Operators
and
Matrices
Volume 13, Number 2 (2019), 527–544
doi:10.7153/oam-2019-13-40

A NEW BANACH SPACE DEFINED BY EULER TOTIENT MATRIX OPERATOR

MERVE İLKHAN* AND EMRAH EVREN KARA

(*Communicated by I. M. Spitkovsky*)

Abstract. The main object of this paper is to introduce a new Banach space derived by using a matrix operator which is comprised of Euler's totient function. Also, we determine α, β, γ-duals of this space and characterize some matrix classes on this new space. Finally, we obtain necessary and sufficient conditions for some matrix operators to be compact.

1. Introduction

By ω, we denote the space of all real valued sequences. Any vector subspace of ω is called a sequence space. $\Psi, \ell_\infty, c, c_0$ and ℓ_p $(1 \leqslant p < \infty)$ are the sets of all finite, bounded, convergent, null sequences and p-absolutely convergent series, respectively. Throughout the study, we assume that $p, q \geqslant 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

A complete normed space is called a B-space. A topological sequence space is called a K-space if all coordinate functionals p_k, $p_k(u) = u_k$, are continuous. A BK-space is a Banach space with continuous coordinates. A BK-space $\Lambda \supset \psi$ is said to have AK if every sequence $u = (u_k) \in \Lambda$ has a unuqiue representation $u = \sum_k u_k e^{(k)}$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in the nth place for each $k \in \mathbb{N}$. For example, the space ℓ_p $(1 \leqslant p < \infty)$ is a BK-space with the norm $\|u\|_p = (\sum_k |u_k|^p)^{1/p}$ and c_0 and ℓ_∞ are BK-spaces with the norm $\|u\|_\infty = \sup_k |u_k|$. Also, the BK-spaces c_0 and ℓ_p have AK but c and ℓ_∞ do not have AK.

The α-, β-, γ-duals of a sequence space Λ are defined by

$$\Lambda^\alpha = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k u_k| < \infty \text{ for all } u = (u_k) \in \Lambda \right\},$$

$$\Lambda^\beta = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k u_k \text{ converges for all } u = (u_k) \in \Lambda \right\},$$

$$\Lambda^\gamma = \left\{ a = (a_k) \in \omega : \sup_n \left| \sum_{k=1}^{n} a_k u_k \right| < \infty \text{ for all } u = (u_k) \in \Lambda \right\},$$

Mathematics subject classification (2010): 40C05, 46A45, 47B37, 47B07.

Keywords and phrases: Euler function, Möbius function, sequence spaces, matrix operators, compact operators, Hausdorff measure of noncompactness.

* Corresponding author.

respectively.

Let Λ and Ω be two sequence spaces and $S=(s_{nk})$ be an infinite matrix of real numbers. Then, we say that S is a matrix mapping from Λ into Ω if for every sequence $u=(u_k)\in\Lambda$, $Su=(S_n(u))$, the S-transform of u, is in Ω, where

$$S_n(u)=\sum_{k=1}^{\infty}s_{nk}u_k$$

provided that the series is convergent for each $n\in\mathbb{N}=\{1,2,...\}$. Throughout the study, S_n will be the sequence of n^{th} row of an infinite matrix $S=(s_{nk})$.

(Λ,Ω) stands for the class of all infinite matrices from a sequence space Λ into another sequence space Ω. Hence, $S\in(\Lambda,\Omega)$ if and only if $S_n\in\Lambda^{\beta}$ for all $n\in\mathbb{N}$.

The matrix domain Λ_S of an infinite matrix S in a sequence space Λ consists of sequences whose S-transforms are in Λ; that is,

$$\Lambda_S=\{u=(u_k)\in\omega:Su\in\Lambda\}.$$

In the literature, there are many papers related to new sequence spaces constructed by means of the matrix domain of a special triangle. See, for this construction and for some triangular matrices [1, 2, 3, 4, 11, 12, 13, 15, 16, 17, 18, 22, 25, 33, 34]. For more details about matrix domains of triangles, one can see [5].

Throughout the paper, φ and μ denote the Euler function and the Möbius function, respectively. For every $m\in\mathbb{N}$ with $m>1$, $\varphi(m)$ is the number of positive integers less than m which are coprime with m and $\varphi(1)=1$. If $p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ is the prime factorization of a natural number $m>1$, then

$$\varphi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_r}).$$

Also, the equality

$$m=\sum_{k|m}\varphi(k)$$

holds for every $m\in\mathbb{N}$ and $\varphi(m_1m_2)=\varphi(m_1)\varphi(m_2)$, where $m_1,m_2\in\mathbb{N}$ are coprime [19]. Given any $m\in\mathbb{N}$ with $m>1$, μ is defined as

$$\mu(m)=\begin{cases}(-1)^r & \text{if } m=p_1p_2...p_r, \text{ where } p_1,p_2,...,p_r \text{ are} \\ & \text{non-equivalent prime numbers} \\ 0 & \text{if } p^2\mid m \text{ for some prime number } p\end{cases}$$

and $\mu(1)=1$. If $p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ is the prime factorization of a natural number $m>1$, then

$$\sum_{k|m}k\mu(k)=(1-p_1)(1-p_2)...(1-p_r).$$

Also, the equality

$$\sum_{k|m}\mu(k)=0 \tag{1}$$

holds except for $m = 1$ and $\mu(m_1 m_2) = \mu(m_1)\mu(m_2)$, where $m_1, m_2 \in \mathbb{N}$ are coprime [19]. One can consult to [31] for more details related to these functions.

Let Λ be a normed space and B_Λ be the unit sphere in Λ. For a BK-space $\Lambda \supset \psi$ and $z = (z_k) \in \omega$, we use the notation

$$\|z\|_\Lambda^* = \sup_{u \in B_\Lambda} \left| \sum_k z_k u_k \right|$$

under the assumption that the supremum is finite. In this case observe that $z \in \Lambda^\beta$.

LEMMA 1. [21, Theorem 1.29] *$\ell_1^\beta = \ell_\infty$, $\ell_p^\beta = \ell_q$ and $\ell_\infty^\beta = \ell_1$, where $1 < p < \infty$. If $\Lambda \in \{\ell_1, \ell_p, \ell_\infty\}$, then $\|z\|_\Lambda^* = \|z\|_{\Lambda^\beta}$ holds for all $z \in \Lambda^\beta$, where $\|.\|_{\Lambda^\beta}$ is the natural norm on Λ^β.*

By $B(\Lambda, \Omega)$, we denote the set of all bounded (continuous) linear operators from Λ into Ω.

LEMMA 2. [21, Theorem 1.23 (a)] *Let Λ and Ω be BK-spaces. Then, for every $S \in (\Lambda, \Omega)$, there exists a linear operator $L_S \in B(\Lambda, \Omega)$ such that $L_S(u) = Su$ for all $u \in \Lambda$.*

LEMMA 3. [21] *Let $\Lambda \supset \psi$ be a BK-space and $\Omega \in \{c_0, c, \ell_\infty\}$. If $S \in (\Lambda, \Omega)$, then*

$$\|L_S\| = \|S\|_{(\Lambda,\Omega)} = \sup_n \|S_n\|_\Lambda^* < \infty.$$

The Hausdorff measure of noncompactness of a bounded set Q in a metric space Λ is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \subset \cup_{i=1}^n B(x_i, r_i), x_i \in \Lambda, r_i < \varepsilon, n \in \mathbb{N}\},$$

where $B(x_i, r_i)$ is the open ball centered at x_i and radius r_i for each $i = 1, 2, \ldots, n$. For more details about the Hausdorff measure of noncompactness, one can consult [21] and references therein.

The following theorem is useful to compute the Hausdorff measure of noncompactness in ℓ_p for $1 \leqslant p < \infty$.

THEOREM 1. [32] *Let Q be a bounded subset in ℓ_p for $1 \leqslant p < \infty$ and $P_r : \ell_p \to \ell_p$ be the operator defined by $P_r(u) = (u_0, u_1, u_2, \ldots, u_r, 0, 0, \ldots)$ for all $u = (u_k) \in \ell_p$ and each $r \in \mathbb{N}$. Then, we have*

$$\chi(Q) = \lim_r \left(\sup_{u \in Q} \|(I - P_r)(u)\|_{\ell_p} \right),$$

where I is the identity operator on ℓ_p.

Let Λ and Ω be Banach spaces. Then, a linear operator $L : \Lambda \to \Omega$ is said to be compact if the domain of L is all of Λ and $L(Q)$ is a totally bounded subset of Ω for every bounded subset Q in Λ. Equivalently, we say that L is compact if its domain is all of Λ and for every bounded sequence $u = (u_n)$ in Λ, the sequence $(L(u_n))$ has a convergent subsequence in Ω.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given as follows.

Let Λ and Ω be Banach spaces and $L \in B(\Lambda, \Omega)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by

$$\|L\|_\chi = \chi(L(B_\Lambda)) \tag{2}$$

and

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \tag{3}$$

One of the applications of the Hausdorff measure of noncompactness is to obtain necessary and sufficient conditions for matrix operators between BK spaces to be compact. Several authors studied compact operators on the sequence spaces. Many significant results are obtained related to the Hausdorff measure of noncompactness of a linear operator. One can see the papers [6, 7, 8, 9, 10, 14, 20, 21, 23, 24, 27, 28, 29, 30, 32] and references therein.

In this paper, we introduce a new BK-space derived by the aid of Euler function. After determining α, β, γ-duals of this space, we obtain necessary and sufficient conditions for some matrix operators to be compact.

2. The sequence space $\ell_p(\Phi)$

In the present section, we introduce the sequence space $\ell_p(\Phi)$ by using the regular matrix Φ, where $1 \leqslant p \leqslant \infty$. Also, we present some theorems which give inclusion relations corcerning this space.

The matrix $\Phi = (\phi_{nk})$ is defined as

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & , \text{ if } k \mid n \\ 0 & , \text{ if } k \nmid n. \end{cases}$$

We call this matrix as *Euler Totient matrix operator*.

The inverse $\Phi^{-1} = (\phi_{nk}^{-1})$ of the matrix Φ is computed in [35] as

$$\phi_{nk}^{-1} = \begin{cases} \frac{\mu(\frac{n}{k})}{\varphi(n)} k & , \text{ if } k \mid n \\ 0 & , \text{ if } k \nmid n. \end{cases}$$

for all $k, n \in \mathbb{N}$.

Now, we introduce the sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ by

$$\ell_p(\Phi) = \left\{ u = (u_n) \in \omega : \sum_n \left| \frac{1}{n} \sum_{k|n} \varphi(k) u_k \right|^p < \infty \right\} \quad (1 \leqslant p < \infty)$$

and

$$\ell_\infty(\Phi) = \left\{ u = (u_n) \in \omega : \sup_n \left| \frac{1}{n} \sum_{k|n} \varphi(k) u_k \right| < \infty \right\}.$$

As the notation of matrix domain, the sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ may be represented by

$$\ell_p(\Phi) = (\ell_p)_\Phi \ \ (1 \leqslant p < \infty) \ \text{ and } \ \ell_\infty(\Phi) = (\ell_\infty)_\Phi.$$

Unless otherwise stated, $v = (v_n)$ will be the Φ-transform of a sequence $u = (u_n)$, that is, $v_n = \Phi_n(u) = \frac{1}{n}\sum_{k|n} \varphi(k) u_k$ for all $n \in \mathbb{N}$.

THEOREM 2. *The spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ are Banach spaces with the norms given by $\|u\|_{\ell_p(\Phi)} = \left(\sum_n \left|\frac{1}{n}\sum_{k|n}\varphi(k)u_k\right|^p\right)^{1/p}$ and $\|u\|_{\ell_\infty(\Phi)} = \sup_n \left|\frac{1}{n}\sum_{k|n}\varphi(k)u_k\right|$, respectively, where $1 \leqslant p < \infty$.*

Proof. We omit the proof which is straightforward. □

COROLLARY 1. *The spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ are BK-spaces, where $1 \leqslant p < \infty$.*

THEOREM 3. *The space $\ell_p(\Phi)$ is linearly isomorphic to ℓ_p, where $1 \leqslant p \leqslant \infty$.*

Proof. Let S be a mapping defined from $\ell_p(\Phi)$ to ℓ_p such that $S(u) = \Phi u$ for all $u \in \ell_p(\Phi)$. It is clear that S is linear. Also it is injective since the kernel of S consists of only zero. To prove that S is surjective consider the sequence $u = (u_n)$ whose terms are

$$u_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} k v_k$$

for all $n \in \mathbb{N}$, where $v = (v_k)$ is any sequence in ℓ_p. It follows from (1) that

$$\begin{aligned}
\Phi_n(u) &= \frac{1}{n}\sum_{k|n}\varphi(k)u_k = \frac{1}{n}\sum_{k|n}\varphi(k)\sum_{j|k}\frac{\mu(\frac{k}{j})}{\varphi(k)} j v_j \\
&= \frac{1}{n}\sum_{k|n}\sum_{j|k}\mu(\frac{k}{j}) j v_j = \frac{1}{n}\sum_{k|n}\left(\sum_{j|k}\mu(j)\right)\frac{n}{k} v_{\frac{n}{k}} = \frac{1}{n}\mu(1) n v_n = v_n
\end{aligned}$$

and so $u = (u_n) \in \ell_p(\Phi)$. S preserves norms since the equality $\|u\|_{\ell_p(\Phi)} = \|Su\|_{\ell_p}$ holds. □

REMARK 1. The space $\ell_2(\Phi)$ is an inner product space with the inner product defined as $\langle u, v\rangle_{\ell_2(\Phi)} = \langle \varphi u, \varphi v\rangle_{\ell_2}$, where $\langle .,.\rangle_{\ell_2}$ is the inner product on ℓ_2 which induces $\|.\|_{\ell_2}$.

THEOREM 4. *The space $\ell_p(\Phi)$ is not an inner product space for $p \neq 2$.*

Proof. Consider the sequences $u=(u_n)$ and $\tilde{u}=(\tilde{u}_n)$, where

$$u_n=\begin{cases}\sum_{k|n}\frac{\mu(\frac{n}{k})}{\varphi(n)}k\ , & \text{if } n \text{ is even}\\ \frac{\mu(n)}{\varphi(n)}\ , & \text{if } n \text{ is odd}\end{cases}$$

and

$$\tilde{u}_n=\begin{cases}\sum_{k|n}(-1)^{k-1}\frac{\mu(\frac{n}{k})}{\varphi(n)}k\ , & \text{if } n \text{ is even}\\ \frac{\mu(n)}{\varphi(n)}\ , & \text{if } n \text{ is odd}\end{cases}$$

for all $n\in\mathbb{N}$. Then, we have $\Phi u=(1,1,0,...,0,...)\in\ell_p$ and $\Phi\tilde{u}=(1,-1,0,...,0,...)\in\ell_p$. Hence, one can easily observe that

$$\|u+\tilde{u}\|_{\ell_p(\Phi)}+\|u-\tilde{u}\|_{\ell_p(\Phi)}\neq 2(\|u\|_{\ell_p(\Phi)}+\|\tilde{u}\|_{\ell_p(\Phi)}).\quad\square$$

THEOREM 5. *The inclusion $\ell_p(\Phi)\subset\ell_q(\Phi)$ strictly holds for $1\leqslant p<q<\infty$.*

Proof. It is clear that the inclusion $\ell_p(\Phi)\subset\ell_q(\Phi)$ holds since $\ell_p\subset\ell_q$ for $1\leqslant p<q<\infty$. Also, $\ell_p\subset\ell_q$ is strict and so there exists a sequence $w=(w_n)$ in $\ell_q\backslash\ell_p$. By defining a sequence $u=(u_n)$ as

$$u_n=\sum_{k|n}\frac{\mu(\frac{n}{k})}{\varphi(n)}kw_k$$

for all $n\in\mathbb{N}$, we conclude that $u\in\ell_q(\Phi)\backslash\ell_p(\Phi)$. Hence, the desired inclusion is strict. $\square$

THEOREM 6. *The inclusion $\ell_p(\Phi)\subset\ell_\infty(\Phi)$ strictly holds for $1\leqslant p<\infty$.*

Proof. The inclusion is obvious since $\ell_p\subset\ell_\infty$ holds for $1\leqslant p<\infty$. Let $u=(u_n)$ be a sequence such that $u_n=\sum_{k|n}(-1)^k\frac{\mu(\frac{n}{k})}{\varphi(n)}k$ for all $n\in\mathbb{N}$. We obtain that $\Phi u=\left(\frac{1}{n}\sum_{k|n}\varphi(k)\sum_{j|k}(-1)^j\frac{\mu(\frac{k}{j})}{\varphi(k)}j\right)=((-1)^n)\in\ell_\infty\backslash\ell_p$ which implies that $u\in\ell_\infty(\Phi)\backslash\ell_p(\Phi)$ for $1\leqslant p<\infty$. $\square$

3. The α-, β- and γ-duals of the space $\ell_p(\Phi)$

In this section, we determine the α-, β- and γ-duals of the sequence space $\ell_p(\Phi)$, where $1\leqslant p\leqslant\infty$. The following lemmas are required to prove our main results in this section. Here and in what follows $\mathscr{K}$ denotes the family of all finite subsets of $\mathbb{N}$.

LEMMA 4. [36] *The following statements hold:*
$S=(s_{nk})\in(\ell_p,\ell_1)$ if and only if

$$\sup_{F\in\mathscr{K}}\sum_k\left|\sum_{n\in F}s_{nk}\right|^q<\infty \tag{4}$$

holds, where $1<p<\infty$.

$S=(s_{nk})\in(\ell_\infty,\ell_1)$ *if and only if* (4) *holds with* $q=1$.

$S=(s_{nk})\in(\ell_1,\ell_1)$ *if and only if*

$$\sup_k\sum_n|s_{nk}|<\infty \tag{5}$$

holds.

$S=(s_{nk})\in(\ell_p,c)$ *if and only if*

$$\lim_{n\to\infty}s_{nk} \text{ exists for each } k\in\mathbb{N} \tag{6}$$

and

$$\sup_n\sum_k|s_{nk}|^q<\infty \tag{7}$$

holds, where $1<p<\infty$.

$S=(s_{nk})\in(\ell_\infty,c)$ *if and only if* (6) *and*

$$\lim_{n\to\infty}\sum_k|s_{nk}|=\sum_k\left|\lim_{n\to\infty}s_{nk}\right| \tag{8}$$

hold.

$S=(s_{nk})\in(\ell_1,c)$ *if and only if* (6) *and*

$$\sup_{n,k}|s_{nk}|<\infty \tag{9}$$

hold.

$S=(s_{nk})\in(\ell_p,c_0)$ *if and only if*

$$\lim_{n\to\infty}s_{nk}=0 \text{ for each } k\in\mathbb{N} \tag{10}$$

and (7) *holds, where* $1<p<\infty$.

$S=(s_{nk})\in(\ell_\infty,c_0)$ *if and only if* (10) *and*

$$\lim_{n\to\infty}\sum_k|s_{nk}|=0 \tag{11}$$

hold.

$S=(s_{nk})\in(\ell_1,c_0)$ *if and only if* (9) *and* (10) *hold.*

$S=(s_{nk})\in(\ell_p,\ell_\infty)$ *if and only if* (7) *holds, where* $1<p<\infty$.

$S=(s_{nk})\in(\ell_\infty,\ell_\infty)$ *if and only if* (7) *holds with* $q=1$.

$S=(s_{nk})\in(\ell_1,\ell_\infty)$ *if and only if* (9) *holds.*

In the following theorem, we determine the α-duals of the spaces $\ell_p(\Phi)$ $(1<p<\infty)$, $\ell_\infty(\Phi)$ and $\ell_1(\Phi)$.

THEOREM 7. *The α-duals of the spaces $\ell_p(\Phi)$ $(1 < p < \infty)$, $\ell_\infty(\Phi)$ and $\ell_1(\Phi)$ are as follows:*

$$(\ell_p(\Phi))^\alpha = \left\{ t = (t_n) \in \omega : \sup_{F \in \mathscr{K}} \sum_k \left| \sum_{n \in F, k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} k t_n \right|^q < \infty \right\},$$

$$(\ell_\infty(\Phi))^\alpha = \left\{ t = (t_n) \in \omega : \sup_{F \in \mathscr{K}} \sum_k \left| \sum_{n \in F, k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} k t_n \right| < \infty \right\}$$

and

$$(\ell_1(\Phi))^\alpha = \left\{ t = (t_n) \in \omega : \sup_k \sum_{n \in \mathbb{N}, k|n} \left| \frac{\mu(\frac{n}{k})}{\varphi(n)} k t_n \right| < \infty \right\}.$$

Proof. Consider the matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} 0 & , k \nmid n \\ \frac{\mu(\frac{n}{k})}{\varphi(n)} k t_n & , k \mid n \end{cases}$$

for any sequence $t = (t_n) \in \omega$. Hence, given any $u = (u_n) \in \ell_p(\Phi)$ for $1 \leqslant p \leqslant \infty$, we have $t_n u_n = C_n(v)$ for all $n \in \mathbb{N}$. This implies that $tu \in \ell_1$ with $u \in \ell_p(\Phi)$ if and only if $Cv \in \ell_1$ with $v \in \ell_p$. It follows that $t \in (\ell_p(\Phi))^\alpha$ if and only if $C \in (\ell_p, \ell_1)$ which completes the proof in view of Lemma 4. □

LEMMA 5. [2, Theorem 3.1] *Let $B = (b_{nk})$ be defined via a sequence $t = (t_k) \in \omega$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} t_j v_{jk}$$

for all $k, n \in \mathbb{N}$. Then,

$$\Lambda_U^\beta = \{t = (t_k) \in \omega : B \in (\Lambda, c)\},$$

and

$$\Lambda_U^\gamma = \{t = (t_k) \in \omega : B \in (\Lambda, \ell_\infty)\}.$$

Consequently, we have the following theorem.

THEOREM 8. *Let define the following sets:*

$$A_1 = \left\{ t = (t_k) \in \omega : \lim_{n \to \infty} \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} k t_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$A_2 = \left\{ t = (t_k) \in \omega : \sup_n \sum_k \left| \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} k t_j \right|^q < \infty \right\},$$

$$A_3=\left\{t=(t_k)\in\omega:\lim_{n\to\infty}\sum_k\left|\sum_{j=k,k|j}^{n}\frac{\mu(\frac{j}{k})}{\varphi(j)}kt_j\right|=\sum_k\left|\sum_{j=k,k|j}^{\infty}\frac{\mu(\frac{j}{k})}{\varphi(j)}kt_j\right|\right\}$$

and

$$A_4=\left\{t=(t_k)\in\omega:\sup_{n,k}\left|\sum_{j=k,k|j}^{n}\frac{\mu(\frac{j}{k})}{\varphi(j)}kt_j\right|<\infty\right\}.$$

The β and γ-duals of the spaces $\ell_p(\Phi)$ $(1<p<\infty)$, $\ell_\infty(\Phi)$ and $\ell_1(\Phi)$ are as follows:
$(\ell_p(\Phi))^\beta=A_1\cap A_2$, $(\ell_\infty(\Phi))^\beta=A_1\cap A_3$ *and* $(\ell_1(\Phi))^\beta=A_1\cap A_4$,
$(\ell_p(\Phi))^\gamma=A_2$, $(\ell_p(\infty))^\gamma=A_2$ *with* $q=1$ *and* $(\ell_1(\Phi))^\gamma=A_4$.

For a sequence $z=(z_k)\in\omega$, we define a sequence $\tilde{z}=(\tilde{z}_k)$ as $\tilde{z}_k=\sum_{j=k,k|j}^{\infty}\frac{\mu(\frac{j}{k})}{\varphi(j)}kz_j$ for all $k\in\mathbb{N}$.

We need the following results in the sequel.

LEMMA 6. *Let $z=(z_k)\in(\ell_p(\Phi))^\beta$, where $1\leqslant p\leqslant\infty$. Then $\tilde{z}=(\tilde{z}_k)\in\ell_q$ and*

$$\sum_k z_k u_k=\sum_k \tilde{z}_k v_k$$

for all $u=(u_k)\in\ell_p(\Phi)$.

LEMMA 7. *The following statements hold.*
(a) $\|a\|^*_{\ell_1(\Phi)}=\sup_k|\tilde{a}_k|<\infty$ *for all* $a=(a_k)\in(\ell_1(\Phi))^\beta$.
(b) $\|a\|^*_{\ell_p(\Phi)}=(\sum_k|\tilde{a}_k|^q)^{1/q}<\infty$ *for all* $a=(a_k)\in(\ell_p(\Phi))^\beta$*, where* $1<p<\infty$.
(c) $\|a\|^*_{\ell_\infty(\Phi)}=\sum_k|\tilde{a}_k|<\infty$ *for all* $a=(a_k)\in(\ell_\infty(\Phi))^\beta$.

Proof. We only prove part (a) and the others can be proved analogously. Choose $a=(a_k)\in(\ell_1(\Phi))^\beta$. Then, by Lemma 6, we have $\tilde{a}=(\tilde{a}_k)\in\ell_\infty$ and $\sum_k a_k u_k=\sum_k\tilde{a}_k v_k$ for all $u=(u_k)\in\ell_1(\Phi)$. Since $\|u\|_{\ell_1(\Phi)}=\|v\|_{\ell_1}$ holds, we obtain that $u\in B_{\ell_1(\Phi)}$ if and only if $v\in B_{\ell_1}$. Hence, we deduce that $\|a\|^*_{\ell_1(\Phi)}=\sup_{u\in B_{\ell_1(\Phi)}}|\sum_k a_k u_k|=\sup_{v\in B_{\ell_1}}|\sum_k\tilde{a}_k v_k|=\|\tilde{a}\|^*_{\ell_1}$. From Lemma 1, it follows that $\|a\|^*_{\ell_1(\Phi)}=\|\tilde{a}\|^*_{\ell_1}=\|\tilde{a}\|_{\ell_\infty}=\sup_k|\tilde{a}_k|$. $\square$

4. Some matrix transformations related to the sequence space $\ell_p(\Phi)$

In this section, we give the characterization of the classes $(\ell_p(\Phi),\Omega)$, where $1\leqslant p\leqslant\infty$ and $\Omega\in\{\ell_\infty,c,c_0,\ell_1\}$. Throughout this section, we write $r(n,k)=\sum_{j=1}^{n}r_{jk}$ for all $n,k\in\mathbb{N}$, where $R=(r_{nk})$ is an infinite matrix.

The following theorem is essential for our results.

THEOREM 9. *Let $1\leqslant p\leqslant\infty$ and Λ be an arbitrary subset of ω. Then, we have $S=(s_{nk})\in(\ell_p(\Phi),\Lambda)$ if and only if*

$$R^{(n)}=\left(r^{(n)}_{mk}\right)\in(\ell_p,c)\ for\ each\ n\in\mathbb{N},\tag{12}$$

$$R = (r_{nk}) \in (\ell_p, \Lambda), \tag{13}$$

where $r_{mk}^{(n)} = \begin{cases} 0 & , k > m \\ \sum_{j=k,k|j}^{m} s_{nj} \frac{\mu(\frac{j}{k})}{\varphi(j)} k & , 1 \leqslant k \leqslant m \end{cases}$ *and* $r_{nk} = \sum_{j=k,k|j}^{\infty} s_{nj} \frac{\mu(\frac{j}{k})}{\varphi(j)} k$ *for all* $k,m,n \in \mathbb{N}$.

Proof. We omit the proof since it follows with the same technique in [18, Theorem 4.1]. □

We obtain the following results by combining Theorem 9 with Lemma 4.

THEOREM 10.
(a) $S = (s_{nk}) \in (\ell_1(\Phi), \ell_\infty)$ *if and only if*

$$\lim_{m\to\infty} r_{mk}^{(n)} \text{ exists for each } n,k \in \mathbb{N}, \tag{14}$$

$$\sup_{m,k} \left| r_{mk}^{(n)} \right| < \infty \text{ for each } n \in \mathbb{N} \tag{15}$$

and (9) *holds with* r_{nk} *instead of* s_{nk}.

(b) $S = (s_{nk}) \in (\ell_1(\Phi), c)$ *if and only if* (14) *and* (15) *hold, and* (6) *and* (9) *also hold with* r_{nk} *instead of* s_{nk}.

(c) $S = (s_{nk}) \in (\ell_1(\Phi), c_0)$ *if and only if* (14) *and* (15) *hold, and* (9) *and* (10) *also hold with* r_{nk} *instead of* s_{nk}.

(d) $S = (s_{nk}) \in (\ell_1(\Phi), \ell_1)$ *if and only if* (14) *and* (15) *hold, and* (5) *also holds with* r_{nk} *instead of* s_{nk}.

THEOREM 11. *Let* $1 < p < \infty$.
(a) $S = (s_{nk}) \in (\ell_p(\Phi), \ell_\infty)$ *if and only if* (14) *and*

$$\sup_m \sum_{k=1}^{m} \left| r_{mk}^{(n)} \right|^q < \infty \text{ for each } n \in \mathbb{N} \tag{16}$$

hold, and (7) *also holds with* r_{nk} *instead of* s_{nk}.

(b) $S = (s_{nk}) \in (\ell_p(\Phi), c)$ *if and only if* (14) *and* (16) *hold, and* (6) *and* (7) *also hold with* r_{nk} *instead of* s_{nk}.

(c) $S = (s_{nk}) \in (\ell_p(\Phi), c_0)$ *if and only if* (14) *and* (16) *hold, and* (10) *and* (7) *also hold with* r_{nk} *instead of* s_{nk}.

(d) $S = (s_{nk}) \in (\ell_p(\Phi), \ell_1)$ *if and only if* (14) *and* (16) *hold, and* (4) *also holds with* r_{nk} *instead of* s_{nk}.

THEOREM 12.
(a) $S = (s_{nk}) \in (\ell_\infty(\Phi), \ell_\infty)$ *if and only if* (14) *and*

$$\lim_{m\to\infty} \sum_{k=1}^{m} \left| r_{mk}^{(n)} \right| = \sum_{k=1}^{m} |r_{nk}| \text{ for each } n \in \mathbb{N} \tag{17}$$

hold, and (7) *also holds with* $q=1$ *and* r_{nk} *instead of* s_{nk} .

(b) $S=(s_{nk})\in(\ell_\infty(\Phi),c)$ *if and only if* (14) *and* (17) *hold, and* (6) *and* (8) *also hold with* r_{nk} *instead of* s_{nk}.

(c) $S=(s_{nk})\in(\ell_\infty(\Phi),c_0)$ *if and only if* (14) *and* (17) *hold, and* (10) *and* (11) *also hold with* r_{nk} *instead of* s_{nk}.

(d) $S=(s_{nk})\in(\ell_\infty(\Phi),\ell_1)$ *if and only if* (14) *and* (17) *hold, and* (4) *holds with* r_{nk} *instead of* s_{nk} *and* $q=1$.

By using Theorems 10-12, we derive the following results:

COROLLARY 2. *The following statements hold:*

(a) $S=(s_{nk})\in(\ell_1(\Phi),bs)$ *if and only if* (14), (15) *hold and* (9) *holds with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

(b) $S=(s_{nk})\in(\ell_1(\Phi),cs)$ *if and only if* (14), (15) *hold and* (6),(9) *hold with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

(c) $S=(s_{nk})\in(\ell_1(\Phi),cs_0)$ *if and only if* (14), (15) *hold and* (9),(10) *hold with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

COROLLARY 3. *Let* $1<p<\infty$. *Then, the following statements hold:*

(a) $S=(s_{nk})\in(\ell_p(\Phi),bs)$ *if and only if* (14), (16) *hold and* (7) *holds with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

(b) $S=(s_{nk})\in(\ell_p(\Phi),cs)$ *if and only if* (14), (16) *hold and* (6),(7) *hold with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

(c) $S=(s_{nk})\in(\ell_p(\Phi),cs_0)$ *if and only if* (14), (16) *hold and* (7),(10) *hold with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

COROLLARY 4. *The following statements hold:*

(a) $S=(s_{nk})\in(\ell_\infty(\Phi),bs)$ *if and only if* (14), (17) *hold and* (7) *holds with* $r(n,k)$ *instead of* s_{nk} *and* $q=1$, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

(b) $S=(s_{nk})\in(\ell_\infty(\Phi),cs)$ *if and only if* (14), (17) *hold and* (6),(8) *hold with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

(c) $S=(s_{nk})\in(\ell_\infty(\Phi),cs_0)$ *if and only if* (14), (17) *hold and* (10),(11) *hold with* $r(n,k)$ *instead of* s_{nk}, *where* $r(n,k)=\sum_{j=1}^{n}r_{jk}$.

5. Compact operators on the spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$

Throughout this section, we use the matrix $\tilde{S}=(\tilde{s}_{nk})$ defined by an infinite matrix $S=(s_{nk})$ via

$$\tilde{s}_{nk}=\sum_{j=k,k|j}^{\infty}\frac{\mu(\frac{j}{k})}{\varphi(j)}ks_{nj}$$

for all $n,k\in\mathbb{N}$ under the assumption that the series is convergent.

LEMMA 8. *Let* Λ *be an arbitrary subset of* ω *and* $S=(s_{nk})$ *be an infinite matrix. If* $S\in(\ell_p(\Phi),\Lambda)$, *then* $\tilde{S}\in(\ell_p,\Lambda)$ *and* $Su=\tilde{S}v$ *for all* $u\in\ell_p(\Phi)$, *where* $1\leqslant p\leqslant\infty$.

Proof. It follows from Lemma 6. □

LEMMA 9. *If $S \in (\ell_1(\Phi), \ell_p)$, then we have*

$$\|L_S\| = \|S\|_{(\ell_1(\Phi),\ell_p)} = \sup_k \left(\sum_n |\tilde{s}_{nk}|^p \right)^{1/p} < \infty,$$

where $1 \leqslant p < \infty$.

LEMMA 10. [26, Theorem 3.7] *Let $\Lambda \supset \psi$ be a BK-space. Then, the following statements hold.*

(a) $S \in (\Lambda, \ell_\infty)$, then $0 \leqslant \|L_S\|_\chi \leqslant \limsup_n \|S_n\|_\Lambda^*$.

(b) $S \in (\Lambda, c_0)$, then $\|L_S\|_\chi = \limsup_n \|S_n\|_\Lambda^*$.

(c) If Λ has AK or $\Lambda = \ell_\infty$ and $S \in (\Lambda, c)$, then

$$\frac{1}{2} \limsup_n \|S_n - s\|_\Lambda^* \leqslant \|L_S\|_\chi \leqslant \limsup_n \|S_n - s\|_\Lambda^*,$$

where $s = (s_k)$ and $s_k = \lim_n s_{nk}$ for each $k \in \mathbb{N}$.

LEMMA 11. [26, Theorem 3.11] *Let $\Lambda \supset \psi$ be a BK-space. If $S \in (\Lambda, \ell_1)$, then*

$$\lim_r \left(\sup_{N \in \mathscr{K}_r} \left\| \sum_{n \in N} S_n \right\|_\Lambda^* \right) \leqslant \|L_S\|_\chi \leqslant 4 \lim_r \left(\sup_{N \in \mathscr{K}_r} \left\| \sum_{n \in N} S_n \right\|_\Lambda^* \right)$$

and L_S is compact if and only if $\lim_r \left(\sup_{N \in \mathscr{K}_r} \| \sum_{n \in N} S_n \|_\Lambda^* \right) = 0$, where $\mathscr{K}_r$ is the subcollection of $\mathscr{K}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

THEOREM 13. *Let* $1 < p < \infty$.

1. *For $S \in (\ell_p(\Phi), \ell_\infty)$,*

$$0 \leqslant \|L_S\|_\chi \leqslant \limsup_n \left(\sum_k |\tilde{s}_{nk}|^q \right)^{1/q}$$

holds.

2. *For $S \in (\ell_p(\Phi), c)$,*

$$\frac{1}{2} \limsup_n \left(\sum_k |\tilde{s}_{nk} - \tilde{s}_k|^q \right)^{1/q} \leqslant \|L_S\|_\chi \leqslant \limsup_n \left(\sum_k |\tilde{s}_{nk} - \tilde{s}_k|^q \right)^{1/q}$$

holds, where $\tilde{s} = (\tilde{s}_k)$ and $\tilde{s}_k = \lim_n \tilde{s}_{nk}$ for each $k \in \mathbb{N}$.

3. *For* $S \in (\ell_p(\Phi), c_0)$,

$$\|L_S\|_\chi = \limsup_n \left(\sum_k |\tilde{s}_{nk}|^q \right)^{1/q}$$

holds.

4. *For* $S \in (\ell_p(\Phi), \ell_1)$,

$$\lim_r \|S\|^{(r)}_{(\ell_p(\Phi),\ell_1)} \leqslant \|L_S\|_\chi \leqslant 4 \lim_r \|S\|^{(r)}_{(\ell_p(\Phi),\ell_1)}$$

holds, where $\|S\|^{(r)}_{(\ell_p(\Phi),\ell_1)} = \sup_{N \in \mathscr{K}_r} \left(\sum_k |\sum_{n \in N} \tilde{s}_{nk}|^q \right)^{1/q}$ $(r \in \mathbb{N})$.

Proof.

1. Let $S \in (\ell_p(\Phi), \ell_\infty)$. Since the series $\sum_{k=1}^\infty s_{nk} u_k$ converges for each $n \in \mathbb{N}$, we have $S_n \in (\ell_p(\Phi))^\beta$. From Lemma 7 (b), we write $\|S_n\|^*_{\ell_p(\Phi)} = \|\tilde{S}_n\|^*_{\ell_p} = \|\tilde{S}_n\|_{\ell_q} = (\sum_k |\tilde{s}_{nk}|^q)^{1/q}$ for each $n \in \mathbb{N}$. By using Lemma 10 (a), we conclude that

$$0 \leqslant \|L_S\|_\chi \leqslant \limsup_n \left(\sum_k |\tilde{s}_{nk}|^q \right)^{1/q}.$$

2. Let $S \in (\ell_p(\Phi), c)$. By Lemma 8, we have $\tilde{S} \in (\ell_p, c)$. Hence, from Lemma 10 (c), we write

$$\frac{1}{2} \limsup_n \|\tilde{S}_n - \tilde{s}\|^*_{\ell_p} \leqslant \|L_S\|_\chi \leqslant \limsup_n \|\tilde{S}_n - \tilde{s}\|^*_{\ell_p},$$

where $\tilde{s} = (\tilde{s}_k)$ and $\tilde{s}_k = \lim_n \tilde{s}_{nk}$ for each $k \in \mathbb{N}$. Moreover, Lemma 1 implies that $\|\tilde{S}_n - \tilde{s}\|^*_{\ell_p} = \|\tilde{S}_n - \tilde{s}\|_{\ell_q} = (\sum_k |\tilde{s}_{nk} - \tilde{s}_k|^q)^{1/q}$ for each $n \in \mathbb{N}$. This completes the proof.

3. Let $S \in (\ell_p(\Phi), c_0)$. Since we have $\|S_n\|^*_{\ell_p(\Phi)} = \|\tilde{S}_n\|^*_{\ell_p} = \|\tilde{S}_n\|_{\ell_q} = (\sum_k |\tilde{s}_{nk}|^q)^{1/q}$ for each $n \in \mathbb{N}$, we conclude from Lemma 10 (b) that

$$\|L_S\|_\chi = \limsup_n \left(\sum_k |\tilde{s}_{nk}|^q \right)^{1/q}.$$

4. Let $S \in (\ell_p(\Phi), \ell_1)$. By Lemma 8, we have $\tilde{S} \in (\ell_p, \ell_1)$. It follows from Lemma 11 that

$$\lim_r \left(\sup_{N \in \mathscr{K}_r} \left\| \sum_{n \in N} \tilde{S}_n \right\|^*_{\ell_p} \right) \leqslant \|L_S\|_\chi \leqslant 4 \lim_r \left(\sup_{N \in \mathscr{K}_r} \left\| \sum_{n \in N} \tilde{S}_n \right\|^*_{\ell_p} \right).$$

Moreover, Lemma 1 implies that $\|\sum_{n \in N} \tilde{S}_n\|^*_{\ell_p} = \|\sum_{n \in N} \tilde{S}_n\|_{\ell_q} = (\sum_k |\sum_{n \in N} \tilde{s}_{nk}|^q)^{1/q}$ which completes the proof. $\square$

As a consequence of this theorem, we have the following corollary which follows from (3).

COROLLARY 5. *Let* $1 < p < \infty$.

1. L_S *is compact for* $S \in (\ell_p(\Phi), \ell_\infty)$ *if*

$$\lim_n \left(\sum_k |\tilde{s}_{nk}|^q \right)^{1/q} = 0.$$

2. L_S *is compact for* $S \in (\ell_p(\Phi), c)$ *if and only if*

$$\lim_n \left(\sum_k |\tilde{s}_{nk} - \tilde{s}_k|^q \right)^{1/q} = 0.$$

3. L_S *is compact for* $S \in (\ell_p(\Phi), c_0)$ *if and only if*

$$\lim_n \left(\sum_k |\tilde{s}_{nk}|^q \right)^{1/q} = 0.$$

4. L_S *is compact for* $S \in (\ell_p(\Phi), \ell_1)$ *if and only if*

$$\lim_m \|S\|^{(m)}_{(\ell_p(\Phi), \ell_1)} = 0,$$

where $\|S\|^{(r)}_{(\ell_p(\Phi), \ell_1)} = \sup_{N \in \mathscr{K}_r} \left(\sum_k |\sum_{n \in N} \tilde{s}_{nk}|^q \right)^{1/q}$.

THEOREM 14.

1. *For* $S \in (\ell_\infty(\Phi), \ell_\infty)$,

$$0 \leqslant \|L_S\|_\chi \leqslant \limsup_n \sum_k |\tilde{s}_{nk}|$$

holds.

2. *For* $S \in (\ell_\infty(\Phi), c)$,

$$\frac{1}{2} \limsup_n \sum_k |\tilde{s}_{nk} - \tilde{s}_k| \leqslant \|L_S\|_\chi \leqslant \limsup_n \sum_k |\tilde{s}_{nk} - \tilde{s}_k|$$

holds.

3. *For* $S \in (\ell_\infty(\Phi), c_0)$,

$$\|L_S\|_\chi = \limsup_n \sum_k |\tilde{s}_{nk}|$$

holds.

4. *For* $S \in (\ell_\infty(\Phi), \ell_1)$,

$$\lim_r \|S\|^{(r)}_{(\ell_\infty(\Phi),\ell_1)} \leqslant \|L_S\|_\chi \leqslant 4 \lim_r \|S\|^{(r)}_{(\ell_\infty(\Phi),\ell_1)}$$

holds, where $\|S\|^{(r)}_{(\ell_\infty(\Phi),\ell_1)} = \sup_{N \in \mathscr{K}_r} \left(\sum_k \left|\sum_{n\in N} \tilde{s}_{nk}\right|\right)$ $(r \in \mathbb{N})$.

Proof. It follows with the same technique in Theorem 13. □

Similarly, we have the following result.

COROLLARY 6.

1. L_S *is compact for* $S \in (\ell_\infty(\Phi), \ell_\infty)$ *if*

$$\lim_n \sum_k |\tilde{s}_{nk}| = 0.$$

2. L_S *is compact for* $S \in (\ell_\infty(\Phi), c)$ *if and only if*

$$\lim_n \sum_k |\tilde{s}_{nk} - \tilde{s}_k| = 0.$$

3. L_S *is compact for* $S \in (\ell_\infty(\Phi), c_0)$ *if and only if*

$$\lim_n \sum_k |\tilde{s}_{nk}| = 0.$$

4. L_S *is compact for* $S \in (\ell_\infty(\Phi), \ell_1)$ *if and only if*

$$\lim_r \|S\|^{(r)}_{(\ell_\infty(\Phi),\ell_1)} = 0,$$

where $\|S\|^{(r)}_{(\ell_\infty(\Phi),\ell_1)} = \sup_{N \in \mathscr{K}_r} \left(\sum_k \left|\sum_{n\in N} \tilde{s}_{nk}\right|\right)$.

THEOREM 15.

1. *For* $S \in (\ell_1(\Phi), \ell_\infty)$,

$$0 \leqslant \|L_S\|_\chi \leqslant \limsup_n \left(\sup_k |\tilde{s}_{nk}|\right)$$

holds.

2. *For* $S \in (\ell_1(\Phi), c)$,

$$\frac{1}{2} \limsup_n \left(\sup_k |\tilde{s}_{nk} - \tilde{s}_k|\right) \leqslant \|L_S\|_\chi \leqslant \limsup_n \left(\sup_k |\tilde{s}_{nk} - \tilde{s}_k|\right)$$

holds.

3. *For* $S \in (\ell_1(\Phi), c_0)$,

$$\|L_S\|_\chi = \limsup_n \left(\sup_k |\tilde{s}_{nk}| \right)$$

holds.

4. *For* $S \in (\ell_1(\Phi), \ell_1)$,

$$\|L_S\|_\chi = \lim_r \left(\sup_k \sum_{n=r}^{\infty} |\tilde{s}_{nk}| \right)$$

holds.

Proof. It follows with the same technique in Theorem 13. □
Similarly, we have the following result.

COROLLARY 7.

1. L_S *is compact for* $S \in (\ell_1(\Phi), \ell_\infty)$ *if*

$$\lim_n \left(\sup_k |\tilde{s}_{nk}| \right) = 0.$$

2. L_S *is compact for* $S \in (\ell_1(\Phi), c)$ *if and only if*

$$\lim_n \left(\sup_k |\tilde{s}_{nk} - \tilde{s}_k| \right) = 0.$$

3. L_S *is compact for* $S \in (\ell_1(\Phi), c_0)$ *if and only if*

$$\lim_n \left(\sup_k |\tilde{s}_{nk}| \right) = 0.$$

4. L_S *is compact for* $S \in (\ell_1(\Phi), \ell_1)$ *if and only if*

$$\lim_r \left(\sup_k \sum_{n=r}^{\infty} |\tilde{s}_{nk}| \right) = 0.$$

REFERENCES

[1] B. ALTAY, F. BAŞAR, *The matrix domain and the fine spectrum of the difference operator* Δ *on the sequence space* ℓ_p, $0 < p < 1$, Communications in Mathematical Analysis 2(2) (2007) 1–11.
[2] B. ALTAY, F. BAŞAR, *Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space*, Journal of Mathematical Analysis and Applications 336 (2007) 632–645.
[3] B. ALTAY, F. BAŞAR, M. MURSALEEN, *On the Euler sequence spaces which include the spaces* ℓ_p *and* ℓ_∞ *I*, Information Sciences 176 (2006) 1450–1462.
[4] C. AYDIN, F. BAŞAR, *Some new sequence spaces which include the spaces* ℓ_p *and* ℓ_∞, Demonstratio Mathematica 38 (2005) 641–656.
[5] F. BAŞAR, *Summability Theory and Its Applications*, Bentham Science Publishers, İstanbul, 2012.

[6] F. BASAR, E. MALKOWSKY, *The characterization of compact operators on spaces of strongly summable and bounded sequences*, Applied Mathematics and Computation 217 (2011) 5199–5207.

[7] M. BASARIR, E. E. KARA, *On compact operators on the Riesz B(m)-difference sequence spaces*, Iranian Journal of Science and Technology 35 (2011) 279–285.

[8] M. BASARIR, E. E. KARA, *On some difference sequence spaces of weighted means and compact operators*, Annals of Functional Analysis 2 (2011) 114–129.

[9] M. BASARIR, E. E. KARA, *On the B-difference sequence space derived by generalized weighted mean and compact operators*, Journal of Mathematical Analysis and Applications 391 (2012) 67–81.

[10] M. BASARIR, E. E. KARA, *On compact operators on the Riesz B^m-difference sequence spaces II*, Iranian Journal of Science and Technology (Special Issue-Mathematics) 36 (2012) 371–376.

[11] M. CANDAN, *Domain of the double sequential band matrix in the classical sequence spaces*, Journal of Inequalities and Applications 281 (2012) 15 pages.

[12] M. CANDAN, *Almost convergence and double sequential band matrix*, Acta Mathematica Scientia 34B(2) (2014) 354–366.

[13] E. E. KARA, *Some topological and geometrical properties of new Banach sequence spaces*, Journal of Inequalities and Applications 2013(38) (2013) 15 pages.

[14] E. E. KARA, M. BASARIR, *On some Euler $B^{(m)}$ difference sequence spaces and compact operators*, Journal of Mathematical Analysis and Applications 379 (2011) 499–511.

[15] E. E. KARA, M. BASARIR, Ş. KONCA, *On some new weighted Euler sequence spaces and compact operators*, Mathematical Inequalities and Applications 17(2) (2014) 649–664.

[16] E. E. KARA, M. ILKHAN, *On some Banach sequence spaces derived by a new band matrix*, British Journal of Mathematics & Computer Science 9(2) (2015) 141–159.

[17] E. E. KARA, M. ILKHAN, *Some properties of generalized Fibonacci sequence spaces*, Linear and Multilinear Algebra 64(11) (2016) 2208–2223.

[18] M. KIRIŞÇI, F. BAŞAR, *Some new sequence spaces derived by the domain of generalized difference matrix*, Computers & Mathematics with Applications 60 (2010) 1299–1309.

[19] E. KOVAC, *On φ convergence and φ density*, Mathematica Slovaca 55 (2005) 329–351.

[20] B. DE MALAFOSSE, V. RAKOCEVIC, *Applications of measure of noncompactness in operators on the spaces*, Journal of Mathematical Analysis and Applications 323 (2006) 131–145.

[21] E. MALKOWSKY, V. RAKOCEVIC, *An introduction into the theory of sequence spaces and measure of noncompactness*, Zbornik radova, Matematicki Inst. SANU, Belgrade, 9(17) (2000), 143–234.

[22] M. MURSALEEN, F. BAŞAR, B. ALTAY, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II*, Nonlinear Analysis TMA 65(3) (2006) 707–717.

[23] M. MURSALEEN, V. KARAKAYA, H. POLAT, N. SIMSEK, *Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means*, Computers & Mathematics with Applications 62 (2011) 814–820.

[24] M. MURSALEEN, S. A. MOHIUDDINE, *Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces*, Nonlinear Analysis 75 (2012) 2111–2115.

[25] M. MURSALEEN, A. K. NOMAN, *On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I*, Filomat 25(2) (2011) 33–51.

[26] M. MURSALEEN, A. K. NOMAN, *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Analysis 73 (2010) 2541–2557.

[27] M. MURSALEEN, A. K. NOMAN, *On σ-conservative matrices and compact operators on the space V_σ*, Applied Mathematics Letters 24 (2011) 1554–1560.

[28] M. MURSALEEN, A. K. NOMAN, *The Hausdorff measure of noncompactness of matrix operators on some BK spaces*, Operators and Matrices 5 (2011) 473–486.

[29] M. MURSALEEN, A. K. NOMAN, *Compactness of matrix operators on some new difference sequence spaces*, Linear Algebra and its Applications 436 (2012) 41–52.

[30] M. MURSALEEN, A. K. NOMAN, *Applications of Hausdorff measure of noncompactness in the spaces of generalized means*, Mathematical Inequalities & Applications 16 (2013) 207–220.

[31] I. NIVEN, H. S. ZUCKERMAN, H. L. MONTGOMERY, *An introduction to the theory of numbers*, (5th edition), Wiley, New York, 1991.

[32] V. RAKOCEVIC, *Measures of noncompactness and some applications*, Filomat 12 (1998) 87–120.

[33] H. ROOPAEI, D. FOROUTANNIA, *The norm of backward difference operator $\Delta^{(n)}$ on certain sequence spaces*, Operators and Matrices 12(3) (2018) 867–880.

[34] E. SAVAŞ, *Matrix transformations between some new sequence spaces*, Tamkang Journal of Mathematics 19 (1988) 75–80.

[35] I. SCHOENBERG, *The integrability of certain functions and related summability methods*, The American Mathematical Monthly 66 (1959) 361–375.

[36] M. STIEGLITZ, H. TIETZ, *Matrix transformationen von folgenraumen eine ergebnisübersicht*, Mathematische Zeitschrift 154 (1977) 1–16.

(Received September 27, 2018)

Merve İlkhan
Department of Mathematics
Düzce University
Düzce, Turkey
e-mail: merveilkhan@gmail.com

Emrah Evren Kara
Department of Mathematics
Düzce University
Düzce, Turkey
e-mail: karaeevren@gmail.com