\varphi(t)\geqslant\Psi(t)\geqslant\int_{\frac{t}{2}}^t\frac{\varphi(x)}{x}dx\geqslant\varphi(\frac{t}{2})$ shows that $\mathscr{B}^\varphi=\mathscr{B}^\Psi$. On the other hand, since φ is convex on $[0,\infty)$, the Minkowski's functional $\|f\|_\varphi=\inf\{k>0:S_\varphi(\frac{f'}{k})\leqslant 1\}$ defines a semi-norm, where $S_\varphi(f):=$

$\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(|f(z)|)$. Moreover, $\mathscr{B}^\varphi$ is a Banach space with the norm $\|f\|_{\mathscr{B}^\varphi} := |f(0)|+\|f\|_\varphi$.

Furthermore, motivated by the same spirit, for $0<\alpha<\infty$, the $\alpha-$Bloch-Orlicz space $\mathscr{B}^\varphi_\alpha$ on the unit disk was considered by Liang in [7] (also see [8]), which is defined by

$$\mathscr{B}^\varphi_\alpha = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \varphi(\lambda|f'(z)|) < \infty\}$$

for some $\lambda>0$ depending of f, where φ also denotes the Young's function. On the one hand, we can further assume without loss of generality that φ^{-1} is differentiable by the same arguments discussed above. On the other hand, the Minkowski's functional $\|f\|_{\varphi,\alpha} = \inf\{k>0 : S_{\varphi,\alpha}(\frac{f'}{k}) \leqslant 1\}$ defines a semi-norm for $\mathscr{B}^\varphi_\alpha$, where $S_{\varphi,\alpha}(f) := \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\varphi(|f(z)|)$. To this end, $\mathscr{B}^\varphi_\alpha$ becomes a Banach space with the norm $\|f\|_{\mathscr{B}^\varphi_\alpha} := |f(0)|+\|f\|_{\varphi,\alpha}$.

The properties of the composition operators on the Bloch-Orlicz space were initiated by Julio C. Ramos Fern$\acute{a}$ndez in [4], where the boundedness and compactness of the composition operators on the Bloch-Orlicz space were investigated. In [7] Liang investigated the boundedness and compactness of the Volterra-type operators from the weighted Bergman-Orlicz space to the $\beta-$Zygmund-Orlicz and the $\gamma-$Bloch-Orlicz spaces, respectively. However, the boundedness and compactness of the weighted differentiation composition operators from the α-Bloch space to the $\alpha-$Bloch-Orlicz space have not been studied yet.

We use the notation $A \preceq B$ for quantities A and B to mean that $A \leqslant CB$ for some constant C since variables indicating the dependency of constants throughout this paper will not be necessarily specified.

2. Auxiliary

In this section, we show some basic results on the $\alpha-$Bloch-Orlicz space $\mathscr{B}^\varphi_\alpha$ with $\alpha>0$ to be used later. Most of them are direct statements from [7] and hence we omit the details.

PROPOSITION 2.1. [7] *For* $\alpha>0$,

$$S_{\varphi,\alpha}\Big(\frac{f'}{\|f\|_{\mathscr{B}^\varphi_\alpha}}\Big) \leqslant S_{\varphi,\alpha}\Big(\frac{f'}{\|f\|_{\varphi,\alpha}}\Big) \leqslant 1$$

holds for each $f\in\mathscr{B}^\varphi_\alpha$.

Proof. The proof is similar with Lemma 2 in [4]. □

REMARK 2.2. Observe that for each $\alpha>0$,

$$|f'(z)| \leqslant \varphi^{-1}\Big(\frac{1}{(1-|z|^2)^\alpha}\Big)\|f\|_{\varphi,\alpha} \tag{2.1}$$

holds for all $f \in \mathscr{B}_\alpha^\varphi$ and $z \in \mathbb{D}$ by Proposition 2.1. In fact, a simple estimation shows that

$$|f(z)| \leqslant |f(0)| + \int_{[0,s]} |f'(s)||ds| \leqslant (1+|z|\varphi^{-1}(\frac{1}{(1-|z|^2)^\alpha}))\|f\|_{\mathscr{B}_\alpha^\varphi} \tag{2.2}$$

since $|s| \leqslant |z|$ for all $s \in [0,z]$ and φ^{-1} is an increasing function on $[0,+\infty)$, which also implies that the evaluation functional defined by $e_z(f) = f(z)$ is continuous on $\mathscr{B}_\alpha^\varphi$, where $z \in \mathbb{D}$ is fixed.

For $\alpha > 0$, the proposition below shows that the $\alpha-$Bloch-Orlicz space is isometrically equal to a special $\mu-$Bloch space.

PROPOSITION 2.3. ([7],*Lemma 1.3*) *For $\alpha > 0$, the $\alpha-$Bloch-Orlicz space is isometrically equal to a $\mu_\alpha-$Bloch space, where*

$$\mu_\alpha(z) = \frac{1}{\varphi^{-1}(\frac{1}{(1-|z|^2)^\alpha})}.$$

In other words,

$$\|f\|_{\mathscr{B}_\alpha^\varphi} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_\alpha(z)|f'(z)|$$

holds for each $f \in \mathscr{B}_\alpha^\varphi$.

REMARK 2.4. From Proposition 2.3, it follows that for $\alpha > 0$, the $\alpha-$Bloch-Orlicz space $\mathscr{B}_\alpha^\varphi$ coincides with the $\frac{\alpha}{p}-$Bloch space if $\varphi(t) = t^p$, $p > 1$.

The equivalent condition below is first appeared in [7]. However, there was a little mistake and hence it is modified as follows.

COROLLARY 2.5. *For $\alpha > 0$, the equivalent condition*

$$S_{\varphi,\alpha}(f) \leqslant 1 \Leftrightarrow \|f\|_{\varphi,\alpha} \leqslant 1$$

holds for each $f \in \mathscr{B}_\alpha^\varphi$.

3. The boundedness of the weighted differentiation composition operator from $\mathscr{B}_\alpha$ to $\mathscr{B}_\alpha^\varphi$ with $\alpha > 0$

In this section we investigate the boundedness of the weighted differentiation composition operators from the $\alpha-$Bloch space to the $\alpha-$Bloch-Orlicz space, where $\alpha > 0$. The method used in the proof of the boundedness is standard (see, e.g., [1]).

We first introduce a well-known result of the $\alpha-$Bloch space with $\alpha > 0$ (see, e.g., [23]).

LEMMA 3.1. *For $\alpha > 0$ and $f \in \mathscr{B}_\alpha$, there exists a constant C_k dependent of $k \in \mathbb{N}$ such that*

$$|f^{(k)}(z)| \leqslant \frac{C_k \|f\|_\alpha}{(1-|z|^2)^{\alpha+k-1}}.$$

THEOREM 3.2. *For $\alpha > 0$, the differentiation weighted composition operator $D^n_{\phi,\psi}$ is bounded from $\mathscr{B}_\alpha$ to $\mathscr{B}^\varphi_\alpha$ if and only if*

$$A_n := \sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n-1}} < \infty$$

and

$$B_n := \sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} < \infty.$$

Proof. Suppose that $A_n < \infty$ and $B_n < \infty$. For each $f \in \mathscr{B}_\alpha \setminus \{0\}$,

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\alpha(z)|(D^n_{\phi,\psi}f)'(z)| \\
\leqslant &\sup_{z\in\mathbb{D}} \mu_\alpha(z)(|\psi'(z)f^{(n)}(\phi(z))| + |\psi(z)f^{(n+1)}(\phi(z))\phi'(z)|) \\
\leqslant &\sup_{z\in\mathbb{D}} \mu_\alpha(z)(|\psi'(z)|\frac{C_n\|f\|_\alpha}{(1-|\phi(z)|^2)^{\alpha+n-1}} + |\psi(z)\phi'(z)|\frac{C_{n+1}\|f\|_\alpha}{(1-|\phi(z)|^2)^{\alpha+n}}) \\
\leqslant &A_nC_n\|f\|_\alpha + B_nC_{n+1}\|f\|_\alpha \leqslant \tilde{C}(A_n+B_n)\|f\|_\alpha,
\end{aligned}$$

where $\tilde{C}$ is chosen in accordance with $C_n + C_{n+1} \leqslant \tilde{C}$ and the second inequality is calculated by Lemma 3.1. Then the boundedness of the weighted differentiation composition operator $D^n_{\phi,\psi}$ on $\mathscr{B}^\varphi_\alpha$ is guaranteed by

$$\|D^n_{\phi,\psi}f\|_{\varphi,\alpha} = \|D^n_{\phi,\psi}f\|_{\mu_\alpha} \lesssim \|f\|_\alpha$$

and

$$|D^n_{\phi,\psi}f(0)| \lesssim \|f\|_\alpha.$$

Conversely, if $D^n_{\phi,\psi} : \mathscr{B}_\alpha \to \mathscr{B}^\varphi_\alpha$ is bounded, then there exists a constant $C \geqslant 0$ such that $\|D^n_{\phi,\psi}f\|_{\varphi,\alpha} \leqslant C\|f\|_{\mathscr{B}_\alpha}$ for each $0 \neq f \in \mathscr{B}_\alpha$.

Taking $h_n(z) = \frac{z^n}{n!} \in \mathscr{B}_\alpha$, it follows by the boundedness of $D^n_{\phi,\psi}$ that

$$\sup_{z\in\mathbb{D}} \mu_\alpha(z)|\psi'(z)| = \sup_{z\in\mathbb{D}} \mu_\alpha(z)|(D^n_{\phi,\psi}h_n)'(z)| \leqslant C\|h_n\|_\alpha. \tag{3.1}$$

Further taking $h_{n+1}(z) = \frac{z^{n+1}}{(n+1)!} \in \mathscr{B}_\alpha$, it follows by the boundedness of $D^n_{\phi,\psi}$ again that

$$\sup_{z\in\mathbb{D}} \mu_\alpha(z)|\psi'(z)\phi(z) + \psi(z)\phi'(z)| = \sup_{z\in\mathbb{D}} \mu_\alpha(z)|(D^n_{\phi,\psi}h_{n+1})'(z)| \leqslant C\|h_{n+1}\|_\alpha.$$

Then we have that by (3.1)

$$\sup_{z\in\mathbb{D}} \mu_\alpha(z)|\psi(z)\phi'(z)| \leqslant C\|h_{n+1}\|_\alpha. \tag{3.2}$$

Consider the function

$$f_{a,k}(z) = \frac{(1-|a|^2)^{k+1}}{(1-\overline{a}z)^{\alpha+k}},$$

where $a \in \mathbb{D}$, $z \in \mathbb{D}$ and $k \in \mathbb{N}$. A simple calculation shows that

$$f'_{a,k}(z) = \frac{(\alpha+k)\overline{a}(1-|a|^2)^{k+1}}{(1-\overline{a}z)^{\alpha+k+1}}$$

and hence

$$\begin{aligned}\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'_{a,k}(z)| &= (\alpha+k)\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\frac{|\overline{a}|(1-|a|^2)^{k+1}}{|1-\overline{a}z|^{\alpha+k+1}}\\ \leqslant&(\alpha+k)\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha}{(1-|z|)^\alpha}\frac{(1-|a|^2)^{k+1}}{(1-|a|)^{k+1}} \leqslant (\alpha+k)2^{\alpha+k+1}.\end{aligned}$$

It follows that

$$\sup_{a\in\mathbb{D}}\|f_{a,k}\|_{\mathscr{B}_\alpha} < \infty,$$

which yields to $f_{a,k} \in \mathscr{B}_\alpha$.

On the one hand, for each $a \in \mathbb{D}$, we define

$$F(z) = \frac{(\alpha+n+2)\alpha!}{(\alpha+n)!}f_{\phi(a),1}(z) - \frac{(\alpha+1)!}{(\alpha+n)!}f_{\phi(a),2}(z), z\in\mathbb{D}.$$

Obviously, $F \in \mathscr{B}_\alpha$. A simple calculation shows that

$$F^{(n)}(z) = (\alpha+n+2)\frac{(1-|\phi(a)|^2)^2\overline{\phi(a)}^n}{(1-\overline{\phi(a)}z)^{\alpha+n+1}} - (\alpha+n+1)\frac{(1-|\phi(a)|^2)^3\overline{\phi(a)}^n}{(1-\overline{\phi(a)}z)^{\alpha+n+2}}$$

and

$$\begin{aligned}F^{(n+1)}(z) =&(\alpha+n+1)(\alpha+n+2)\frac{(1-|\phi(a)|^2)^2\overline{\phi(a)}^{n+1}}{(1-\overline{\phi(a)}z)^{\alpha+n+2}}\\ &-(\alpha+n+1)(\alpha+n+2)\frac{(1-|\phi(a)|^2)^3\overline{\phi(a)}^{n+1}}{(1-\overline{\phi(a)}z)^{\alpha+n+3}}.\end{aligned}$$

Thus we have that $F^{(n)}(\phi(a)) = \frac{\overline{\phi(a)}^n}{(1-|\phi(a)|^2)^{\alpha+n-1}}$ and $F^{(n+1)}(\phi(a)) = 0$. Note that for each $z \in \mathbb{D}$,

$$\frac{\mu_\alpha(z)|\psi'(z)||\overline{\phi(z)}|^n}{(1-|\phi(z)|^2)^{\alpha+n-1}} = \mu_\alpha(z)|(D^n_{\phi,\psi}F)'(z)| \leqslant C\|F\|_\alpha,$$

which yields to

$$\sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi'(z)||\phi(z)|^n}{(1-|\phi(z)|^2)^{\alpha+n-1}} \lesssim \|F\|_\alpha.$$

Hence,

$$\sup_{|\phi(z)|>\frac{1}{2}} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{n-1}} \leqslant \sup_{|\phi(z)|>\frac{1}{2}} \frac{\mu_\alpha(z)|\psi'(z)||2\phi(z)|^n}{(1-|\phi(z)|^2)^{\alpha+n-1}} \leqslant 2^n\|F\|_\alpha < \infty.$$

Furthermore, observe that by (3.1)

$$\sup_{|\phi(z)|\leqslant\frac{1}{2}} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{n-1}} \lesssim \sup_{|\phi(z)|\leqslant\frac{1}{2}} \mu_\alpha(z)|\psi'(z)| < \infty.$$

Then we conclude that

$$A_n = \sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n-1}} < \infty.$$

On the other hand, for each $a\in\mathbb{D}$, we define

$$G(z) = -\frac{\alpha!}{(\alpha+n)!} f_{\phi(a),1}(z) + \frac{(\alpha+1)!}{(\alpha+n+1)!} f_{\phi(a),2}(z), z\in\mathbb{D}.$$

Obviously, $G\in\mathscr{B}_\alpha$. A simple calculation shows that

$$G^{(n)}(z) = -\frac{(1-|\phi(a)|^2)^2\overline{\phi(a)}^n}{(1-\overline{\phi(a)}z)^{\alpha+n+1}} + \frac{(1-|\phi(a)|^2)^3\overline{\phi(a)}^n}{(1-\overline{\phi(a)}z)^{\alpha+n+2}}$$

and

$$G^{(n+1)}(z) = (\alpha+n+2)\frac{(1-|\phi(a)|^2)^2\overline{\phi(a)}^{n+1}}{(1-\overline{\phi(a)}z)^{\alpha+n+2}} - (\alpha+n+1)\frac{(1-|\phi(a)|^2)^3\overline{\phi(a)}^{n+1}}{(1-\overline{\phi(a)}z)^{\alpha+n+3}}.$$

Thus we have that $G^{(n+1)}(\phi(a)) = \frac{\overline{\phi(a)}^{n+1}}{(1-|\phi(a)|^2)^{\alpha+n}}$ and $G^{(n)}(\phi(a)) = 0$. Note that for each $z\in\mathbb{D}$,

$$\frac{\mu_\alpha(z)|\psi(z)||\phi'(z)||\overline{\phi(z)}|^{n+1}}{(1-|\phi(z)|^2)^{\alpha+n}} = \mu_\alpha(z)|(D^n_{\phi,\psi}G)'(z)| \leqslant C\|G\|_\alpha,$$

which yields to

$$\sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi(z)||\phi'(z)||\phi(z)|^{n+1}}{(1-|\phi(z)|^2)^{\alpha+n}} \lesssim C\|G\|_\alpha.$$

Hence,

$$\sup_{|\phi(z)|>\frac{1}{2}} \frac{\mu_\alpha(z)|\psi(z)||\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} \leqslant \sup_{|\phi(z)|>\frac{1}{2}} \frac{\mu_\alpha(z)|\psi(z)||\phi'(z)||2\phi(z)|^{n+1}}{(1-|\phi(z)|^2)^{\alpha+n}} \leqslant 2^{n+1}C\|G\|_\alpha < \infty.$$

Furthermore, observe that by (3.2) and the fact $\|\phi\|_\infty \leqslant 1$,

$$\sup_{|\phi(z)|\leqslant\frac{1}{2}} \frac{\mu_\alpha(z)|\psi(z)||\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} \lesssim \sup_{|\phi(z)|\leqslant\frac{1}{2}} \mu_\alpha(z)|\psi(z)||\phi'(z)| < \infty.$$

Then we conclude that

$$B_n = \sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi(z)||\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} < \infty.$$

This completes the proof. □

4. The compactness of the weighted differentiation composition operator from $\mathscr{B}_\alpha$ to $\mathscr{B}_\alpha^\varphi$ with $\alpha > 0$

In this section we investigate the compactness of the weighted differentiation composition operators from the $\alpha-$Bloch space to the $\alpha-$Bloch-Orlicz space with $\alpha > 0$, where the method we used in the proof is also standard (see, e.g., [1]).

THEOREM 4.1. *For $\alpha > 0$, the weighted differentiation composition operator $D^n_{\phi,\psi}$ is compact from $\mathscr{B}_\alpha$ to $\mathscr{B}_\alpha^\varphi$ if and only if $\psi \in \mathscr{B}_\alpha^\varphi$,*

$$J := \sup_{z\in\mathbb{D}} \mu_\alpha(z)|\psi(z)\phi'(z)| < \infty, \tag{4.1}$$

$$\lim_{|\phi(z)|\to 1^-} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n-1}} = 0 \tag{4.2}$$

and

$$\lim_{|\phi(z)|\to 1^-} \frac{\mu_\alpha(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} = 0. \tag{4.3}$$

Proof. Suppose that $\psi \in \mathscr{B}_\alpha^\varphi$, (4.1), (4.2) and (4.3) hold. We firstly prove that $D^n_{\phi,\psi} : \mathscr{B}_\alpha \to \mathscr{B}_\alpha^\varphi$ is bounded. For every $\varepsilon > 0$, there exists a $0 < r < 1$ such that for $|\phi(z)| > r$,

$$\frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n-1}} < \frac{\varepsilon}{2}$$

and

$$\frac{\mu_\alpha(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} < \frac{\varepsilon}{2}$$

hold. It follows that by the conditions $\psi \in \mathscr{B}_\alpha^\varphi$ and (4.1),

$$A_n = \sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n-1}} \leqslant \frac{\varepsilon}{2} + \frac{\|\psi\|_{\mathscr{B}_\alpha^\varphi}}{(1-r^2)^{\alpha+n-1}}$$

and

$$B_n = \sup_{z\in\mathbb{D}} \frac{\mu_\alpha(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} \leqslant \frac{\varepsilon}{2} + \frac{J}{(1-r^2)^{\alpha+n-1}}.$$

Then we conclude that $D^n_{\phi,\psi} : \mathscr{B}_\alpha \to \mathscr{B}^\varphi_\alpha$ is bounded.

For a chosen sequence $\{f_j\}_j \subset \mathscr{B}_\alpha$ which satisfies that $\sup_{j\in\mathbb{N}} \|f_j\|_{\mathscr{B}_\alpha} \leqslant K$ and $\{f_j\}$ converges to zero uniformly on any compact subsets of the unit disk as $j\to\infty$, where K is a fixed constant, we are only supposed to check that $\lim_{n\to\infty} \|D^n_{\phi,\psi} f_j\|_{\mathscr{B}^\varphi_\alpha} = 0$ to establish the compactness of $D^n_{\phi,\psi}$. Note that $\lim_{j\to\infty} f_j(0) = 0$ implies that $\lim_{j\to\infty} f_j^{(k)}(0) = 0$ for each $k\in\mathbb{N}$ uniformly on any compact subsets of the unit disk.

It follows by Proposition 2.3 that

$$\begin{aligned}
&\|D^n_{\phi,\psi} f_j\|_{\mathscr{B}^\varphi_\alpha} = \|D^n_{\phi,\psi} f_j\|_{\mu_\alpha} \\
\leqslant &|D^n_{\phi,\psi} f_j(0)| + \sup_{z\in\mathbb{D}} \mu_\alpha(z)(|\psi'(z) f_j^{(n)}(\phi(z))| + |\psi(z)\phi'(z) f_j^{(n+1)}(\phi(z))|) \\
\leqslant &|D^n_{\phi,\psi} f_j(0)| + \sup_{\{z\in\mathbb{D}:|\phi(z)|\leqslant r\}} \mu_\alpha(z)(|\psi'(z) f_j^{(n)}(\phi(z))| + |\psi(z)\phi'(z) f_j^{(n+1)}(\phi(z))|) \\
&+ \sup_{\{z\in\mathbb{D}:|\phi(z)|> r\}} \mu_\alpha(z)(|\psi'(z) f_j^{(n)}(\phi(z))| + |\psi(z)\phi'(z) f_j^{(n+1)}(\phi(z))|) \\
\leqslant &|D^n_{\phi,\psi} f_j(0)| + \|\psi\|_{\mathscr{B}^\varphi_\alpha} \sup_{\{z\in\mathbb{D}:|z|\leqslant r\}} |f_j^{(n)}(z)| + J \sup_{\{z\in\mathbb{D}:|z|\leqslant r\}} |f_j^{(n+1)}(z)| \\
&+ \sup_{\{z\in\mathbb{D}:|\phi(z)|> r\}} \mu_\alpha(z)|\psi'(z) f_j^{(n)}(\phi(z))| + \sup_{\{z\in\mathbb{D}:|\phi(z)|> r\}} \mu_\alpha(z)|\psi(z)\phi'(z) f_j^{(n+1)}(\phi(z))| \\
\leqslant &|D^n_{\phi,\psi} f_j(0)| + \|\psi\|_{\mathscr{B}^\varphi_\alpha} \sup_{\{z\in\mathbb{D}:|z|\leqslant r\}} |f_j^{(n)}(z)| + J \sup_{\{z\in\mathbb{D}:|z|\leqslant r\}} |f_j^{(n+1)}(z)| \\
&+ \sup_{\{z\in\mathbb{D}:|\phi(z)|> r\}} \mu_\alpha(z)|\psi'(z)| \frac{C_n \|f_j\|_\alpha}{(1-|\phi(z)|^2)^{\alpha+n-1}} \\
&+ \sup_{\{z\in\mathbb{D}:|\phi(z)|> r\}} \mu_\alpha(z)|\psi(z)\phi'(z)| \frac{C_{n+1} \|f_j\|_\alpha}{(1-|\phi(z)|^2)^{\alpha+n}} \\
\leqslant &|D^n_{\phi,\psi} f_j(0)| + \|\psi\|_{\mathscr{B}^\varphi_\alpha} \sup_{\{z\in\mathbb{D}:|z|\leqslant r\}} |f_j^{(n)}(z)| + J \sup_{\{z\in\mathbb{D}:|z|\leqslant r\}} |f_j^{(n+1)}(z)| + K\tilde{C}\varepsilon,
\end{aligned}$$

where $\tilde{C}$ is chosen in accordance with $C_n + C_{n+1} \leqslant \tilde{C}$ and the third inequality from the bottom is calculated by Lemma 3.1. It follows that $\lim_{j\to\infty} \|D^n_{\phi,\psi} f_j\|_{\mathscr{B}^\varphi_\alpha} = 0$. Then we conclude that $D^n_{\phi,\psi} : \mathscr{B}_\alpha \to \mathscr{B}^\varphi_\alpha$ is compact.

Conversely, suppose that $D^n_{\phi,\psi} : \mathscr{B}_\alpha \to \mathscr{B}^\varphi_\alpha$ is compact and hence $D^n_{\phi,\psi} : \mathscr{B}_\alpha \to \mathscr{B}^\varphi_\alpha$ is bounded. By (3.1) and (3.2), $\psi \in \mathscr{B}^\varphi_\alpha$ and $J < \infty$ hold. We prove (4.2) and (4.3) hold as follows. Set $\{z_j\}_j$ be a sequence in the unit disk satisfying $\lim_{j\to\infty} |\phi(z_j)| = 1$. If such sequence does not exist, then the proof is completed.

On the one hand, we define the function

$$F_{\phi(z_j)}(z) = \frac{(\alpha+n+2)\alpha!}{(\alpha+n)!} \frac{(1-|\phi(z_j)|^2)^2}{(1-\overline{\phi(z_j)}z)^{\alpha+1}} - \frac{(\alpha+1)!}{(\alpha+n)!} \frac{(1-|\phi(z_j)|^2)^3}{(1-\overline{\phi(z_j)}z)^{\alpha+2}},$$

where $j \in \mathbb{N}$ and $z \in \mathbb{D}$. Obviously, $F_{\phi(z_j)} \in \mathscr{B}_\alpha$ and $F_{\phi(z_j)} \to 0$ uniformly on any compact subset of the unit disk as $j \to \infty$. By the compactness of $D^n_{\phi,\psi}$, it follows that

$$\lim_{j\to\infty} \|D^n_{\phi,\psi} F_{\phi(z_j)}\|_{\mu_\alpha} = \lim_{j\to\infty} \|D^n_{\phi,\psi} F_{\phi(z_j)}\|_{\mathscr{B}^\varphi_\alpha} = 0.$$

Note that $F^{(n)}_{\phi(z_j)}(\phi(z_j)) = \frac{\overline{\phi(z_j)}^n}{(1-|\phi(z_j)|^2)^{\alpha+n-1}}$ and $F^{(n+1)}_{\phi(z_j)}(0) = 0$. Thus we have

$$\lim_{j\to\infty} \frac{\mu_\alpha(z_j)|\psi'(z_j)|}{(1-|\phi(z_j)|^2)^{\alpha+n-1}} = 0,$$

which yields to

$$\lim_{|\phi(z)|\to 1^-} \frac{\mu_\alpha(z)|\psi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n-1}} = 0.$$

On the other hand, we define the function

$$G_{\phi(z_j)}(z) = -\frac{\alpha!}{(\alpha+n)!}\frac{(1-|\phi(z_j)|^2)^2}{(1-\overline{\phi(z_j)}z)^{\alpha+1}} - \frac{(\alpha+1)!}{(\alpha+n+1)!}\frac{(1-|\phi(z_j)|^2)^3}{(1-\overline{\phi(z_j)}z)^{\alpha+2}},$$

where $j \in \mathbb{N}$ and $z \in \mathbb{D}$. Obviously, $F_{\phi(z_j)} \in \mathscr{B}_\alpha$ and $F_{\phi(z_j)} \to 0$ uniformly on any compact subset of the unit disk as $j \to \infty$. By the compactness of $D^n_{\phi,\psi}$, it follows that

$$\lim_{j\to\infty} \|D^n_{\phi,\psi} F_{\phi(z_j)}\|_{\mu_\alpha} = \lim_{j\to\infty} \|D^n_{\phi,\psi} F_{\phi(z_j)}\|_{\mathscr{B}^\varphi_\alpha} = 0.$$

Note that $G^{(n+1)}_{\phi(z_j)}(\phi(z_j)) = \frac{\overline{\phi(z_j)}^n}{(1-|\phi(z_j)|^2)^{\alpha+n}}$ and $G^{(n)}_{\phi(z_j)}(0) = 0$. Thus we have

$$\lim_{j\to\infty} \frac{\mu_\alpha(z_j)|\psi(z_j)||\phi'(z_j)|}{(1-|\phi(z_j)|^2)^{\alpha+n}} = 0,$$

which yields to

$$\lim_{|\phi(z)|\to 1^-} \frac{\mu_\alpha(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{\alpha+n}} = 0.$$

This completes the proof. □

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