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NUMERICAL RADIUS INEQUALITIES RELATED TO THE GEOMETRIC MEANS OF NEGATIVE POWER

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Abstract. The norm inequalities related to the geometric means are discussed by many researchers. We discuss numerical radius inequalities related to the geometric means. Though the operator norm is unitarily invariant one, the numerical radius is not so and unitarily similar. In this paper, we show numerical radius inequalities related to the geometric means of negative power for positive invertible operators.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and $B(H)$ denote the algebra of all bounded linear operators on H. An operator A on H is said to be positive (in symbol; $A \geqslant 0$) if $\langle Ax,x\rangle \geqslant 0$ for all $x \in H$. We write $A > 0$ if A is positive and invertible. For selfadjoint operators A and B, we write $A \geqslant B$ if $A-B$ is positive, i.e., $\langle Ax,x\rangle \geqslant \langle Bx,x\rangle$ for all $x \in H$. For any operator $A \in B(H)$, the operator norm $\|A\|$ and the numerical radius $w(A)$ are defined by

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1, x \in H\} \quad \text{and} \quad w(A) = \sup\{|\langle Ax,x\rangle| : \|x\| = 1, x \in H\},$$

respectively. Let $r(A)$ be the spectral radius of A. Then it is known that

$$r(A) \leqslant w(A) \leqslant \|A\|, \qquad \text{for all operators } A \in B(H),$$

and if A is a selfadjoint operator, then $r(A) = w(A) = \|A\|$. Let A and B be positive invertible operators on H. In [5], the α-geometric mean $A \sharp_\alpha B$ is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}}, \qquad \text{for all } \alpha \in [0,1].$$

In [3], the chaotic geometric mean $A \diamondsuit_\alpha B$ is defined by

$$A \diamondsuit_\alpha B = e^{(1-\alpha)\log A + \alpha \log B}, \qquad \text{for all } \alpha \in \mathbb{R}.$$

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If A and B commute, then $A \sharp_\alpha B = A \diamondsuit_\alpha B = A^{1-\alpha}B^\alpha$ for all $\alpha \in [0,1]$. However, we have no relation among $A \sharp_\alpha B$, $A \diamondsuit_\alpha B$ and $A^{1-\alpha}B^\alpha$ for all $\alpha \in [0,1]$ under the Löwner partial order. In [6], we showed the following norm inequality:

$$\|A \sharp_\alpha B\| \leqslant \|A \diamondsuit_\alpha B\| \leqslant \left\|A^{1-\alpha}B^\alpha\right\|, \tag{1.1}$$

for all $\alpha \in [0,1]$. Moreover, we pay attention to the geometric means of negative power: In [1], the quasi β-geometric mean $A \natural_\beta B$ is defined by

$$A \natural_\beta B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\beta A^{\frac{1}{2}}, \qquad \text{for all } \beta \in [-1,0),$$

whose formula is the same as $\sharp_\alpha$. Though $A \natural_\beta B$ for $\beta \in [-1,0)$ are not operator mean in the sense of Kubo-Ando theory [5], $A \natural_\beta B$ have operator mean like properties for any positive invertible operators A and B. For more detail, see [1]. By a similar method in [4], we have the following norm inequality in the framework of Hilbert space operators:

$$\left\|A \diamondsuit_\beta B\right\| \leqslant \left\|A \natural_\beta B\right\| \leqslant \left\|A^{1-\beta}B^\beta\right\|, \qquad \text{for all } \beta \in [-1,-\tfrac{1}{2}]. \tag{1.2}$$

In fact, by a similar way in [4, Theorem 3.1], we have the following Ando-Hiai type inequality: For positive invertible operators A and B, and $\beta \in [-1,0)$,

$$A \natural_\beta B \leqslant I \quad \text{implies} \quad A^r \natural_\beta B^r \leqslant I, \qquad \text{for all } 0 < r \leqslant 1,$$

and so

$$\left\|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right\| \leqslant \left\|(A^q \natural_\beta B^q)^{\frac{1}{q}}\right\|, \qquad \text{for all } 0 < p < q. \tag{1.3}$$

Since a chaotic geometric mean version of the Lie-Trotter formula holds:

$$A \diamondsuit_\beta B = \lim_{p\to 0}(A^p \natural_\beta B^p)^{\frac{1}{p}} \qquad \text{in the operator norm topology,}$$

we have $\left\|A \diamondsuit_\beta B\right\| \leqslant \left\|A \natural_\beta B\right\|$. For $\beta \in [-1,-\frac{1}{2}]$, it follows from Araki-Cordes inequality [2, Theorem 5.9] that

$$\left\|A \natural_\beta B\right\| \leqslant \left\|A^{\frac{\beta-1}{2\beta}}B^{-1}A^{\frac{\beta-1}{2\beta}}\right\|^{-\beta} \leqslant \left\|A^{1-\beta}B^{2\beta}A^{1-\beta}\right\|^{\frac{1}{2}} = \left\|A^{1-\beta}B^\beta\right\|$$

and so we have the desired inequality (1.2).

Moreover, the following norm inequality holds:

$$\left\|A \diamondsuit_\beta B\right\| \leqslant \left\|A^{\frac{1-\beta}{2}}B^\beta A^{\frac{1-\beta}{2}}\right\| \leqslant \left\|A \natural_\beta B\right\|, \qquad \text{for all } \beta \in [-1,0). \tag{1.4}$$

By (1.2), we would expect that the numerical radius inequality

$$w(A \natural_\beta B) \leqslant w(A^{1-\beta}B^\beta), \qquad \text{for all } \beta \in [-1,-\tfrac{1}{2}]. \tag{1.5}$$

However, though the operator norm $\|\cdot\|$ is unitarily invariant norm, the numerical radius $w(\cdot)$ is not so and unitarily similar. In fact, we have the following counterexample for (1.5): We consider the case of $\beta = -\frac{1}{2}$. Put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix},$$

and we have

$$A \natural_{-\frac{1}{2}} B = A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A^{\frac{3}{2}}B^{-\frac{1}{2}} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}.$$

Then we have $w(A^{\frac{3}{2}}B^{-\frac{1}{2}}) = \frac{1}{2}(\frac{2}{3} + \frac{\sqrt{5}}{3}) < w(A \natural_{-\frac{1}{2}} B) = \frac{\sqrt{5}}{3}$.

In this paper, we show the numerical radius inequalities related to the geometric means of negative power for positive invertible operators.

2. Numerical radius inequalities related to the geometric means

First of all, we show the following numerical radius inequality related to (1.1):

THEOREM 2.1. *Let A and B be positive invertible operators on H. Then*

$$w(A \sharp_\alpha B) \leqslant w(A \diamondsuit_\alpha B) \leqslant w(A^{1-\alpha}B^\alpha), \tag{2.1}$$

for all $\alpha \in [0,1]$.

Proof. The former inequality of (2.1) follows from (1.1). Since it follows from [2, Corollary 5.4] that $\|A \diamondsuit_\alpha B\| \leqslant \left\|A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}\right\|$ for all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} w(A \diamondsuit_\alpha B) = \|A \diamondsuit_\alpha B\| &\leqslant \left\|A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}\right\| = r(A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}) = r(A^{1-\alpha}B^\alpha) \\ &\leqslant w(A^{1-\alpha}B^\alpha), \end{aligned}$$

for all $\alpha \in [0,1]$, where $r(A)$ is the spectral radius of A, and so we have the latter inequality of (2.1). $\square$

We show the following numerical radius inequalities related to the geometric means of negative power:

THEOREM 2.2. *Let A and B be positive invertible operators on H. Then*

$$w(A \natural_\beta B) \leqslant w(A^{2(1-\beta)q}B^{2\beta q})^{\frac{1}{2q}}, \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}] \text{ and } q \geqslant 1. \tag{2.2}$$

Proof. We recall the Araki-Cordes inequality for the operator norm [2, Theorem 5.9]: If $A, B \geqslant 0$, then

$$\|B^p A^p B^p\| \leqslant \|BAB\|^p, \qquad \text{for all } 0 < p \leqslant 1. \tag{2.3}$$

For $\beta \in [-1, -\frac{1}{2}]$, by (1.3), we have $\left\|A \natural_\beta B\right\| \leqslant \left\|A^q \natural_\beta B^q\right\|^{\frac{1}{q}}$ for all $q \geqslant 1$ and this implies

$$\begin{aligned}
w(A \natural_\beta B) = \left\|A \natural_\beta B\right\| &\leqslant \left\|A^q \natural_\beta B^q\right\|^{\frac{1}{q}} \quad \text{for all } q \geqslant 1 \\
&= \left\|A^{\frac{q}{2}}(A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}})^\beta A^{\frac{q}{2}}\right\|^{\frac{1}{q}} \\
&= \left\|A^{\frac{q}{2}}(A^{\frac{q}{2}} B^{-q} A^{\frac{q}{2}})^{-\beta} A^{\frac{q}{2}}\right\|^{\frac{1}{q}} \\
&\leqslant \left\|A^{\frac{-(1-\beta)q}{2\beta}} B^{-q} A^{\frac{-(1-\beta)q}{2\beta}}\right\|^{\frac{-\beta}{q}} \quad \text{for all } \tfrac{1}{2} \leqslant -\beta \leqslant 1 \\
&\leqslant \left\|A^{(1-\beta)q} B^{2\beta q} A^{(1-\beta)q}\right\|^{\frac{1}{2q}} \quad \text{for all } \tfrac{1}{2} \leqslant -\tfrac{1}{2\beta} \leqslant 1 \\
&= r(A^{(1-\beta)q} B^{2\beta q} A^{(1-\beta)q})^{\frac{1}{2q}} \\
&= r(A^{2(1-\beta)q} B^{2\beta q})^{\frac{1}{2q}} \\
&\leqslant w(A^{2(1-\beta)q} B^{2\beta q})^{\frac{1}{2q}}
\end{aligned}$$

and so we have the desired inequality (2.2). □

Though the inequality $w(A \natural_\beta B) \leqslant w(A^{1-\beta} B^\beta)$ does not always hold for all $\beta \in [-1, -\frac{1}{2}]$, we have the following estimate from above for $w(A \natural_\beta B)$ by Theorem 2.2:

COROLLARY 2.3. *Let A and B be positive invertible operators on H. Then*

$$w(A \diamondsuit_\beta B) \leqslant w(A^{\frac{1-\beta}{2}} B^\beta A^{\frac{1-\beta}{2}}) \leqslant w(A \natural_\beta B) \leqslant w(A^{2(1-\beta)} B^{2\beta})^{\frac{1}{2}},$$

for all $\beta \in [-1, -\frac{1}{2}]$.

Proof. This corollary follows from the case of $q = 1$ in Theorem 2.2 and (1.4). □

Finally, we consider the relation between $w(A^{1-\beta} B^\beta)$ and $w(A^{2(1-\beta)} B^{2\beta})^{\frac{1}{2}}$. To show inequalities related to $w(A^{1-\beta} B^\beta)$ for $\beta \in [-1, 0)$, we need the following Cordes type inequality related to the numerical radius:

LEMMA 2.4. *Let A and B be positive invertible operators on H. Then*

$$w(AB) \leqslant w(A^{\frac{2}{p}} B^{\frac{2}{p}})^{\frac{p}{2}}, \qquad \textit{for all } p \in (0, 1],$$

or equivalently

$$w(A^{\frac{1}{2}} B^{\frac{1}{2}})^2 \leqslant w(A^p B^p)^{\frac{1}{p}}, \qquad \textit{for all } p \geqslant 1.$$

Proof. By the Araki-Cordes inequality for the operator norm (2.3), we have

$$w(A^pB^p)^2 \leqslant \|A^pB^p\|^2 = \|B^pA^p\|^2 = \left\|A^pB^{2p}A^p\right\| \leqslant \left\|AB^2A\right\|^p = r(AB^2A)^p = r(A^2B^2)^p \leqslant w(A^2B^2)^p,$$

for all $0 < p \leqslant 1$ and we have $w(AB) \leqslant w(A^{\frac{2}{p}}B^{\frac{2}{p}})^{\frac{p}{2}}$. $\square$

THEOREM 2.5. *Let A and B be positive invertible operators on H. Then*

$$w(A \diamondsuit_\beta B) \leqslant w(A^{\frac{1-\beta}{2}}B^\beta A^{\frac{1-\beta}{2}}) \leqslant w(A^{1-\beta}B^\beta) \leqslant w(A^{2(1-\beta)}B^{2\beta})^{\frac{1}{2}},$$

for all $\beta \in \mathbb{R}$.

Proof. It follows that

$$w(A^{\frac{1-\beta}{2}}B^\beta A^{\frac{1-\beta}{2}}) = r(A^{\frac{1-\beta}{2}}B^\beta A^{\frac{1-\beta}{2}}) = r(A^{1-\beta}B^\beta) \leqslant w(A^{1-\beta}B^\beta)$$

and by Lemma 2.4

$$w(A^{1-\beta}B^\beta) \leqslant w(A^{2(1-\beta)}B^{2\beta})^{\frac{1}{2}},$$

for all $\beta \in \mathbb{R}$. $\square$

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