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STRONGLY DRAZIN INVERSE IN RINGS

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Abstract. An element $a\in R$ has strongly Drazin inverse if there exists $a'\in R$ such that $aa'=a'a, a'=a'aa'$ and $a-aa'\in R$ is nilpotent, i.e., a is strongly nil-clean. Additive results for strongly Drazin inverse in a ring are presented. The explicit formulas for such generalized inverse of $a+b$ are given. These extend the results on s-Drazin and Drazin inverses of Wang (Filomat, **31**(2017), 1781–1789), Yang and Liu (J. Comput. Appl. Math., **235**(2011), 1412–1417). As an application, we give various conditions under which a 2×2 block matrix has s-Drazin inverse.

1. Introduction

Let R be a ring. An element a in R has Drazin inverse if there exists unique $a^D\in R$ such that

$$aa^D=a^Da, a^D=a^Daa^D, a^k=a^{k+1}a^D,$$

where $k=i(a)$ the index of a, i.e., the smallest nonnegative integer such the preceding conditions hold. If a is nilpotent, $i(a)$ is just the nilpotent index of a. The Drazin inverse plays an important role in various fields like matrix theory, operator theory, Markov chains, singular differential and difference equations, iterative methods, etc. (see [1, 4, 11, 13, 19, 20]).

Following Wang, an element $a\in R$ has s-Drazin inverse, i.e., strongly Drazin inverse, if there exists some $x\in R$ such that

$$ax=xa, x=xax, a-ax\in R$$

is nilpotent. If the s-Drazin inverse exists, it is unique and denote by a'. As is well known, a complex $n\times n$ matrix A has s-Drazin inverse if it is the sum of an idempotent and a nilpotent matrices that commute if and only if eigenvalues of A are only 0 and 1 (see [16, Example 2.5]). The motivation of this paper is to further explore properties of s-Drazin inverse in a ring.

In Section 2, additive results for s-Drazin inverse are presented. Let $a,b\in R$ have s-Drazin inverse. If $aba^2=0, abab=0$ and $ab^2=0$, then $a+b$ has s-Drazin inverse. This extends the results on s-Drazin and Drazin inverses of Wang ([16, Theorem 4.2])

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and Yang and Liu ([20, Theorem 2.1]). Certain explicit formulas of s-Drazin inverse are thereby obtained.

In Section 3, We consider the s-Drazin inverse for a 2×2 block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

where $A \in M_m(R), D \in M_n(R)$ have s-Drazin inverses. As applications, we give various conditions under which a 2×2 block matrix has s-Drazin inverse.

Let $A \in M_n(R)$ have s-Drazin inverse. We use A^{π} to denote the spectral idempotent $I_n - AA'$. Finally, in Section 4, we consider the multiplicative perturbation and give the s-Drazin inverse of a block matrix under some spectral conditions.

Throughout the paper, all rings are associative with an identity. We use $U(R)$ to denote the set of all units in R. R' indicates the set of all s-Drazin invertible elements in R. $M_m(R)$ denotes the set of all $m\times m$ matrices over R. $\mathbb{C}$ and $\mathbb{N}$ stand for the field of complex numbers and the set of all natural numbers, respectively.

2. Additive results

The aim of this section is to establish the new additive results for the s-Drazin inverse of the sum $a+b$ which will be used in the sequel. The explicit formulas for the s-Drazin inverse of $a+b$ are illustrated as well. We begin with

THEOREM 2.1. *Let R be a ring, and let $a\in R$. Then the following are equivalent:*

(1) *a has s-Drazin inverse .*

(2) *a is the sum of an idempotent e and a nilpotent w that commute. i.e., a is strongly nil-clean.*

(3) $a - a^2 \in N(R)$.

In the above cases the s-Drazin inverse of a is given by

$$a' = a^D = (1+w)^{-1}e = (a + \sum_{i=0}^{n} \binom{2n}{i} a^i(1-a)^{2n-i})^{-1} \sum_{i=n+1}^{2n} \binom{2n}{i} a^i(1-a)^{2n-i},$$

where $n = i(a-a^2)$.

Proof. $(1)\Rightarrow(3)$ Let $a\in R$ have s-Drazin inverse. Then $a'aa' = a'$ and $w := a - aa' \in N(R)$. This implies that $a - a^2 = aa' + w - (aa'+w)^2 = (1-2aa'-w)w \in N(R)$, as desired.

$(3)\Rightarrow(2)$ Write $(a-a^2)^n = 0$ with $n = i(a-a^2)$. Then $a^n(1-a)^n = 0$; hence, we have

$$1 = (a+(1-a))^{2n} = \sum_{i=0}^{n} \binom{2n}{i} a^i(1-a)^{2n-i} + \sum_{i=n+1}^{2n} \binom{2n}{i} a^i(1-a)^{2n-i}.$$

Set

$$e=\sum_{i=n+1}^{2n}\binom{2n}{i}a^i(1-a)^{2n-i} \text{ and } f=\sum_{i=0}^{n}\binom{2n}{i}a^i(1-a)^{2n-i}.$$

Then $ef=fe=0$, and so $e+f=1, e^2=e$ and $ea=ae$. Moreover,

$$\begin{aligned} a-e &= a-a^{2n}-\sum_{i=n+1}^{2n-1}\binom{2n}{i}a^i(1-a)^{2n-i} \\ &= \sum_{i=1}^{n}(a^{2(i-1)}-a^{2i})-\sum_{i=n+1}^{2n-1}\binom{2n}{i}a^i(1-a)^{2n-i} \end{aligned}$$

is nilpotent, as required.

$(2)\Rightarrow(1)$ By [16, Lemma 2.2], we easily see that a has s-Drazin inverse.

Since $a'=a'aa'$ and $a-a^2a'=(1-aa')(a-aa')\in N(R)$, by the uniqueness of Drazin inverse of a, we get $a'=a^D$. By the preceding discussion, we have $a+1-e=1+(a-e)$ is invertible and $a(1-e)$ is nilpotent. This implies that $1-e$ is a spectral idempotent of a. In light of the preceding discussion, we have

$$a'=a^D=(a+1-e)^{-1}e(a+\sum_{i=0}^{n}\binom{2n}{i}a^i(1-a)^{2n-i})^{-1}\sum_{i=n+1}^{2n}\binom{2n}{i}a^i(1-a)^{2n-i},$$

as desired. □

The following corollaries are generalizations of Cline's formula and Jacobson's lemma for s-Drazin inverse (see [16, Theorem 3.1 and Theorem 3.4]).

COROLLARY 2.2. *Let R be a ring, and let $a,b,c\in R$ with $aba=aca$. If ac has s-Drazin inverse, then ba has s-Drazin inverse. In this case, $(ba)'=b((ac)')^2a$.*

Proof. In view of Theorem 2.1, $ac-(ac)^2\in N(R)$. Write $(ac-(ac)^2)^k=0$ for some $k\in\mathbb{N}$. Then we check that

$$(ba-(ba)^2)^{k+1}=b(ac-(ac)^2)^k(a-aca)=0,$$

and so $ba-(ba)^2$ is nilpotent. By using Theorem 2.1 again, ba has s-Drazin inverse. As the s-Drazin inverse of a is just the Drazin inverse if exists. By using Cline's formula for Drazin inverse, we get $(ba)'=b((ac)')^2a$, as asserted. □

COROLLARY 2.3. *Let R be a ring, and let $a,b,c\in R$ with $aba=aca$. If $1-ac$ has s-Drazin inverse, then $1-ba$ has s-Drazin inverse. In this case,*

$$(1-ba)'=\sum_{i=0}^{n}(ba)^i-b\Big(\sum_{i=0}^{n}\sum_{j=0}^{n-1}(ac)^i(1-ac)^j\Big)\big(1-(1-ac)(1-ac)'\big)a,$$

where $n=i(1-ac)$.

Proof. Since $1-ac\in R$ has s-Drazin inverse, it follows by [16, Lemma 3.3] that $ac\in R$ has s-Drazin inverse. According to Theorem 2.2, $ba\in R$ has s-Drazin inverse. In view of [16, Lemma 3.3], we see that

$$(ac)' = \Big(\sum_{j=0}^{n-1}(1-ac)^j\Big)\big(1-(1-ac)(1-ac)'\big),$$

where $n=i(ac)$. Therefore we have

$$\begin{aligned}
(1-ba)' &= \Big(\sum_{i=0}^{n}(ba)^i\Big)\big(1-(ba)(ba)'\big) = \Big(\sum_{i=0}^{n}(ba)^i\Big)\big(1-((ba)b((ac)')^2a)\big)\\
&= \Big(\sum_{i=0}^{n}(ba)^i\Big)\big(1-(b(ab)((ac)')^2a\big) = \Big(\sum_{i=0}^{n}(ba)^i\Big)\big(1-(bacac((ac)')^3a\big)\\
&= \Big(\sum_{i=0}^{n}(ba)^i\Big)\big(1-(b(ac)'a\big)\\
&= \Big(\sum_{i=0}^{n}(ba)^i\Big)\big(1-(b\Big(\sum_{j=0}^{n-1}(1-ac)^j\Big)\big(1-(1-ac)(1-ac)'\big)a\big)\\
&= \sum_{i=0}^{n}(ba)^i - b\Big(\sum_{i=0}^{n}\sum_{j=0}^{n-1}(ac)^i(1-ac)^j\Big)\big(1-(1-ac)(1-ac)'\big)a.
\end{aligned}$$

This completes the proof. $\square$

EXAMPLE 2.4. Let $A=\begin{pmatrix}-2 & 2 & -1\\ -4 & 4 & -2\\ -1 & 1 & 0\end{pmatrix}\in M_3(\mathbb{Z})$. Then the characteristic polynomial $\chi(A)=t(t-1)^2$. By the Caylay-Hamilton's Theorem, $A(A-I_3)^2=0$, and so $(A-A^2)^2=0$. Hence $i(A-A^2)=2$, and so A has s-Drazin inverse. Thus, A is the sum of an idempotent and a nilpotent matrix that commute. In fact, we have a corresponding factorization $A=E+W$ with $EA=AE$, where

$$E=\begin{pmatrix}-1 & 1 & 0\\ -2 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}, W=\begin{pmatrix}-1 & 1 & -1\\ -2 & 2 & -2\\ -1 & 1 & -1\end{pmatrix}.$$

Therefore

$$A'=A^D=(I_3+W)^{-1}E=\Big(I_3+\begin{pmatrix}-1 & 1 & -1\\ -2 & 2 & -2\\ -1 & 1 & -1\end{pmatrix}\Big)^{-1}\begin{pmatrix}-1 & 1 & 0\\ -2 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 2\\ 1 & -1 & 2\end{pmatrix}.$$

We come now to consider the s-Drazin inverse of the sum of two elements in a ring. The following lemma is crucial.

LEMMA 2.5. *Let $a,b\in N(R)$. If $ab=0$, then $a+b\in N(R)$.*

Proof. See [16, Lemma 4.3]. $\square$

LEMMA 2.6. *Let* $a,b\in N(R)$. *If* $aba=0$ *and* $ab^2=0$, *then* $a+b\in N(R)$.

Proof. Clearly, $(a+b)^2=(a^2+ab)+(ba+b^2)$. Since $a^2b,ab,ba,b^2\in N(R)$, it follows by Lemma 2.5 that $a^2+ab,ba+b^2\in N(R)$. It is easy to verify that $(a^2+ab)(ba+b^2)=0$, by using Lemma 2.5 again, $a+b$ is nilpotent. $\square$

THEOREM 2.7. *Let* $a,b\in R$ *have s-Drazin inverse. If* $aba^2=0,abab=0$ *and* $ab^2=0$, *then* $a+b\in R$ *has s-Drazin inverse.*

Proof. It is easy to verify that

$$(a+b)-(a+b)^2=(a-a^2)-ba+(b-b^2)-ab.$$

Set $s=a-a^2,t=-ba,c=b-b^2,d=-ab,x=s+t$ and $y=c+d$.

Claim 1. $x\in N(R)$. As a has s-Drazin inverse, then by Theorem 2.1, $a-a^2$ is in $N(R)$, so s is in $N(R)$. Also $abab=0$ implies that $-b(abab)a=(-ba)^3=0$ then $t=-ba\in N(R)$. We have $sts=(a-a^2)(-ba)(a-a^2)=-aba^2-aba^3+a^2ba^2-a^2ba^3=0$. Also $st^2=-ababa+a^2baba=0$. Hence by Lemma 2.6, $x=s+t\in N(R)$.

Claim 2. $y\in N(R)$. Since b has s-Drazin inverse, it follows by Theorem 2.1 that $b-b^2$ is in $N(R)$, i.e., $c\in N(R)$. Also $abab=0$ implies that $-(abab)=(-ab)^2=0$ then $d=-ab\in N(R)$. We have $cdc=(b-b^2)(-ab)(b-b^2)=-bab^2-bab^3+b^2ab^2-b^2ab^3=0$ and $cd^2=(b-b^2)(-ab)(b-b^2)=-bab^2-b^2ab^3=0$. Thus $y=c+d\in N(R)$, as desired.

Claim 3. $x+y\in N(R)$. One directly computes that $xyx^2=xyxy=xy^2=0$. Clearly, we see that

$$\begin{pmatrix}1 & y\end{pmatrix}\begin{pmatrix}x\\ 1\end{pmatrix}=x+y.$$

Also we have

$$\begin{pmatrix}x\\ 1\end{pmatrix}\begin{pmatrix}1 & y\end{pmatrix}=\begin{pmatrix}x & xy\\ 1 & y\end{pmatrix}.$$

We see that

$$\begin{pmatrix}x & xy\\ 1 & y\end{pmatrix}^2=\begin{pmatrix}x^2+xy & x^2y\\ x+y & xy+y^2\end{pmatrix}=\begin{pmatrix}0 & x^2y\\ 0 & 0\end{pmatrix}+\begin{pmatrix}x^2y+xy & 0\\ x+y & xy+y^2\end{pmatrix}.$$

Let $P=\begin{pmatrix}0 & x^2y\\ 0 & 0\end{pmatrix}$ and $Q=\begin{pmatrix}x^2+xy & 0\\ x+y & xy+y^2\end{pmatrix}$. It is clear by computing that $P^2=0$, so P is nilpotent. By virtue of Lemma 2.6, we see that $x^2+xy,xy+y^2\in N(R)$. Hence Q is nilpotent. We easily check that

$$PQP=\begin{pmatrix}0 & x^2y\\ 0 & 0\end{pmatrix}\begin{pmatrix}x^2y+xy & 0\\ x+y & xy+y^2\end{pmatrix}\begin{pmatrix}0 & x^2y\\ 0 & 0\end{pmatrix}=0;$$

$$PQ^2=\begin{pmatrix}0 & x^2y\\ 0 & 0\end{pmatrix}\begin{pmatrix}x^2y+xy & 0\\ x+y & xy+y^2\end{pmatrix}^2=0.$$

By using Lemma 2.6 again, $P+Q$ is nilpotent. Therefore $(a+b)-(a+b)^2 = x+y \in N(R)$. According to Theorem 2.1, $a+b \in R'$. $\square$

We now extend [11, Corollary 2.3] to the s-Drazin inverse as follows.

COROLLARY 2.8. *Let $P,Q \in M_n(\mathbb{C})$ have s-Drazin inverses. If $PQP^2 = 0, PQPQ = 0$ and $PQ^2 = 0$, then $P+Q$ has s-Drazin inverse. In this case,*

$$\begin{aligned}(P+Q)' =& Y_1 + Y_2 + \big(Y_1(P')^2 + (Q')^2 Y_2 - \sum_{i=1}^{2}(Q')^i (P')^{3-i}\big)PQ \\ &+ \big(Y_1(P')^3 + (Q')^3 Y_2 - \sum_{i=1}^{3}(Q')^i (P')^{4-i}\big)PQP,\end{aligned}$$

where

$$r = i(P), s = i(Q), Y_1 = \sum_{i=0}^{s-1} Q^{\pi} Q^i (P')^{i+1}, Y_2 = \sum_{i=0}^{r-1} (Q')^{i+1} P^i P^{\pi}.$$

Proof. Clearly, $P' = P^D$ and $Q' = Q^D$. In view of Theorem 2.7, $P+Q$ has s-Drazin inverse. By virtue of Theorem 2.1, $(P+Q)' = (P+Q)^D$, and so the result follows by [11, Corollary 2.3]. $\square$

We note that Corollary 2.8 can not be proved by the technique in [11, Theorem 2.1] for Drazin inverse as for any element $a \in R$, a has Drazin inverse if and only if a^2 has Drazin inverse, but it is not the case for the s-Drazin inverse.

COROLLARY 2.9. *Let $P,Q \in M_n(\mathbb{C})$ have s-Drazin inverses. If $PQP = 0$ and $PQ^2 = 0$, then $P+Q$ has s-Drazin inverse. In this case,*

$$\begin{aligned}(P+Q)' = & Q^{\pi} \sum_{i=0}^{s-1} Q^i (P')^{i+1} + \sum_{i=0}^{r-1} (Q')^{i+1} P^i P^{\pi} + Q^{\pi} \sum_{i=0}^{s-1} Q^i (P')^{i+2} Q \\ &+ \sum_{i=0}^{r-2} (Q')^{i+3} P^{i+1} P^{\pi} Q - Q'P'Q - (Q')^2 PP'Q,\end{aligned}$$

where $r = i(P), s = i(Q)$.

Proof. In view of Corollary 2.8, we have $(P+Q)' = (P+Q)^D$, and so the result follows by [20, Theorem 2.1]. $\square$

We now illustrate the preceding results by given a numerical example on complex matrices.

EXAMPLE 2.10. Let

$$a = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, b = \begin{pmatrix} 1&0&0&0 \\ -1&-1&1&1 \\ 1&0&0&0 \\ -1&-1&1&1 \end{pmatrix} \in M_4(\mathbb{C}).$$

Then $a^2 = 0$ and $b^D = \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}$; hence, $b - bb^D = \begin{pmatrix} 0&0&0&0\\ -1&-1&1&1\\ 0&0&0&0\\ -1&-1&1&1 \end{pmatrix} \in N(M_4(\mathbb{C}))$.
Thus a, b have s-Drazin inverses. Moreover, we check that

$$aba^2 = 0, abab = 0, ab^2 = 0,$$

then $a+b \in R$ has s-Drazin inverse by Theorem 2.9. But $aba, bab \neq 0$. In this case,

$$(a+b)' = \begin{pmatrix} 0&-1&1&2\\ 0&0&0&0\\ 0&-1&1&2\\ 0&0&0&0 \end{pmatrix}.$$

3. Block matrices

It is of interest to investigate when a matrix has s-Drazin inverse, i.e., it is the sum of an idempotent and nilpotent matrices that commutate (see [2, 5, 10]). In [2], the author determined when a matrix over local rings has s-Drazin inverse by the decomposition of its characteristic polynomial. The aim of this section is to use our previous results to determine when a 2×2 block matrix has s-Drazin inverse. By the splitting approach, we now derive

LEMMA 3.1. *If $CBCB = 0$, then $\begin{pmatrix} 0&B\\ C&0 \end{pmatrix}$ has s-Drazin inverse.*

Proof. Write

$$\begin{pmatrix} 0&B\\ C&0 \end{pmatrix} = \begin{pmatrix} 0&0\\ C&0 \end{pmatrix} + \begin{pmatrix} 0&B\\ 0&0 \end{pmatrix}.$$

Let $p = \begin{pmatrix} 0&0\\ C&0 \end{pmatrix}$ and $q = \begin{pmatrix} 0&B\\ 0&0 \end{pmatrix}$. In view of [16, Lemma 4.1], p, q have s-Drazin inverses. By hypothesis, we easily check that $pqpq = 0, pqp^2 = 0$ and $q^2 = 0$. Therefore M has s-Drazin inverse by Theorem 2.7. □

LEMMA 3.2. *If $ABC = 0$ and $CBCB = 0$, then $\begin{pmatrix} A&B\\ C&0 \end{pmatrix}$ has s-Drazin inverse.*

Proof. Write

$$\begin{pmatrix} A&B\\ C&0 \end{pmatrix} = \begin{pmatrix} A&0\\ 0&0 \end{pmatrix} + \begin{pmatrix} 0&B\\ C&0 \end{pmatrix}.$$

Let $p = \begin{pmatrix} A&0\\ 0&0 \end{pmatrix}$ and $q = \begin{pmatrix} 0&B\\ C&0 \end{pmatrix}$. Obviously, p has s-Drazin inverse. By virtue of Lemma 3.1, q has s-Drazin inverse. It is obvious that $pqp^2 = 0, pqpq = 0$ and $pq^2 = 0$. In light of Theorem 2.9, M has s-Drazin inverse. □

THEOREM 3.3. *If* $ABC = 0, DCA = 0, DCB = 0$ *and* $CBCB = 0$*, then* M *has s-Drazin inverse.*

Proof. Write
$$M = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Let $p = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ and $q = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. In light of [16, Lemma 4.1], p has s-Drazin inverse. By virtue of Lemma 3.2, q has s-Drazin inverse. We easily check that $pqp^2 = 0, pqpq = 0$ and $pq^2 = 0$. Therefore M has s-Drazin inverse by Theorem 2.9. □

COROLLARY 3.4. *If* $ABC = 0, CBC = 0, DCA = 0$ *and* $DCB = 0$*, then* M *has s-Drazin inverse.*

Proof. This is obvious by Theorem 3.3. □

LEMMA 3.5. *If* $DCB = 0$ *and* $CBCB = 0$*, then* $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ *has s-Drazin inverse.*

Proof. Write
$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = p + q,$$

where $p = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Clearly, p has s-Drazin inverse. According to Lemma 3.1, q has s-Drazin inverse. Also $pqp^2 = 0, pqpq = 0$ and $pq^2 = 0$. Then by Theorem 2.7, $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ has s-Drazin inverse. □

THEOREM 3.6. *If* $ABC = 0, ABD = 0, DCB = 0$ *and* $CBCB = 0$*, then* M *has s-Drazin inverse.*

Proof. Write
$$M = p + q, p = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}.$$

Then p has s-Drazin inverse. In view of Lemma 3.5, q has s-Drazin inverse. Since $ABC = 0$ and $ABD = 0$ we check that $pqp^2 = 0, pqpq = 0$ and $pq^2 = 0$. Therefore M has s-Drazin inverse by Theorem 2.7. □

As an immediate consequence, we derive

COROLLARY 3.7. *If* $ABC = 0, ABD = 0, BCB = 0$ *and* $DCB = 0$*, then* M *has s-Drazin inverse.*

4. Multiplicative perturbations

The goal of this section is to investigate s-Drazin inverse for 2×2 block matrices with multiplicative perturbations. Let M be a block matrix given by (1.1). It is of interest to consider the Drzain inverse of M under generalized Schur condition $D = CA'B$ (see [1]). We now consider s-Drazin inverse for such matrices.

THEOREM 4.1. *Let $A\in M_m(R), D\in M_n(R)$ have s-Drazin inverses and M be given by (1.1). If $AA^{\pi}BCA=0, AA^{\pi}BCB=0, CA^{\pi}BCA=0, CA^{\pi}BCB=0, ABCA'=0$ and $D=CA'B$, then $M\in M_{m+n}(R)$ has s-Drazin inverse.*

Proof. We easily see that

$$M=\begin{pmatrix} A & B\\ C & CA'B\end{pmatrix}=P+Q,$$

where

$$P=\begin{pmatrix} A & AA'B\\ C & CA'B\end{pmatrix}, Q=\begin{pmatrix} 0 & A^{\pi}B\\ 0 & 0\end{pmatrix}.$$

Then we check that $PQP^2=0, PQPQ=0, PQ^2=0$. Clearly, Q has s-Drazin inverse. Moreover, we have

$$P=P_1+P_2,\ P_1=\begin{pmatrix} AA^{\pi} & 0\\ CA^{\pi} & 0\end{pmatrix},\ P_2=\begin{pmatrix} A^2A' & AA'B\\ CAA' & CA'B\end{pmatrix},$$

$P_1P_2=0$ and P_1 is nilpotent. Hence, P_1 has s-Drazin inverse. It is easy to verify that

$$P_2=\begin{pmatrix} AA'\\ CA'\end{pmatrix}\begin{pmatrix} A & AA'B\end{pmatrix}.$$

Obviously, we have

$$\begin{pmatrix} A & AA'B\end{pmatrix}\begin{pmatrix} AA'\\ CA'\end{pmatrix}=A^2A'+AA'BCA'.$$

Since AA' has s-Drazin inverse. by [16, Lemma 4.4], A^2A' has s-Drazin inverse. Since $(CA')(AA'B)=CA'B=D$ has s-Drazin inverse, it follows by [16, Theorem 3.1] that $AA'BCA'$ has s-Drazin inverse. Moreover, We easily check that

$$(A^2A')(AA'BCA')=A^2(A'BCA')=ABCA'=0,$$

it follows by Theorem 2.7 that $A^2A'+AA'BCA'$ has s-Drazin inverse. By using Corollary 2.2, P_2 has s-Drazin inverse. According to Theorem 2.7, $M\in M_{m+n}(R)$ has s-Drazin inverse. $\square$

COROLLARY 4.2. *Let $A\in M_m(R), D\in M_n(R)$ have s-Drazin inverses and M be given by (1.1). If $BCA=0, BCB=0$ and $D=CA'B$, then $M\in M_{m+n}(R)$ has s-Drazin inverse.*

Proof. Since $BCA = 0$ and $BCB = 0$, we have $AA^{\pi}BCA = 0, AA^{\pi}BCB = 0$, $CA^{\pi}BCA = 0, CA^{\pi}BCB = 0$. Moreover, we see that $ABCA' = 0 = A(BCA)(A')^2 = 0$. In light of Theorem 4.1, we complete the proof. □

THEOREM 4.3. *Let $A \in M_m(R), D \in M_n(R)$ have s-Drazin inverses and M be given by (1.1). If $BCA^{\pi}A^2 = 0, BCA^{\pi}AB = 0, BCA^{\pi}BC = 0, ABCA' = 0$ and $D = CA'B$, then $M \in M_{m+n}(R)$ has s-Drazin inverse.*

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA'B \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & B \\ CAA' & CA'B \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}.$$

By assumption, we verify that $PQP^2 = 0, PQPQ = 0, PQ^2 = 0$. Clearly, Q is nilpotent, and then it has s-Drazin inverse. Moreover, we see that

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A' & AA'B \\ CAA' & CA'B \end{pmatrix},\ P_2 = \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}$$

and $P_1P_2 = 0$. We easily check that P_2 is nilpotent. Furthermore, we see that

$$P_1 = \begin{pmatrix} AA' \\ CA' \end{pmatrix} \begin{pmatrix} A & AA'B \end{pmatrix}.$$

Clearly, we have

$$\begin{pmatrix} A & AA'B \end{pmatrix} \begin{pmatrix} AA' \\ CA' \end{pmatrix} = A^2A' + AA'BCA'.$$

In view of [16, Lemma 4.4], $A^2A' = A(AA')$ has s-Drazin inverse. Since $D = CA'B$ has s-Drazin inverse, it follows by Corollary 2.2 that $AA'BCA'$ has s-Drazin inverse. Since $ABCA' = 0$, as in the proof of Theorem 4.1, we prove that $A^2A' + AA'BCA'$ has s-Drazin inverse. Thus, Q_1 has s-Drazin inverse. Therefore Q has s-Drazin inverse. By Theorem 2.7, M has s-Drazin inverse, as asserted. □

COROLLARY 4.4. *Let $A \in M_m(R), D \in M_n(R)$ have s-Drazin inverses and M be given by (1.1). If $BCA^{\pi} = 0, ABCA' = 0$ and $D = CA'B$, then $M \in M_{m+n}(R)$ has s-Drazin inverse.*

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