Operators
and
Matrices
Volume 13, Number 2 (2019), 507–525
doi:10.7153/oam-2019-13-39

SPECTROGRAMS AND TIME–FREQUENCY LOCALIZED FUNCTIONS IN THE HANKEL SETTING

SAIFALLAH GHOBBER*, SIWAR HKIMI AND SLIM OMRI

(*Communicated by D. Han*)

Abstract. The uncertainty principle in Fourier analysis sets a limit to the possible simultaneous concentration of a function and its Hankel transform. Nevertheless, signals that have highly concentrated time–frequency content have many applications in quantum mechanics, PDE, engineering and in signal analysis. We use here time–frequency localization operators in the Hankel setting to measure the time–frequency content of functions on a subset of finite measure Σ within the time–frequency plane. Then, using eigenfunctions and eigenvalues of these operators, we prove a characterization of functions that are time–frequency concentrated in Σ, and we obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions, since they are maximally time–frequency-concentrated in the region of interest.

1. Introduction

Hankel transforms are integral transformations whose kernels are Bessel functions. They are sometimes referred to as Fourier-Bessel transforms. When we are dealing with problems that show circular symmetry, Hankel transforms may be very useful. For example, the Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function, and this plays an important role in optical data processing. Moreover the Hankel transform arises as a generalization of the Fourier transform of a radial integrable function on Euclidean d-space as well as from the eigenvalues expansion of a Schrödinger operator. For $\alpha \geqslant -\frac{1}{2}$, the Hankel transform (see [23] and [25] as reference sources for some definitions and properties that are useful in the harmonic analysis associated with the Hankel transform) is defined on $L^1_\alpha(\mathbb{R}_+)\cap L^2_\alpha(\mathbb{R}_+)$ by,

$$\mathscr{H}_\alpha(f)(\xi) = \int_0^\infty f(x) j_\alpha(x\xi)\,\mathrm{d}\mu_\alpha(x), \qquad \xi \in \mathbb{R}_+ = [0,\infty),$$

and it is extended from $L^1_\alpha(\mathbb{R}_+)\cap L^2_\alpha(\mathbb{R}_+)$ to $L^2_\alpha(\mathbb{R}_+)$ in the usual way, where $\mathrm{d}\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}\,\mathrm{d}x$ is a weight measure, the Bessel function j_α is given by:

$$j_\alpha(x) := \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n},$$

Mathematics subject classification (2010): 94A12, 45P05, 42C25, 42C40.

Keywords and phrases: Time-frequency concentration, localized functions, windowed Hankel transform, localization operator, spectrogram.

* Corresponding author.

and, for $1 \leqslant p < \infty$, $L^p_\alpha(\mathbb{R}_+)$ is the Banach space consisting of measurable functions f on $\mathbb{R}_+$ equipped with the norms:

$$\|f\|_{L^p_\alpha} = \left(\int_0^\infty |f(x)|^p \, \mathrm{d}\mu_\alpha(x)\right)^{1/p}.$$

Notice that, $j_{-1/2}(x) = \cos(x)$ and $\mathrm{d}\mu_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\,\mathrm{d}x$ is the Lebesgue measure, then

$$\mathscr{H}_{-1/2}(f)(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(x)\,\mathrm{d}x$$

is the Fourier-cosine transform, which is the Fourier transform restricted to even functions in $L^2(\mathbb{R})$. More generally, if $f \in L^2(\mathbb{R}^d)$, $d > 1$, is a radial function on $\mathbb{R}^d$, such that $f(x) = \tilde{f}(|x|)$, with $\tilde{f} \in L^2_{d/2-1}(\mathbb{R}_+)$, then

$$\mathscr{F}(f)(\xi) = \mathscr{H}_{d/2-1}(\tilde{f})(|\xi|), \qquad \xi \in \mathbb{R}^d,$$

where $|\cdot| = \sqrt{\langle \cdot,\cdot \rangle}$ is the Euclidean norm on $\mathbb{R}^d$, and $\mathscr{F}$ is the usual Fourier transform defined by

$$\mathscr{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x,\xi \rangle}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

The uncertainty principle, in its several forms, sets a restriction on the time–frequency behavior of a function. For example a signal f and its Hankel transform $\mathscr{H}_\alpha(f)$ cannot both be concentrated in subsets of finite measure (see [10, 11, 14]). The Heisenberg-type [5, 13] and the Shapiro-type uncertainty principles [13, 15] provides another quantitative restriction on the joint time–frequency behavior of functions and orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$. These facts indicate that a signal cannot have all its energy concentrated in a finite region of the time–frequency plane.

Signals that have highly concentrated time–frequency content are very useful in many applications, and time–frequency resolution is usually associated with the windowed Fourier transform, also known as the (continuous) Gabor transform, or the short-time Fourier transform, which is defined on $L^2(\mathbb{R}^d)$ by

$$\mathscr{F}_g(f)(x,\xi) = \mathscr{F}\left[f\,\overline{g(\cdot - x)}\right](\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{g(t-x)e^{i\langle t,\xi \rangle}}\,\mathrm{d}t,$$

where $g \in L^2(\mathbb{R}^d)$ is a nonzero window function.

In the present paper, we will study functions whose time–frequency content are concentrated in a some region in phase space using time–frequency localization operators associated with the windowed Hankel transform, as a main tool. To be more precise, for $y \in \mathbb{R}_+$, we define the translation operator τ^α_y on $L^2_\alpha(\mathbb{R}_+)$ by:

$$\tau^\alpha_y f(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)(\sin\theta)^{2\alpha}\,\mathrm{d}\theta, \quad x \in \mathbb{R}_+,$$

and for $\xi \in \mathbb{R}_+$ we define the modulation operator $\mathscr{M}_\xi^\alpha$ on $L^2_\alpha(\mathbb{R}_+)$ by:

$$\mathscr{M}_\xi^\alpha g := \mathscr{H}_\alpha\left(\sqrt{\tau_\xi^\alpha |\mathscr{H}_\alpha(g)|^2}\right), \qquad \xi \in \widehat{\mathbb{R}}_+,$$

where $\widehat{\mathbb{R}}_+$ denote the half real line thought of as the frequency axis.

For a nonzero window function $g \in L^2_\alpha(\mathbb{R}_+)$, the windowed Hankel transform of $f \in L^2_\alpha(\mathbb{R}_+)$ with respect to the window g is given by:

$$\mathscr{V}_g^\alpha(f)(x,\xi) = \int_0^\infty f(t)\overline{\tau_x^\alpha \mathscr{M}_\xi^\alpha g(t)}\, \mathrm{d}\mu_\alpha(t), \qquad (x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+.$$

Then the time-frequency localization operator with window g and symbol χ_Σ is formally defined as,

$$\mathscr{A}_\Sigma^g f(t) = \int_\Sigma \mathscr{V}_g^\alpha(f)(x,\xi)\tau_x^\alpha \mathscr{M}_\xi^\alpha g(t)\, \mathrm{d}\nu_\alpha(x,\xi), \tag{1.1}$$

where $\mathrm{d}\nu_\alpha(x,\xi) = \mathrm{d}\mu_\alpha(x)\,\mathrm{d}\mu_\alpha(\xi)$.

Contrary to the Hankel transform, the windowed Hankel transform cannot be obtained from the windowed Fourier transform by taking spherical averages, *i.e.* if $\tilde{f},\tilde{g} \in L^2_{d/2-1}(\mathbb{R}_+)$ are the radial parts of $f,g \in L^2(\mathbb{R}^d)$, then it is not true in general that

$$\mathscr{V}_{\tilde{g}}^{d/2-1}(\tilde{f})(|x|,|\xi|) = \mathscr{F}_g(f)(x,\xi), \qquad x,\xi \in \mathbb{R}^d, \tag{1.2}$$

because, generally $\mathscr{F}_g(f)$ is not radial in any of the two variables. So that the windowed Hankel transform $\mathscr{V}_g^\alpha$, and then the localization operator $\mathscr{A}_\Sigma^g$ are two new objects and not just an average of the standard windowed Fourier transform or the standard localization operator (see *e.g.* Example 1 and Example 2 in [7] showing that (1.2) is not true, even in dimension $d = 1$).

In this paper, we study the localization operator $\mathscr{A}_\Sigma^g$, which leads to a compact self-adjoint operator whose eigenfunctions with large eigenvalues span a subspace that can be used to determine the component of a general signal that is concentrated within the given region of the time–frequency plane.

The time-frequency localization operators were first introduced and studied by Daubechies in [8], and Ramanathan-Topiwala in [21]. These operators can be used to extract and localize components of a signal from its representation in the time–frequency plane. They have appeared in physics as tools in quantization procedures [4] called anti-Wick operators, and in the approximation of pseudo-differential operators [6]. The method of Daubechies extended the work of Landau, Pollak, and Slepian (see [18, 19, 26]), who developed the study of bandlimited functions that are concentrated on a finite time interval. They made use of compositions of time- and band-limiting operators and considered the eigenvalue problem associated with these operators. The resulting operators yield the well-known prolate spheroidal functions as eigenfunctions. These functions satisfy some optimality in concentration in a rectangular region in the time-frequency domain. The study of localization operators associated to windowed Hankel transforms is a contribution to the topic of radial time-frequency analysis [23, 24, 25]. As it was done for the prolate spheroidal wave functions (PSWFs for short), the author in [20] showed that the radial part of the bi-dimensional PSWFs, called circular prolate spheroidal wave function solve the energy concentration problem of Hankel bandlimited functions. Recently, the author in [28] showed that the generalized prolate spheroidal wave functions are the solution of the energy concentration problem in reproducing kernel Hilbert space.

Here, we make use of time–frequency localization operators to describe functions that have time–frequency content in a subset of finite measure. As in the case of the prolate spheroidal wave functions, the eigenfunctions of the time–frequency localization operators $\mathscr{A}_\Sigma^g$ are maximally time–frequency concentrated in the region of interest and we will use these eigenfunctions to approximate time–frequency localized functions (see Theorem 3.3).

The remainder of the paper is organized as follows. Next section is devoted to some preliminaries results and in Section 3, we prove our main results.

2. Preliminaries

2.1. Notation

If A is a subset of $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$, then we denote by $A^c = \left(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+\right) \backslash A$ the complement of A in $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ and the characteristic function of A will be denoted by χ_A.

For $1 \leqslant p < \infty$, $L^p_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ will be the Banach space consisting of measurable functions F on $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ equipped with the norms

$$\|F\|_{L^p_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)} = \left(\int_{\mathbb{R}_+ \times \widehat{\mathbb{R}}_+} |F(x,\xi)|^p \, \mathrm{d}\nu_\alpha(x,\xi)\right)^{1/p}.$$

The singular values $\{e_n(\mathscr{A})\}_{n=1}^\infty$ of a compact operator $\mathscr{A} \in B\left(L^2_\alpha(\mathbb{R}_+)\right)$ are the eigenvalues of the positive self-adjoint operator $|\mathscr{A}| = \sqrt{\mathscr{A}^*\mathscr{A}}$. For $1 \leqslant p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in ℓ_p. Hence S_p is equipped with the norm

$$\|\mathscr{A}\|_{S_p} = \left(\sum_{n=1}^\infty (e_n(\mathscr{A}))^p\right)^{1/p}. \tag{2.1}$$

In particular, S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators. It is well known that, the trace of an operator $\mathscr{A} \in S_1$ is defined by (see *e.g.* [27, Theorem 2.6]):

$$\operatorname{tr}(\mathscr{A}) = \sum_{n=1}^\infty \langle \mathscr{A}\varphi_n, \varphi_n\rangle_{\mu_\alpha}, \tag{2.2}$$

where $\{\varphi_n\}_{n=1}^\infty$ is any orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$. Moreover, if $\mathscr{A}$ is positive, then (see *e.g.* [27, Theorem 2.7]):

$$\operatorname{tr}(\mathscr{A}) = \|\mathscr{A}\|_{S_1}. \tag{2.3}$$

Moreover, if a compact operator $\mathscr{A}$ on the Hilbert space $L^2_\alpha(\mathbb{R}_+)$ is Hilbert-Schmidt, then the positive operator $\mathscr{A}^*\mathscr{A}$ is in the space of trace class S_1 and

$$\|\mathscr{A}\|^2_{HS} := \|\mathscr{A}\|^2_{S_2} = \|\mathscr{A}^*\mathscr{A}\|_{S_1} = \operatorname{tr}(\mathscr{A}^*\mathscr{A}) = \sum_{n=1}^{\infty} \|\mathscr{A}\varphi_n\|^2_{L^2_\alpha}, \tag{2.4}$$

for any orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ for $L^2_\alpha(\mathbb{R}_+)$.

For consistency, we define $S_\infty := B\left(L^2_\alpha(\mathbb{R}_+)\right)$ to be the space of bounded operators from $L^2_\alpha(\mathbb{R}_+)$ into $L^2_\alpha(\mathbb{R}_+)$, equipped with norm,

$$\|\mathscr{A}\|_{S_\infty} = \sup_{f\,:\,\|f\|_{L^2_\alpha}\leqslant 1} \|\mathscr{A}f\|_{L^2_\alpha}. \tag{2.5}$$

It is obvious that $S_p \subseteq S_q$, $1 \leqslant p \leqslant q \leqslant \infty$.

2.2. Generalities

For $\alpha > -1/2$, let us recall the *Poisson representation formula*

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1-s^2)^{\alpha-1/2}\cos(sx)\,\mathrm{d}x.$$

Therefore, j_α is bounded with $|j_\alpha(x)| \leqslant j_\alpha(0) = 1$. As a consequence,

$$\|\mathscr{H}_\alpha(f)\|_\infty \leqslant \|f\|_{L^1_\alpha}. \tag{2.6}$$

Here $\|.\|_\infty$ is the usual essential supremum norm and $L^\infty(\mathbb{R}_+)$ and $L^\infty(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ will denote the usual spaces of essentially bounded functions.

It is also well known that the Hankel transform extends to an isometry on $L^2_\alpha(\mathbb{R}_+)$,

$$\|\mathscr{H}_\alpha(f)\|_{L^2_\alpha} = \|f\|_{L^2_\alpha}. \tag{2.7}$$

2.3. Generalized translation

Following Levitan [16], for any function $f \in C^2(\mathbb{R}_+)$ we define the generalized Bessel translation operator

$$\tau_y^\alpha f(x) = u(x,y); \qquad x,\ y \in \mathbb{R}_+,$$

as a solution of the following Cauchy problem:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x}\right)u(x,y) = \left(\frac{\partial^2}{\partial y^2} + \frac{2\alpha+1}{y}\frac{\partial}{\partial y}\right)u(x,y),$$

with initial conditions $u(x,0) = f(x)$ and $\frac{\partial}{\partial x}u(x,0) = 0$, here $\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x}$ is the differential Bessel operator. The solution of the Cauchy problem can be written out in explicit form:

$$\tau_x^\alpha f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2+y^2-2xy\cos\theta})(\sin\theta)^{2\alpha}\,\mathrm{d}\theta. \tag{2.8}$$

The operator τ_x^α can be also written by the formula

$$\tau_x^\alpha f(y) = \int_0^\infty f(t) W(x,y,t)\,\mathrm{d}\mu_\alpha(t), \tag{2.9}$$

where $W(x,y,t)\,\mathrm{d}\mu_\alpha(t)$ is a probability measure and $W(x,y,t)$ is defined by

$$W(x,y,t) = \begin{cases} \dfrac{2\pi^{\alpha+1/2}\Gamma(\alpha+1)^2}{\Gamma\left(\alpha+\frac{1}{2}\right)} \dfrac{\Delta(x,y,t)^{2\alpha-1}}{(xyt)^{2\alpha}}, & \text{if } |x-y| < t < x+y; \\ 0, & \text{otherwise}; \end{cases}$$

where

$$\Delta(x,y,t) = \left((x+y)^2 - t^2\right)^{1/2}\left(t^2 - (x-y)^2\right)^{1/2}$$

is the area of the triangle with side length x,y,t. Further, $W(x,y,t)\,\mathrm{d}\mu_\alpha(t)$ is a probability measure, so that, for $p \geqslant 1$, $|\tau_x^\alpha f|^p \leqslant \tau_x^\alpha |f|^p$ thus

$$\|\tau_x^\alpha f\|_{L^p_\alpha} \leqslant \|f\|_{L^p_\alpha}. \tag{2.10}$$

This allows to extend the definition of $\tau_x^\alpha f$ to functions $f \in L^p_\alpha(\mathbb{R}_+)$.

The Bessel convolution $f *_\alpha g$ of two functions f and g in $L^1_\alpha(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is defined by

$$f *_\alpha g(x) = \int_0^\infty f(t)\tau_x^\alpha g(t)\,\mathrm{d}\mu_\alpha(t) = \int_0^\infty \tau_x^\alpha f(t) g(t)\,\mathrm{d}\mu_\alpha(t), \;\; x \geqslant 0.$$

Then, if $1 \leqslant p,q,r \leqslant \infty$ are such that $1/p + 1/q - 1 = 1/r$, $f *_\alpha g \in L^r_\alpha(\mathbb{R}_+)$ and

$$\|f *_\alpha g\|_{L^r_\alpha} \leqslant \|f\|_{L^p_\alpha}\|g\|_{L^q_\alpha}.$$

This then allows to define $f *_\alpha g$ for $f \in L^p_\alpha(\mathbb{R}_+)$ and $g \in L^q_\alpha(\mathbb{R}_+)$. In particular, if $f \in L^1_\alpha(\mathbb{R}_+)$ and $g \in L^q_\alpha(\mathbb{R}_+)$, $q = 1$ or 2 then

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g). \tag{2.11}$$

Moreover for $f, g \in L^2_\alpha(\mathbb{R}_+)$, the function $f *_\alpha g$ belongs to $L^2_\alpha(\mathbb{R}_+)$ if and only if the function $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ belongs to $L^2_\alpha(\mathbb{R}_+)$ and then (2.11) holds.

2.4. The windowed Hankel transform

Following [7], for every $g \in L^2_\alpha(\mathbb{R}_+)$, the modulation of g by $\xi \in \widehat{\mathbb{R}}_+$ is defined by:

$$\mathscr{M}_\xi^\alpha g = g_\xi^\alpha := \mathscr{H}_\alpha\left(\sqrt{\tau_\xi^\alpha |\mathscr{H}_\alpha(g)|^2}\right). \tag{2.12}$$

Then for every $g \in L^2_\alpha(\mathbb{R}_+)$ and $\xi \in \widehat{\mathbb{R}}_+$, we have:

$$\left\|g_\xi^\alpha\right\|_{L^2_\alpha} = \|g\|_{L^2_\alpha} \quad \text{and} \quad \left\|\mathscr{H}_\alpha(g_\xi^\alpha)\right\|_\infty \leqslant \|\mathscr{H}_\alpha(g)\|_\infty. \tag{2.13}$$

By $g^{\alpha}_{x,\xi}$ we denote the phase-space shift of $g \in L^2_\alpha(\mathbb{R}_+)$ by $(x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+$,

$$g^{\alpha}_{x,\xi} = \tau^{\alpha}_x g^{\alpha}_{\xi}. \tag{2.14}$$

From (2.10), (2.13), the phase-space shift satisfies

$$\left\| g^{\alpha}_{x,\xi} \right\|_{L^2_\alpha} \leqslant \|g\|_{L^2_\alpha}. \tag{2.15}$$

For any function $f \in L^2_\alpha(\mathbb{R}_+)$, we define its windowed Hankel transform with respect to the window g by:

$$\mathscr{V}^{\alpha}_g(f)(x,\xi) = \int_0^{\infty} f(t)\overline{g^{\alpha}_{x,\xi}(t)}\,\mathrm{d}\mu_\alpha(s) = \left\langle f, g^{\alpha}_{x,\xi} \right\rangle_{\mu_\alpha}, \tag{2.16}$$

which can also be written in the form

$$\mathscr{V}^{\alpha}_g(f)(x,\xi) = f *_\alpha g^{\alpha}_{\xi}(x), \qquad (x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+. \tag{2.17}$$

Here $\langle \cdot,\cdot \rangle_{\mu_\alpha}$ is the usual inner product in the Hilbert space $L^2_\alpha(\mathbb{R}_+)$. Thus from the Cauchy-Schwartz inequality and (2.10), (2.13) we have:

$$\left\| \mathscr{V}^{\alpha}_g(f) \right\|_{\infty} \leqslant \|f\|_{L^2_\alpha} \|g\|_{L^2_\alpha}. \tag{2.18}$$

Moreover the windowed Hankel transform satisfies the following properties (see [3, 7]).

PROPOSITION 2.1. *Let $g \in L^2_\alpha(\mathbb{R}_+)$ be a nonzero window function. Then we have:*

1. *A Plancherel-type theorem: for every $f \in L^2_\alpha(\mathbb{R}_+)$:*

$$\left\| \mathscr{V}^{\alpha}_g(f) \right\|_{L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)} = \|f\|_{L^2_\alpha} \|g\|_{L^2_\alpha}. \tag{2.19}$$

2. *An inversion formula : for every $f \in L^2_\alpha(\mathbb{R}_+)$,*

$$f(\cdot) = \frac{1}{\|g\|^2_{L^2_\alpha}} \int_{\mathbb{R}_+ \times \widehat{\mathbb{R}}_+} \mathscr{V}^{\alpha}_g(f)(x,\xi) g^{\alpha}_{x,\xi}(\cdot)\,\mathrm{d}\nu_\alpha(x,\xi). \tag{2.20}$$

3. *An orthogonality relation : for every $f, h \in L^2_\alpha(\mathbb{R}_+)$,*

$$\left\langle \mathscr{V}^{\alpha}_g(f), \mathscr{V}^{\alpha}_g(h) \right\rangle_{\nu_\alpha} = \|g\|^2_{L^2_\alpha} \langle f, h \rangle_{\mu_\alpha}, \tag{2.21}$$

where $\langle \cdot,\cdot \rangle_{\nu_\alpha}$ is the usual inner product in the Hilbert space $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$.

The inversion formula for the windowed Hankel transform (2.20) is well defined in the weak sense for all $f \in L^2_\alpha(\mathbb{R}_+)$ and $0 \neq g \in L^2_\alpha(\mathbb{R}_+)$.

3. Time–frequency concentration via the windowed Hankel transform

In the remainder of this section, g will be a non-zero window function such that $\|g\|_{L^2_\alpha} = 1$ and $\Sigma \subset \mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ will be a subset of finite measure $0 < \nu_\alpha(\Sigma) < \infty$.

3.1. Time–frequency localization operators

For F a function in $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$, we define the formal adjoint $(\mathscr{V}_g^\alpha)^*$ by

$$(\mathscr{V}_g^\alpha)^* F(t) = \int_{\mathbb{R}_+ \times \widehat{\mathbb{R}}_+} F(x,\xi) g^\alpha_{x,\xi}(t) \, \mathrm{d}\nu_\alpha(x,\xi), \tag{3.1}$$

where the integral is defined weakly by

$$\left\langle (\mathscr{V}_g^\alpha)^* F, f \right\rangle_{\mu_\alpha} = \left\langle F, \mathscr{V}_g^\alpha(f) \right\rangle_{\nu_\alpha}. \tag{3.2}$$

DEFINITION 3.1. Let σ be a bounded nonnegative function on $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$. The time-frequency localization operator $\mathscr{A}_\sigma^g$ with window g and symbol σ is formally defined as

$$\mathscr{A}_\sigma^g f(\cdot) = \int_{\mathbb{R}_+ \times \widehat{\mathbb{R}}_+} \sigma(x,\xi) \mathscr{V}_g^\alpha(f)(x,\xi) g^\alpha_{x,\xi}(\cdot) \, \mathrm{d}\nu_\alpha(x,\xi) = (\mathscr{V}_g^\alpha)^* \sigma \mathscr{V}_g^\alpha f(\cdot).$$

Note that, if $\sigma \equiv 1$, then by the inversion formula (2.20), $\mathscr{A}_\sigma^g f = f$. Moreover, if σ is supported on Σ, then $\mathscr{A}_\sigma^g f$ is interpreted as the part of f that lives essentially in Σ.

For the purpose of this research, we shall keep our focus on time–frequency localization operators $\mathscr{A}_\sigma^g$ with symbol $\sigma = \chi_\Sigma$. In this case, we also write the localization operator as $\mathscr{A}_\Sigma^g$. Note that, $\mathscr{A}_\Sigma^g : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is bounded, with

$$\left\| \mathscr{A}_\Sigma^g \right\|_{B\left(L^2_\alpha(\mathbb{R}_+)\right)} \leqslant 1. \tag{3.3}$$

Indeed, since $(\mathscr{V}_g^\alpha)^*$ is a bounded operator from $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ to $L^2_\alpha(\mathbb{R}_+)$ with operator norm less than 1, then for every $f \in L^2_\alpha(\mathbb{R}_+)$,

$$\left\| \mathscr{A}_\Sigma^g f \right\|_{L^2_\alpha} = \left\| (\mathscr{V}_g^\alpha)^* (\chi_\Sigma \mathscr{V}_g^\alpha f) \right\|_{L^2_\alpha} \leqslant \left\| (\mathscr{V}_g^\alpha)^* \right\| \left\| \mathscr{V}_g^\alpha f \right\|_{L^2_\alpha} \leqslant \|f\|_{L^2_\alpha}.$$

It is usually more convenient to use the alternative weak definition of $\mathscr{A}_\sigma^g$ given by

$$\left\langle \mathscr{A}_\sigma^g f, h \right\rangle_{\mu_\alpha} = \left\langle \sigma \, \mathscr{V}_g^\alpha(f), \mathscr{V}_g^\alpha(h) \right\rangle_{\nu_\alpha} = \left\langle \sigma, \overline{\mathscr{V}_g^\alpha(f)} \mathscr{V}_g^\alpha(h) \right\rangle_{\nu_\alpha}. \tag{3.4}$$

Let $\mathbb{H} = \mathscr{V}_g^\alpha \left[L^2_\alpha(\mathbb{R}_+) \right] \subset L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ be the (closed) range of the $\mathscr{V}_g^\alpha$, and let P_g (or $P_{\mathbb{H}}$) be the orthogonal projection from $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ onto $\mathbb{H}$. The orthogonal projector P_g is an integral operator explicitly given by,

$$P_g F(z) = \int_{\mathbb{R}_+ \times \widehat{\mathbb{R}}_+} F(z') \mathscr{K}_g(z; z') \, \mathrm{d}\nu_\alpha(z'), \qquad z = (x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+. \tag{3.5}$$

Using this description, it follows that (see [12, Proposition 4.1]) $\mathbb{H}$ is a reproducing kernel Hilbert space in $L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$ with kernel function $\mathscr{K}_g$ defined by:

$$\mathscr{K}_g\Big((x',\xi');(x,\xi)\Big)=g^\alpha_{x,\xi}*_\alpha g^\alpha_{\xi'}(x')=\mathscr{V}^\alpha_g(g^\alpha_{x,\xi})(x',\xi'). \tag{3.6}$$

This means that each function $F\in\mathbb{H}$ is continuous and satisfies:

$$F(x,\xi)=\Big\langle F,\mathscr{K}_g\Big(\cdot;(x,\xi)\Big)\Big\rangle_{\nu_\alpha},\quad (x,\xi)\in\mathbb{R}_+\times\widehat{\mathbb{R}}_+. \tag{3.7}$$

Since $\mathscr{K}_g$ is the integral kernel of an orthogonal projection, it satisfies

$$\overline{\mathscr{K}_g(z';z)}=\mathscr{K}_g(z;z'),\qquad z=(x,\xi),\ \ z'=(x',\xi')\in\mathbb{R}_+\times\widehat{\mathbb{R}}_+, \tag{3.8}$$

and

$$\mathscr{K}_g(z;z')=\int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+}\mathscr{K}_g(z;z'')\mathscr{K}_g(z'';z')\,\mathrm{d}\nu_\alpha(z''). \tag{3.9}$$

We introduce the orthogonal projections P_Σ on $L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$, known as the time–frequency limiting operator defined by:

$$P_\Sigma F=F\chi_\Sigma;\quad F\in L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+).$$

DEFINITION 3.2. Let $0<\varepsilon<1$. Then,

1. a nonzero function $f\in L^2_\alpha(\mathbb{R}_+)$ is ε-time–frequency concentrated inside Σ if,

$$\|P_{\Sigma^c}\mathscr{V}^\alpha_g(f)\|^2_{L^2(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\leqslant\varepsilon\|f\|^2_{L^2_\alpha}, \tag{3.10}$$

2. a nonzero function $f\in L^2_\alpha(\mathbb{R}_+)$ is ε-localized with respect to an operator L if

$$\|Lf-f\|^2_{L^2_\alpha}\leqslant\varepsilon\|f\|^2_{L^2_\alpha}. \tag{3.11}$$

The definition (3.10) is the analogous for our setting of that given in reference [9] for the windowed Fourier transform, and the definition (3.11) has been introduced in [2] to refine the Landau-Pollak degrees of freedom estimate. We will use and compare these to measures (see Proposition 3.2). Notice also that when $\varepsilon=0$, then Σ will be the exact support of $\mathscr{V}^\alpha_g(f)$, so that Inequality (3.10) with $0<\varepsilon<1$ means that $\mathscr{V}^\alpha_g(f)$ is "practically zero" outside Σ and then Σ may be considered as the "essential" support of $\mathscr{V}^\alpha_g(f)$.

Since $\mathbb{H}$ is a reproducing kernel Hilbert space in $L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$, then $P_\Sigma P_g$ is a Hilbert-Schmidt operator with (see [12, Inequality (4.8)]),

$$\|P_\Sigma P_g\|^2_{HS}\leqslant\nu_\alpha(\Sigma). \tag{3.12}$$

DEFINITION 3.3. We define the Gabor-Toeplitz operator $T_{g,\Sigma} : \mathbb{H} \to \mathbb{H}$ by,

$$T_{g,\Sigma} F = P_g P_\Sigma F. \tag{3.13}$$

PROPOSITION 3.1. *The time–frequency operator $\mathscr{A}_\Sigma^g$ is Hilbert-Schmidt.*

Proof. Since $\mathscr{A}_\Sigma^g = (\mathscr{V}_g^\alpha)^* P_\Sigma \mathscr{V}_g^\alpha$, then

$$\left(\mathscr{V}_g^\alpha \mathscr{A}_\Sigma^g (\mathscr{V}_g^\alpha)^*\right) F = P_g P_\Sigma F = T_{g,\Sigma} F, \quad F \in \mathbb{H}. \tag{3.14}$$

Therefore the time–frequency operator $\mathscr{A}_\Sigma^g$ and the Gabor-Toeplitz operator $T_{g,\Sigma}$ are related by

$$(\mathscr{V}_g^\alpha)^* T_{g,\Sigma} \mathscr{V}_g^\alpha = \mathscr{A}_\Sigma^g. \tag{3.15}$$

Consequently $T_{g,\Sigma}$ and $\mathscr{A}_\Sigma^g$ enjoy the same spectral properties, in particular $\mathscr{A}_\Sigma^g$ is Hilbert-Schmidt. □

Boundedness and Schatten class properties of time–frequency localization operators in terms of properties of the symbol σ have been studied in [3]. More precisely the authors in [3] have proved the following result.

THEOREM 3.1. *Let σ be symbol in $L_\alpha^p(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$, $1 \leqslant p \leqslant \infty$. Then the localization operator $\mathscr{A}_\sigma^g : L_\alpha^2(\mathbb{R}_+) \to L_\alpha^2(\mathbb{R}_+)$ is in S_p with,*

$$\|\mathscr{A}_\sigma^g\|_{S_p} \leqslant 4^{\frac{1}{p}} \|\sigma\|_{L_\alpha^p(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)}. \tag{3.16}$$

Since the time–frequency localization operator $\mathscr{A}_\Sigma^g = (\mathscr{V}_g^\alpha)^* \chi_\Sigma \mathscr{V}_g^\alpha$ that we consider is a compact and self–adjoint operator, the spectral theorem gives the following spectral representation:

$$\mathscr{A}_\Sigma^g f = \sum_{n=1}^\infty s_n(\Sigma) \langle f, \varphi_n^\Sigma \rangle_{\mu_\alpha} \varphi_n^\Sigma, \qquad f \in L_\alpha^2(\mathbb{R}_+), \tag{3.17}$$

where $\{s_n(\Sigma)\}_{n=1}^\infty$ are the positive eigenvalues arranged in a non-increasing manner and $\{\varphi_n^\Sigma\}_{n=1}^\infty$ is the corresponding orthonormal set of eigenfunctions. Note that $s_n(\Sigma) \searrow 0$, and by (3.3), we have for all $n \geqslant 1$,

$$s_n(\Sigma) \leqslant s_1(\Sigma) \leqslant 1. \tag{3.18}$$

This, together with (3.15), we deduce that the Gabor-Toeplitz operator $T_{g,\Sigma}$ can be diagonalized as

$$T_{g,\Sigma} F = \sum_{n=1}^\infty s_n(\Sigma) \langle F, \phi_n^\Sigma \rangle_{\nu_\alpha} \phi_n^\Sigma, \qquad F \in \mathbb{H}, \tag{3.19}$$

where $\phi_n^\Sigma = \mathscr{V}_g^\alpha(\varphi_n^\Sigma)$. The functions φ_n^Σ and the eigenvalues $s_n(\Sigma)$ depend on the choice of the window g, but we do not make this dependence explicit in the notation.

LEMMA 3.1. *For all* $z = (x, \xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+$, *let* $\Theta_\Sigma(z) = \|\mathscr{V}_g^\alpha(g_{x,\xi})\|_{L^2(\Sigma,\nu_\alpha)}$. *Then*

$$\Theta_\Sigma(z) = \sum_{n=1}^{\infty} s_n(\Sigma)\left|\phi_n^\Sigma(z)\right|^2. \tag{3.20}$$

Proof. For $z = (x, \xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+$, we have from (3.7), that for all $F \in \mathbb{H}$,

$$F(z) = \left\langle F, \mathscr{K}_g(\cdot\,;z)\right\rangle_{\nu_\alpha}. \tag{3.21}$$

Therefore from (3.6)

$$\begin{aligned}\left\langle T_{g,\Sigma}\mathscr{K}_g(\cdot\,;z), \mathscr{K}_g(\cdot\,;z)\right\rangle_{\nu_\alpha} &= \left\langle P_\Sigma\mathscr{K}_g(\cdot\,;z), \mathscr{K}_g(\cdot\,;z)\right\rangle_{\nu_\alpha} \\ &= \int_\Sigma \mathscr{K}_g(z';z)\,\overline{\mathscr{K}_g(z';z)}\,\mathrm{d}\nu_\alpha(z') \\ &= \Theta_\Sigma(z).\end{aligned}$$

Let $\{\psi^\Sigma\}_{n=1}^\infty \subset \mathbb{H}$ be an orthonormal basis of $\mathrm{Ker}(T_{g,\Sigma})$ (eventually empty). Hence $\{\phi^\Sigma\}_{n=1}^\infty \cup \{\psi^\Sigma\}_{n=1}^\infty$ is an orthonormal basis of $\mathbb{H}$ and therefore the reproducing kernel $\mathscr{K}_g$ can be written as

$$\mathscr{K}_g(z;z') = \overline{\mathscr{K}_g(z';z)} = \sum_{n=1}^{\infty} \phi_n^\Sigma(z)\overline{\phi_n^\Sigma(z')} + \sum_{n=1}^{\infty} \psi_n^\Sigma(z)\overline{\psi_n^\Sigma(z')}. \tag{3.22}$$

Using this we compute again

$$\begin{aligned}\left\langle T_{g,\Sigma}\mathscr{K}_g(\cdot\,;z), \mathscr{K}_g(\cdot\,;z)\right\rangle_{\nu_\alpha} &= \left\langle T_{g,\Sigma}\sum_{n=1}^{\infty}\overline{\phi_n^\Sigma(z)}\phi_n^\Sigma, \sum_{k=1}^{\infty}\overline{\phi_k^\Sigma(z)}\phi_k^\Sigma\right\rangle_{\nu_\alpha} \\ &= \sum_{n,k}\overline{\phi_n^\Sigma(z)}\phi_k^\Sigma(z)\left\langle T_{g,\Sigma}\phi_n^\Sigma, \phi_k^\Sigma\right\rangle_{\nu_\alpha} \\ &= \sum_{n=1}^{\infty} s_n(\Sigma)\left|\phi_n^\Sigma(z)\right|^2,\end{aligned}$$

and the conclusion follows. □

3.2. Functions time–frequency concentrated in Σ

We denote by $\mathscr{C}(\varepsilon, \Sigma, g)$ the set of functions in $L^2_\alpha(\mathbb{R}_+)$ that are ε-time–frequency-concentrated in Σ and by relation (3.4), a function $f \in L^2_\alpha(\mathbb{R}_+)$ is in $\mathscr{C}(\varepsilon, \Sigma, g)$ if,

$$\left\langle \mathscr{A}_\Sigma^g f, f\right\rangle_{\mu_\alpha} \geqslant (1-\varepsilon)\|f\|^2_{L^2_\alpha}, \tag{3.23}$$

or equivalently

$$\left\langle \left(I - \mathscr{A}_\Sigma^g\right) f, f\right\rangle_{\mu_\alpha} \leqslant \varepsilon\|f\|^2_{L^2_\alpha}, \tag{3.24}$$

where I is the identity operator.

Since

$$\left\langle \mathscr{A}_\Sigma^g f, f\right\rangle_{\mu_\alpha} = \sum_{n=1}^{\infty} s_n(\Sigma)\left|\left\langle f, \varphi_n^\Sigma\right\rangle_{\mu_\alpha}\right|^2 = \left\|\mathscr{V}_g^\alpha(f)\right\|^2_{L^2(\Sigma,\nu_\alpha)}, \tag{3.25}$$

then the operator $\mathscr{A}_\Sigma^g$ is useful in studying the following optimization problem

$$\text{Maximize} \quad \left\|\mathscr{V}_g^\alpha(f)\right\|^2_{L^2(\Sigma,\nu_\alpha)}, \quad \|f\|_{L^2_\alpha} = 1,$$

which aims to look for the function that has a spectrogram that is well concentrated in Σ. Consequently, φ_1^Σ, the first eigenfunction of the compact self-adjoint operator $\mathscr{A}_\Sigma^g$, solves the problem:

$$s_1(\Sigma) = \left\|\mathscr{V}_g^\alpha(\varphi_1^\Sigma)\right\|_{L^2(\Sigma,\nu_\alpha)} = \max\left\{\left\langle \mathscr{A}_\Sigma^g f, f\right\rangle_{\mu_\alpha} \,:\, \|f\|_{L^2_\alpha} = 1\right\}. \tag{3.26}$$

Moreover, the min-max lemma for self-adjoint operators states that (see *e.g.* [22, Section 95]),

$$s_n(\Sigma) = \max\left\{\left\langle \mathscr{A}_\Sigma^g f, f\right\rangle_{\mu_\alpha} \,:\, \|f\|_{L^2_\alpha} = 1,\ f \perp \varphi_1^\Sigma, \ldots, \varphi_{n-1}^\Sigma\right\}. \tag{3.27}$$

So the eigenvalues of $\mathscr{A}_\Sigma^g$ determines the number of orthogonal functions that have a well-concentrated spectrogram in Σ. Moreover if φ_n^Σ is an eigenfunction of $\mathscr{A}_\Sigma^g$ with eigenvalue $s_n(\Sigma) \geqslant (1-\varepsilon)$, then from the spectral representation,

$$\left\langle \mathscr{A}_\Sigma^g \varphi_n^\Sigma, \varphi_n^\Sigma\right\rangle_{\mu_\alpha} = s_n(\Sigma) \geqslant (1-\varepsilon). \tag{3.28}$$

Hence φ_n^Σ is in $\mathscr{C}(\varepsilon,\Sigma,g)$. In addition if we denote by V_N the span of the first N eigenfunctions of the time-frequency localization operator $\mathscr{A}_\Sigma^g$, then for $f \in V_N$,

$$\begin{aligned}\left\langle \mathscr{A}_\Sigma^g f, f\right\rangle_{\mu_\alpha} &= \sum_{n=1}^{N} s_n(\Sigma)\left|\left\langle f, \varphi_n^\Sigma\right\rangle_{\mu_\alpha}\right|^2 \\ &\geqslant s_N(\Sigma)\sum_{n=1}^{N}\left|\left\langle f, \varphi_n^\Sigma\right\rangle_{\mu_\alpha}\right|^2 = s_N(\Sigma)\|f\|^2_{L^2_\alpha}.\end{aligned} \tag{3.29}$$

This implies that a function f in V_N is in $\mathscr{C}\left(1-s_N(\Sigma),\Sigma,g\right)$. So for a properly chosen N, functions in V_N are in $\mathscr{C}(\varepsilon,\Sigma,g)$. Moreover the quantity

$$n(\varepsilon,\Sigma) = \operatorname{card}\{n \,:\, s_n(\Sigma) \geqslant 1-\varepsilon\}$$

determines the maximum dimension of a subspace $V \subset L^2_\alpha(\mathbb{R}_+)$ of signals $f \in V$ that are in $\mathscr{C}(\varepsilon,\Sigma,g)$.

Landau in [17] introduced the notion of ε-approximated eigenvalues and eigenfunctions, that is, ρ is said to be an ε-approximated eigenvalue of L if there exists a unit L^2_α-norm function f in $L^2_\alpha(\mathbb{R}_+)$, such that

$$\|Lf - \rho f\|_{L^2_\alpha} \leqslant \varepsilon. \tag{3.30}$$

Then f is called an ε-approximated eigenfunction corresponding to ρ. So a function $f \in L^2_\alpha(\mathbb{R}_+)$ that is ε-localized with respect to L is a $\sqrt{\varepsilon}$-approximated eigenfunction of L corresponding to 1. Moreover we have the following comparison.

PROPOSITION 3.2. *If $f \in L^2_\alpha(\mathbb{R}_+)$ is in $\mathscr{C}(\varepsilon,\Sigma,g)$, then f is also ε-localized with respect to $\mathscr{A}^g_\Sigma$. On the other hand, if $f \in L^2_\alpha(\mathbb{R}_+)$ is ε-localized with respect to $\mathscr{A}^g_\Sigma$, then f is in $\mathscr{C}(\varepsilon+\sqrt{\varepsilon},\Sigma,g)$.*

Proof. The time–frequency operator $\mathscr{A}^g_\Sigma$ is bounded with $\|\mathscr{A}^g_\Sigma\|_{B\left(L^2_\alpha(\mathbb{R}_+)\right)} \leqslant 1$, then

$$\left\langle(\mathscr{A}^g_\Sigma)^2 f,f\right\rangle_{\mu_\alpha} \leqslant \left\langle\mathscr{A}^g_\Sigma f,f\right\rangle_{\mu_\alpha}, \tag{3.31}$$

or equivalently,

$$\left\langle(I-\mathscr{A}^g_\Sigma)^2 f,f\right\rangle_{\mu_\alpha} \leqslant \left\langle(I-\mathscr{A}^g_\Sigma) f,f\right\rangle_{\mu_\alpha}. \tag{3.32}$$

Since $\mathscr{A}^g_\Sigma$ is self-adjoint, the left-hand side is equal to $\left\|\mathscr{A}^g_\Sigma f-f\right\|^2_{L^2_\alpha}$, so by (3.24) the first statement is obtained.

For the second statement, we observe that

$$\begin{aligned}
2\left\langle(I-\mathscr{A}^g_\Sigma) f,f\right\rangle_{\mu_\alpha} &= \left\|\mathscr{A}^g_\Sigma f-f\right\|^2_{L^2_\alpha}+\|f\|^2_{L^2_\alpha}-\left\|\mathscr{A}^g_\Sigma f\right\|^2_{L^2_\alpha}\\
&\leqslant \left\|\mathscr{A}^g_\Sigma f-f\right\|^2_{L^2_\alpha}+\left(\left\|\mathscr{A}^g_\Sigma f-f\right\|_{L^2_\alpha}+\left\|\mathscr{A}^g_\Sigma f\right\|_{L^2_\alpha}\right)^2-\left\|\mathscr{A}^g_\Sigma f\right\|^2_{L^2_\alpha}\\
&= 2\left\|\mathscr{A}^g_\Sigma f-f\right\|^2_{L^2_\alpha}+2\left\|\mathscr{A}^g_\Sigma f-f\right\|_{L^2_\alpha}\left\|\mathscr{A}^g_\Sigma f\right\|_{L^2_\alpha}.
\end{aligned}$$

So, by (3.3) we have

$$\left\langle(I-\mathscr{A}^g_\Sigma) f,f\right\rangle_{\mu_\alpha} \leqslant \left\|\mathscr{A}^g_\Sigma f-f\right\|^2_{L^2_\alpha}+\left\|\mathscr{A}^g_\Sigma f-f\right\|_{L^2_\alpha}\|f\|_{L^2_\alpha}, \tag{3.33}$$

and the result follows. □

Relation (3.29) implies that any function f in $V_N=\operatorname{span}\{\varphi^\Sigma_n\}_{n=1}^N$ is in $\mathscr{C}(1-s_N(\Sigma),\Sigma,g)$. In contrast, functions which are in $\mathscr{C}(1-s_N(\Sigma),\Sigma,g)$ need not lie in V_N. Nevertheless based on an idea from the recent paper [9], we obtain the following theorem that characterizes functions that are in $\mathscr{C}(1-s_N(\Sigma),\Sigma,g)$.

THEOREM 3.2. *Let N_0 be the integer such that and $s_{1+N_0}(\Sigma)<1-\varepsilon\leqslant s_{N_0}(\Sigma)$. Furthermore, let $f_{\ker}$ denote the orthogonal projection of f onto the kernel $\operatorname{Ker}(\mathscr{A}^g_\Sigma)$ of $\mathscr{A}^g_\Sigma$. A function $f\in L^2_\alpha(\mathbb{R}_+)$ is in $\mathscr{C}(\varepsilon,\Sigma,g)$ if and only if,*

$$\sum_{n=1}^{N_0}(s_n(\Sigma)+\varepsilon-1)\left|\left\langle f,\varphi^\Sigma_n\right\rangle_{\mu_\alpha}\right|^2 \geqslant (1-\varepsilon)\|f_{\ker}\|^2_{L^2_\alpha} + \sum_{n=N_0+1}^{\infty}(1-\varepsilon-s_n(\Sigma))\left|\left\langle f,\varphi^\Sigma_n\right\rangle_{\mu_\alpha}\right|^2.$$

Proof. The eigenfunctions $\left\{\varphi^\Sigma_n\right\}_{n=1}^\infty$ form an orthonormal subset in $L^2_\alpha(\mathbb{R}_+)$, possibly incomplete if $\operatorname{Ker}(\mathscr{A}^g_\Sigma)\neq\{0\}$; hence, we can write

$$f=\sum_{n=1}^{\infty}\left\langle f,\varphi^\Sigma_n\right\rangle_{\mu_\alpha}\varphi^\Sigma_n+f_{\ker}, \tag{3.34}$$

where $f_{\ker} \in \operatorname{Ker}(\mathscr{A}_\Sigma^g)$. Then

$$\left\langle \mathscr{A}_\Sigma^g f, f \right\rangle_{\mu_\alpha} = \sum_{n=1}^{\infty} s_n(\Sigma) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2. \tag{3.35}$$

So the function f is ε-time–frequency concentrated on Σ if and only if

$$\sum_{n=1}^{\infty} s_n(\Sigma) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 \geqslant (1-\varepsilon) \left(\|f_{\ker}\|_{L_\alpha^2}^2 + \sum_{n=1}^{\infty} \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 \right), \tag{3.36}$$

and the conclusion follows. □

REMARK 3.1.

1. A function $f \in L_\alpha^2(\mathbb{R}_+)$ is in $\mathscr{C}(1 - s_N(\Sigma), \Sigma, g)$ if and only if,

$$\sum_{n=1}^{N-1} (s_n(\Sigma) - s_N(\Sigma)) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 \geqslant s_N(\Sigma) \|f_{\ker}\|_{L_\alpha^2}^2 + \sum_{n=N+1}^{\infty} (s_N(\Sigma) - s_n(\Sigma)) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2.$$

2. Despite the interpretation of $\mathscr{A}_\Sigma^g$ as the part of f that essentially lives in Σ, it is possible that the resulting function $\mathscr{A}_\Sigma^g f$ is not ε-time–frequency concentrated on Σ. In fact, for every eigenfunction φ_n^Σ with corresponding eigenvalue $s_n(\Sigma) < 1 - \varepsilon$,

$$\left\langle \mathscr{A}_\Sigma^g \left(\mathscr{A}_\Sigma^g \varphi_n^\Sigma \right), \mathscr{A}_\Sigma^g \varphi_n^\Sigma \right\rangle_{\mu_\alpha} = s_n^3(\Sigma) = s_n(\Sigma) \|\mathscr{A}_\Sigma^g \varphi_n^\Sigma\|_{L_\alpha^2}^2 < (1-\varepsilon) \|\mathscr{A}_\Sigma^g \varphi_n^\Sigma\|_{L_\alpha^2}^2.$$

   Therefore $\mathscr{A}_\Sigma^g \varphi_n^\Sigma$ is not ε-time–frequency concentrated on Σ.

While a function f that is ε-time–frequency concentrated on Σ does not necessarily lies in some subspace V_N of eigenfunctions of $\mathscr{A}_\Sigma^g$, it can be approximated using a finite number of such eigenfunctions. Let $\mathscr{P}_{V_N}$ denote the orthogonal projection onto the subspace V_N.

THEOREM 3.3. *Let f be a function in $\mathscr{C}(\varepsilon, \Sigma, g)$. For fixed $c > 1$, let $\left\{ \varphi_n^\Sigma \right\}_{n=1}^N$ be the system of all eigenfunctions of $\mathscr{A}_\Sigma^g$ corresponding to eigenvalues $s_n(\Sigma) > 1 - \frac{1}{c}$. Then*

1. $\|\mathscr{P}_{V_N} f\|_{L_\alpha^2}^2 \geqslant (1 - c\varepsilon) \|f\|_{L_\alpha^2}^2$,
2. $\|f - \mathscr{P}_{V_N} f\|_{L_\alpha^2}^2 < c\varepsilon \|f\|_{L_\alpha^2}^2$,
3. $\left\langle \mathscr{A}_\Sigma^g \mathscr{P}_{V_N} f, \mathscr{P}_{V_N} f \right\rangle_{\mu_\alpha} \geqslant s_N(\Sigma)(1 - c\varepsilon) \|f\|_{L_\alpha^2}^2$.

Proof. We have, by assumption:

$$\left\langle \mathscr{A}_\Sigma^g f, f \right\rangle_{\mu_\alpha} = \sum_{n=1}^{\infty} s_n(\Sigma) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 \geqslant (1-\varepsilon)\|f\|_{L^2_\alpha}^2. \tag{3.37}$$

Assume towards a contradiction that

$$\left\| \mathscr{P}_{V_N} f = \sum_{n=1}^{N} \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \varphi_n^\Sigma \right\|_{L^2_\alpha}^2 = \sum_{n=1}^{N} \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 < (1-c\varepsilon)\|f\|_{L^2_\alpha}^2. \tag{3.38}$$

Since

$$\|f\|_{L^2_\alpha}^2 = \|f_{\ker}\|_{L^2_\alpha}^2 + \sum_{n=1}^{\infty} \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2, \tag{3.39}$$

then

$$\sum_{n=N+1}^{\infty} \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 = \|f\|_{L^2_\alpha}^2 - \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 - \|f_{\ker}\|_{L^2_\alpha}^2. \tag{3.40}$$

Therefore

$$\sum_{n=N+1}^{\infty} s_n(\Sigma) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 \leqslant \frac{c-1}{c} \left(\|f\|_{L^2_\alpha}^2 - \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 - \|f_{\ker}\|_{L^2_\alpha}^2 \right), \tag{3.41}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} s_n(\Sigma) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 &\leqslant s_1(\Sigma) \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 \\ &\quad + \frac{c-1}{c} \left(\|f\|_{L^2_\alpha}^2 - \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 - \|f_{\ker}\|_{L^2_\alpha}^2 \right). \end{aligned}$$

Moreover, by (3.18),

$$\begin{aligned} \left\langle \mathscr{A}_\Sigma^g f, f \right\rangle_{\mu_\alpha} &\leqslant \frac{c-1}{c} \|f\|_{L^2_\alpha}^2 + \frac{1}{c} \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 - \frac{c-1}{c} \|f_{\ker}\|_{L^2_\alpha}^2 \\ &< \frac{c-1}{c} \|f\|_{L^2_\alpha}^2 + \frac{1}{c} (1-c\varepsilon) \|f\|_{L^2_\alpha}^2 - \frac{c-1}{c} \|f_{\ker}\|_{L^2_\alpha}^2 \\ &= (1-\varepsilon) \|f\|_{L^2_\alpha}^2 - \frac{c-1}{c} \|f_{\ker}\|_{L^2_\alpha}^2 < (1-\varepsilon) \|f\|_{L^2_\alpha}^2. \end{aligned}$$

This contradicts (3.37) and we conclude for the first inequality.

For the second inequality we have,

$$\|f\|_{L^2_\alpha}^2 = \|\mathscr{P}_{V_N} f + (f - \mathscr{P}_{V_N} f)\|_{L^2_\alpha}^2 = \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 + \|f - \mathscr{P}_{V_N} f\|_{L^2_\alpha}^2. \tag{3.42}$$

It follows then,

$$\begin{aligned} \|f - \mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 &= \|f\|_{L^2_\alpha}^2 - \|\mathscr{P}_{V_N} f\|_{L^2_\alpha}^2 \\ &\leqslant \|f\|_{L^2_\alpha}^2 - (1-c\varepsilon)\|f\|_{L^2_\alpha}^2 = c\varepsilon \|f\|_{L^2_\alpha}^2. \end{aligned}$$

Finally for the third inequality,

$$\begin{aligned}\left\langle \mathscr{A}^g_\Sigma \mathscr{P}_{V_N} f, \mathscr{P}_{V_N} f \right\rangle_{\mu_\alpha} &= \sum_{n=1}^{N} s_n(\Sigma) \left| \left\langle f, \varphi_n^\Sigma \right\rangle_{\mu_\alpha} \right|^2 \\ &\geqslant s_N(\Sigma) \|\mathscr{P}_{V_N} f\|^2_{L^2_\alpha} \\ &\geqslant s_N(\Sigma)(1-c\varepsilon)\|f\|^2_{L^2_\alpha}.\end{aligned}$$

This completes the proof of the proposition. □

3.3. Spectrogram of a subspace

Given an N-dimensional subspace V of $L^2_\alpha(\mathbb{R}_+)$, $\mathscr{P}_V$ the orthogonal projection onto V with projection kernel κ_V, i.e.

$$\mathscr{P}_V f(\cdot) = \int_{\mathbb{R}_+} \kappa_V(\cdot,t) f(t)\,\mathrm{d}\mu_\alpha(t). \tag{3.43}$$

Recall that if $\{e_n\}_{n=1}^N$ is an orthonormal basis of V, then

$$\kappa_V(x,t) = \sum_{n=1}^{N} e_n(x)\overline{e_n(t)}. \tag{3.44}$$

The kernel κ_V is independent of the choice of orthonormal basis for V.

DEFINITION 3.4. The spectrogram of the subspace V with window function g is defined as:

$$\mathrm{SPEC}_g V(x,\xi) := \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \kappa_V(t,y)\overline{g^\alpha_{x,\xi}(t)} g^\alpha_{x,\xi}(y)\,\mathrm{d}\nu_\alpha(t,y). \tag{3.45}$$

Then we have the following result.

LEMMA 3.2. *The spectrogram* $\mathrm{SPEC}_g V$ *is given by,*

$$\mathrm{SPEC}_g V(x,\xi) = \sum_{n=1}^{N} |\mathscr{V}^\alpha_g(e_n)(x,\xi)|^2. \tag{3.46}$$

Proof. We have

$$\begin{aligned}\mathrm{SPEC}_g V(x,\xi) &= \int_0^\infty \int_0^\infty \sum_{n=1}^{N} e_n(t)\overline{e_n(y)} g^\alpha_{x,\xi}(y)\overline{g^\alpha_{x,\xi}(t)}\,\mathrm{d}\nu_\alpha(t,y) \\ &= \sum_{n=1}^{N} \int_0^\infty e_n(t)\overline{g^\alpha_{x,\xi}(t)}\,\mathrm{d}\mu_\alpha(t) \int_0^\infty \overline{e_n(y)} g^\alpha_{x,\xi}(y)\,\mathrm{d}\mu_\alpha(y) \\ &= \sum_{n=1}^{N} \mathscr{V}^\alpha_g(e_n)(x,\xi)\overline{\mathscr{V}^\alpha_g(e_n)(x,\xi)}.\end{aligned}$$

This allows as to conclude. □

Let $\rho_\Sigma := \mathrm{SPEC}_g V_{n(\varepsilon,\Sigma)}$ (called the accumulated spectrogram, see *e.g.* [1]), where we assume that $n(\varepsilon,\Sigma) = \lceil \nu_\alpha(\Sigma) \rceil$ is the smallest integer greater than or equal to $\nu_\alpha(\Sigma)$ and

$$V_{\lceil \nu_\alpha(\Sigma) \rceil} = \operatorname{span}\left\{\varphi_n^\Sigma\right\}_{n=1}^{\lceil \nu_\alpha(\Sigma) \rceil}.$$

Then

$$\rho_\Sigma = \sum_{n=1}^{\lceil \nu_\alpha(\Sigma) \rceil} |\mathscr{V}_g^\alpha(\varphi_n^\Sigma)|^2 = \sum_{n=1}^{\lceil \nu_\alpha(\Sigma) \rceil} \left|\phi_n^\Sigma\right|^2. \tag{3.47}$$

LEMMA 3.3. *Define the quantity*

$$E(\Sigma) := 1 - \frac{\sum_{n=1}^{n(\varepsilon,\Sigma)} s_n(\Sigma)}{\nu_\alpha(\Sigma)}. \tag{3.48}$$

Then

$$0 \leqslant E(\Sigma) \leqslant \varepsilon. \tag{3.49}$$

Proof.
Since, for all $1 \leqslant n \leqslant n(\varepsilon,\Sigma)$,

$$s_n(\Sigma) \geqslant 1 - \varepsilon, \tag{3.50}$$

then

$$\sum_{n=1}^{n(\varepsilon,\Sigma)} s_n(\Sigma) \geqslant (1-\varepsilon) n(\varepsilon,\Sigma). \tag{3.51}$$

Therefore,

$$0 \leqslant E(\Sigma) \leqslant 1 - (1-\varepsilon)\frac{n(\varepsilon,\Sigma)}{\nu_\alpha(\Sigma)}. \tag{3.52}$$

Since $n(\varepsilon,\Sigma) \geqslant \nu_\alpha(\Sigma)$, we obtain the desired result. □

The following result bounds the error between ρ_Σ and Θ_Σ.

THEOREM 3.4. *Assume that Σ is a bounded set of finite measure. Then*

$$\frac{1}{\nu_\alpha(\Sigma)} \|\rho_\Sigma - \Theta_\Sigma\|_{L^1_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)} \leqslant \frac{1}{\nu_\alpha(\Sigma)} + 2E(\Sigma), \tag{3.53}$$

and

$$\limsup_{r\to\infty} \frac{1}{\nu_\alpha(\Sigma_r)} \|\rho_{\Sigma_r} - \Theta_{\Sigma_r}\|_{L^1_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)} \leqslant 2\varepsilon. \tag{3.54}$$

Proof. From Lemma 3.1 we have,

$$\rho_\Sigma(z) - \Theta_\Sigma(z) = \sum_{n=1}^{\infty} (\ell_n - s_n(\Sigma)) \left|\phi_n^\Sigma(z)\right|^2, \tag{3.55}$$

where $\ell_n = 1$ if $n \leqslant \lceil \nu_\alpha(\Sigma) \rceil$ and 0 otherwise. By Plancherel theorem (2.19),

$$\begin{aligned}
\|\rho_\Sigma - \Theta_\Sigma\|_{L^1_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} &= \sum_{n=1}^{\infty} |\ell_n - s_n(\Sigma)| = \sum_{n=1}^{\lceil \nu_\alpha(\Sigma)\rceil} (1 - s_n(\Sigma)) + \sum_{n>\lceil \nu_\alpha(\Sigma)\rceil} s_n(\Sigma) \\
&= \lceil \nu_\alpha(\Sigma)\rceil + \sum_{n=1}^{\infty} s_n(\Sigma) - 2\sum_{n=1}^{\lceil \nu_\alpha(\Sigma)\rceil} s_n(\Sigma) \\
&\leqslant \lceil \nu_\alpha(\Sigma)\rceil + \nu_\alpha(\Sigma) - 2\sum_{n=1}^{\lceil \nu_\alpha(\Sigma)\rceil} s_n(\Sigma) \\
&= (\lceil \nu_\alpha(\Sigma)\rceil - \nu_\alpha(\Sigma)) + 2\left(\nu_\alpha(\Sigma) - \sum_{n=1}^{\lceil \nu_\alpha(\Sigma)\rceil} s_n(\Sigma)\right) \\
&\leqslant 1 + 2\left(\nu_\alpha(\Sigma) - \sum_{n=1}^{\lceil \nu_\alpha(\Sigma)\rceil} s_n(\Sigma)\right),
\end{aligned}$$

and the estimate (3.53) follows. Now since $\limsup_{r\to\infty} \nu_\alpha(\Sigma_r) = \infty$, then with (3.49) we obtain (3.54). □

Acknowledgement. The first author acknowledges the Deanship of Scientific Research at King Faisal University for the financial support under Nasher Track (Grant No. 186011).

REFERENCES

[1] L. D. ABREU, K. GRÖCHENIG, AND J. L. ROMERO, *On accumulated spectrograms*, Trans. Amer. Math. Soc. **368** (2016), 3629–3649.

[2] L. D. ABREU AND J. M. PEREIRA, *Measures of localization and quantitative Nyquist densities*, Appl. Comput. Harmon. Anal. **38** (2015), 524–534.

[3] C. BACCAR, N. B. HAMADI AND H. HERCH, *Time-frequency analysis of localization operators associated to the windowed Hankel transform*, Integral Transforms Spec. Funct. **27** (2016), 245–258.

[4] F. A. BEREZIN, *Wick and anti–Wick operator symbols*, Math. USSR-Sb. **15** 1971.

[5] P. C. BOWIE, *Uncertainty inequalities for Hankel transforms*, SIAM J. Math. Anal. **2** (1971), 601–606.

[6] A. CORDOBA AND C. FEFFERMAN, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations, **3** (1978), 979–1005.

[7] W. CZAJA AND G. GIGANTE, *Continuous Gabor transform for strong hypergroups*, J. Fourier Anal. Appl. **9** (2003), 321–339.

[8] I. DAUBECHIES, *Time–frequency localization operators: a geometric phase space approach*, IEEE Trans. Inform. Theory, **34** (1988), 605–612.

[9] G. A. M. VELASCO AND M. DÖRFLER, *Sampling time-frequency localized functions and constructing localized time-frequency frames*, European J. Appl. Math. **28** (2017), 854–876.

[10] S. GHOBBER AND P. JAMING, *Strong annihilating pairs for the Fourier-Bessel transform*, J. Math. Anal. Appl. **377** (2011), 501–515.

[11] S. GHOBBER AND P. JAMING, *The Logvinenko-Sereda theorem for the Fourier-Bessel transform*, Integral Transforms Spec. Funct. **24** (2013), 470–484.

[12] S. GHOBBER AND S. OMRI, *Time–frequency concentration of the windowed Hankel transform*, Integral Transforms Spec. Funct. **25** (2014), 481–496.

[13] S. GHOBBER, *Phase space localization of orthonormal sequences in* $L^2_\alpha(\mathbb{R}_+)$, J. Approx. Theory, **189** (2015), 123–136.

[14] S. GHOBBER, *A variant of the Hankel multiplier*, Banach J. Math. Anal. **12** (2018), 144–166.

[15] S. GHOBBER, *A dispersion inequality in the Hankel setting*, Czech. Math. J. **68** (2018), 227–241.

[16] B. M. LEVITAN, *Series expansion in Bessel functions and Fourier integrals*, Uspekhi Mat. Nauk. **6** (1951), 102–143.

[17] H. J. LANDAU, *On Szegö's eigenvalue distribution theorem and non-Hermitian kernels*, J. Anal. Math. **28** (1975), 335–357.

[18] H. J. LANDAU AND H. O. POLLACK, *Prolate spheroidal wavefunctions, fourier analysis and uncertainty II*, Bell Syst. Tech. J. **40** (1961), 65–84.

[19] H. J. LANDAU AND H. O. POLLACK, *Prolate spheroidal wavefunctions, fourier analysis and uncertainty III: The dimension of the space of essentially time- and band-limited signals*, Bell Syst. Tech. J. **41** (1962), 1295–1336.

[20] T. MOUMNI, *On essentially time and Hankel band-limited functions*, Integral Transforms Spec. Funct. **23** (2012), 83–95.

[21] J. RAMANATHAN AND P. TOPIWALA, *Time-frequency localization and the spectrogram*, Appl. Comput. Harmon. Anal. **1** (1994), 209–215.

[22] F. RIESZ AND B. SZ.-NAGY, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955.

[23] H. RAUHUT, *Time Frequency and Wavelet Analysis of Functions with Symmetry Properties*, Logos Verlag Berlin, 2005.

[24] H. RAUHUT, *Banach frames in coorbit spaces consisting of elements which are invariant under symmetry groups*, Appl. Comput. Harmon. Anal. **18** (2005), 94–122.

[25] H. RAUHUT AND M. RÖSLER, *Radial multiresolution in dimension three*, Constr. Approx. **2** (2005), 193–218.

[26] D. SLEPIAN AND O. POLLACK, *Prolate spheroidal wavefunctions, fourier analysis and uncertainty I*, Bell Syst. Tech. J. **40** (1961), 43–63.

[27] M. W. WONG, *Wavelet transforms and localization operators*, Birkhäuser, 2002.

[28] A. ZAYED, *Solution of the energy concentration problem in reproducing-kernel Hilbert space*, SIAM J. Appl. Math. **75** (2015), 21–37.

(Received February 5, 2018)

Saifallah Ghobber
Department of Mathematics and Statistics
College of Science, King Faisal University
P.O.Box 400, Al-Hassa 31982, Kingdom of Saudi Arabia
LR11ES11 Analyse Mathématiques et Applications
Faculté des Sciences de Tunis, Université de Tunis El Manar
2092 Tunis, Tunisia
e-mail: sghobber@kfu.edu.sa

Siwar Hkimi
Faculté des Sciences de Tunis
Université de Tunis El Manar
Tunis 2092, Tunisia
e-mail: siwar.hkimi@fst.utm.tn

Slim Omri
Faculté des Sciences de Tunis
Université de Tunis El Manar
Tunis 2092, Tunisia
e-mail: slim.omri@ipein.rnu.tn