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WEIGHTED COMPOSITION OPERATORS BETWEEN LIPSCHITZ SPACES ON POINTED METRIC SPACES

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Abstract. In this paper, we study weighted composition operators between Banach spaces of scalar-valued Lipschitz functions on pointed metric spaces, not necessarily compact. We give necessary and sufficient conditions for the injectivity and the surjectivity of these operators. We also obtain sufficient and necessary conditions for a weighted composition operator between these spaces to be compact.

1. Introduction and preliminaries

The symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a Hausdorff space. We denote by $C_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$-valued continuous functions on X. Then $C_{\mathbb{K}}(X)$ is a commutative algebra over $\mathbb{K}$ with unit 1_X, the constant function on X with value 1. The set of all bounded functions in $C_{\mathbb{K}}(X)$ is denoted by $C^b_{\mathbb{K}}(X)$. It is known that $C^b_{\mathbb{K}}(X)$ is a unital commutative Banach algebra over $\mathbb{K}$ with unit 1_X when equipped with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C^b_{\mathbb{K}}(X)).$$

Let X and Y be Hausdorff spaces and let $S(X)$ and $S(Y)$ be linear subspaces of $C_{\mathbb{K}}(X)$ and $C_{\mathbb{K}}(Y)$ over $\mathbb{K}$, respectively. A map $T : S(X) \to S(Y)$ is called *a weighted composition operator* if there exist a $\mathbb{K}$-valued function u on Y, not necessarily continuous, and a map $\varphi : Y \to X$ such that $T(f)(y) = u(y)f(\varphi(y))$ for all $f \in S(X)$ and $y \in Y$. Then T is denoted by uC_φ and called the weighted composition operator induced by u and φ. Clearly, uC_φ is a linear operator. In the case $u = 1_Y$, the weighted composition operator uC_φ reduces to the composition operator C_φ.

Let (X,d) and (Y,ρ) be metric spaces and K be a nonempty subset of Y. A map $\varphi : K \to X$ is called a *Lipschitz mapping* from (K,ρ) to (X,d) if there exists a constant C such that $d(\varphi(x),\varphi(y)) \leqslant C\rho(x,y)$ for all $x,y \in K$. A map $\varphi : K \to X$ is called a *supercontractive mapping* from (K,ρ) to (X,d) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\varphi(x),\varphi(y))/\rho(x,y) < \varepsilon$ whenever $x,y \in K$ with $0 < \rho(x,y) < \delta$.

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A Lipschitz mapping φ from (Y,ρ) to (X,d) is called a *Lipschitz homeomorphism* if φ is bijective and φ^{-1} is a Lipschitz mapping from (X,d) to (Y,ρ).

Let (X,d) be a metric space. A function $f : X \to \mathbb{K}$ is called a $\mathbb{K}$*-valued Lipschitz function* on (X,d) if f is a Lipschitz mapping from (X,d) to the Euclidean metric space $\mathbb{K}$. We denote by $p_{(X,d)}(f)$ the constant Lipschitz of f, i.e. $p_{(X,d)}(f) = \sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x,y \in X, x \neq y\}$. We denote by $\mathrm{Lip}(X,d)$ the set of all $\mathbb{K}$-valued bounded Lipschitz functions on (X,d). Then $\mathrm{Lip}(X,d)$ is a subalgebra of $C^b_{\mathbb{K}}(X)$ over $\mathbb{K}$ containing 1_X. Moreover, $\mathrm{Lip}(X,d)$ is a commutative unital Banach algebra over $\mathbb{K}$ with the Lipschitz algebra norm

$$\|f\|_{\mathrm{Lip}(X,d)} = \|f\|_X + p_{(X,d)}(f) \qquad (f \in \mathrm{Lip}(X,d)).$$

These algebras were first introduced by Sherbert in [6].

Let (X,d) be a pointed metric space with a basepoint designated by x_0. We denote by $\mathrm{Lip}_0(X,d)$ the set of all $\mathbb{K}$-valued Lipschitz functions f on (X,d) for which $f(x_0) = 0$. Then $\mathrm{Lip}_0(X,d)$ is a Banach space with the $p_{(X,d)}(\cdot)$-norm. If $\mathrm{diam}(X) < \infty$, then every $f \in \mathrm{Lip}_0(X,d)$ is a bounded function on X and $\|f\|_X \leqslant \mathrm{diam}(X)p_{(X,d)}(f)$, where $\mathrm{diam}(X)$ denotes the diameter of X in (X,d). Therefore, in the case $\mathrm{diam}(X) < \infty$, $\mathrm{Lip}_0(X,d)$ is a maximal ideal of $\mathrm{Lip}(X,d)$. For further general facts about Lipschitz spaces $\mathrm{Lip}(X,d)$ and $\mathrm{Lip}_0(X,d)$, we refer to [7].

Kamowitz and Scheinberg in [5] proved that a composition endomorphism C_φ of $\mathrm{Lip}(X,d)$ is compact if and only if φ is a supercontraction from (X,d) to (X,d) whenever (X,d) is a compact metric space. Jiménez-Vargas and Villegas-Vallecillos in [4] generalized some results of [5] by omitting the compactness condition of considered metric spaces. Botelho and Jamison in [1] and Esmaeili and Mahyar in [2] studied weighted composition operators between spaces of vector-valued Lipschitz functions. Golbaharan and Mahyar in [3] studied weighted composition operators on the Lipschitz algebras $\mathrm{Lip}(X,d)$ whenever (X,d) is a compact metric space.

In this paper, we study weighted composition operators between Banach spaces of $\mathbb{K}$-valued Lipschitz functions on pointed metric spaces, not necessarily compact. Let (X,d) and (Y,ρ) be pointed metric spaces, u be a $\mathbb{K}$-valued function on Y and φ be a map from Y to X. In section 2, we discuss some properties of the weighted composition operators uC_φ from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. In section 3, we give necessary and sufficient conditions for the injectivity and the surjectivity of the weighted composition operators uC_φ from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. In section 4, we obtain necessary and sufficient conditions for a weighted composition operator uC_φ from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$ to be compact.

2. Some properties of weighted composition operators

For a $\mathbb{K}$-valued function u on a nonempty set Y, we denote by $\mathrm{coz}(u)$ the set of all $y \in Y$ for which $u(y) \neq 0$.

Let (X,d) and (Y,ρ) be pointed metric spaces. It is interesting to know under which conditions on functions u and φ, the operator uC_φ is a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. It is clear that if either u is a $\mathbb{K}$-valued Lipschitz function on (Y,ρ) and $\varphi : Y \to X$ is a basepoint-preserving Lipschitz mapping or $u \in \mathrm{Lip}_0(Y,\rho)$ and $\varphi : Y \to X$ is a Lipschitz mapping, then uC_φ is a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. However, the following example shows that there exists a nonzero weighted composition operator uC_φ from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$ where φ is not a basepoint-preserving Lipschitz mapping from (Y,ρ) to (X,d).

EXAMPLE 1. Let X be a subset of $(-\infty,\infty)$ containing $\{-1,0,1\}$, let d be the Euclidean metric on X and let 1 be the basepoint of X. Let $Y=(-\infty,\infty)$, let ρ be the Euclidean metric on Y and let 1 be the basepoint of Y. Define the map $\varphi : Y \to X$ by

$$\varphi(y) = \mathrm{sgn}(1-y) \qquad (y \in Y).$$

Then φ is not a Lipschitz mapping from (Y,ρ) to (X,d) since

$$\frac{d(\varphi(1-\frac{1}{n}),\varphi(1+\frac{1}{n}))}{\rho(1-\frac{1}{n},1+\frac{1}{n})} = \frac{2}{\frac{2}{n}} = n,$$

for all $n \in \mathbb{N}$ with $n \geqslant 2$. Moreover, φ is not basepoint-preserving since $\varphi(1)=0 \neq 1$. Define the function $u : Y \to \mathbb{K}$ by

$$u(y) = 1-y \qquad (y \in Y).$$

Set $T = uC_\varphi$. We show that $T(f) \in \mathrm{Lip}_0(Y,\rho)$ for all $f \in \mathrm{Lip}_0(X,d)$. Let $f \in \mathrm{Lip}_0(X,d)$. Then

$$T(f)(1) = u(1)f(\varphi(1)) = 0.$$

It is easy to see that

$$\frac{|T(f)(x)-T(f)(y)|}{\rho(x,y)} \leqslant |f(-1)|,$$

for all $x,y \in Y$ with $x \neq y$. Hence, $T(f) \in \mathrm{Lip}_0(Y,\rho)$. Therefore, $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$.

Here, we give a sufficient condition for the operator $T = uC_\varphi$ to be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$.

THEOREM 2.1. *Let (X,d) and (Y,ρ) be pointed metric spaces and let* $\mathrm{diam}(X) < \infty$. *Suppose that u is a $\mathbb{K}$-valued function on Y and $\varphi : Y \to X$ is a basepoint-preserving map such that* $\sup\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} < \infty$. *If $u \in \mathrm{Lip}(Y,\rho)$, then $T = uC_\varphi$ is a weighted composition operator from* $\mathrm{Lip}_0(X,d)$ *to* $\mathrm{Lip}_0(Y,\rho)$.

Proof. Suppose that $u \in \mathrm{Lip}(Y,\rho)$ and take

$$C = \sup\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\}.$$

Let $f \in \mathrm{Lip}_0(X,d)$. Then for each $x,y \in Y$ with $\varphi(x) \neq \varphi(y)$, we have

$$\begin{aligned}\frac{|T(f)(x)-T(f)(y)|}{\rho(x,y)} =& \frac{|u(x)f(\varphi(x))-u(y)f(\varphi(y))|}{\rho(x,y)} \\ &\leqslant |u(x)|\frac{|f(\varphi(x))-f(\varphi(y))|}{d(\varphi(x),\varphi(y))}\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \\ &+ |f(\varphi(y))|\frac{|u(x)-u(y)|}{\rho(x,y)} \leqslant Cp_{(X,d)}(f) + \|f\|_X p_{(Y,\rho)}(u).\end{aligned}$$

Moreover, for each $x,y \in Y$ with $x \neq y$ and $\varphi(x) = \varphi(y)$, we have

$$\frac{|T(f)(x)-T(f)(y)|}{\rho(x,y)} = \frac{|u(x)-u(y)|}{\rho(x,y)}|f(\varphi(y))| \leqslant p_{(Y,\rho)}(u)\|f\|_X.$$

Therefore, $T(f)$ is a Lipschitz function on (Y,ρ).

On the other hand, if x_0 and y_0 are the basepoints of X and Y, respectively, then

$$T(f)(y_0) = u(y_0)f(\varphi(y_0)) = u(y_0)f(x_0) = u(y_0)0 = 0.$$

Hence, $T(f) \in \mathrm{Lip}_0(Y,\rho)$. This completes the proof. □

The following example shows that for a weighted composition operator $T = uC_\varphi$ from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$ it is not necessary that u be a Lipschitz function. In fact, the following example reveals that the converse of Theorem 2.1 does not hold in general.

EXAMPLE 2. Let X be the open interval $(-1,3)$, let d be the Euclidean metric on X and let 1 be the basepoint of X. Let Y be the open interval $(-1,2)$, let ρ be the Euclidean metric on Y and let 1 be the basepoint of Y. Define the $\mathbb{K}$-valued function u on Y by $u = \chi_{(-1,1]} - \chi_{(1,2)}$, where $\chi_{(-1,1]} : Y \to \mathbb{K}$ and $\chi_{(1,2)} : Y \to \mathbb{K}$ are the characteristic functions of the sets $(-1,1]$ and $(1,2)$, respectively. Then u is not a Lipschitz function on (Y,ρ) since u is not continuous. Define the map $\varphi : Y \to X$ by

$$\varphi(y) = y \qquad (y \in Y).$$

Let $f \in \mathrm{Lip}_0(X,d)$. If $x,y \in (-1,1]$ with $x \neq y$, then

$$|T(f)(x)-T(f)(y)| = |u(x)f(\varphi(x))-u(y)f(\varphi(y))| = |f(x)-f(y)| \leqslant p_{(X,d)}(f)|x-y|.$$

If $x \in (-1,1]$ and $y \in (1,2)$, then

$$\begin{aligned}|T(f)(x)-T(f)(y)| &= |u(x)f(\varphi(x))-u(y)f(\varphi(y))| = |f(x)+f(y)| \leqslant |f(x)|+|f(y)| \\ &= |f(x)-f(1)|+|f(1)-f(y)| \\ &\leqslant p_{(X,d)}(f)|x-1| + p_{(X,d)}(f)|1-y| = p_{(X,d)}(f)|x-y|.\end{aligned}$$

If $x,y \in (1,2)$ with $x \neq y$, then

$$\begin{aligned}|T(f)(x)-T(f)(y)| &= |u(x)f(\varphi(x))-u(y)f(\varphi(y))| = |-f(x)+f(y)| \\ &\leqslant p_{(X,d)}(f)|x-y|.\end{aligned}$$

Hence,

$$\frac{|T(f)(x)-T(f)(y)|}{|x-y|} \leqslant p_{(X,d)}(f)$$

for all $x,y \in Y$ with $x \neq y$. This implies that $T(f)$ is a Lipschitz function on (Y,ρ).

On the other hand, $T(f)(1) = u(1)f(\varphi(1)) = (1-0)f(1) = 0$. Therefore, $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$.

THEOREM 2.2. *Let (X,d) and (Y,ρ) be pointed metric spaces, let u be a $\mathbb{K}$-valued bounded function on Y, let $\varphi : Y \to X$ be a basepoint-preserving map and let $T = uC_\varphi$ be a weighted composition operator from* $\mathrm{Lip}_0(X,d)$ *to* $\mathrm{Lip}_0(Y,\rho)$. *Then*

(i) *T is bounded,*

(ii) $\sup\{|u(x)|\dfrac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} \leqslant 2\|T\|.$

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{Lip}_0(X,d)$ that converges to the function 0 in $(\mathrm{Lip}_0(X,d), p_{(X,d)}(\cdot))$ and the sequence $\{T(f_n)\}_{n=1}^{\infty}$ converges to a function $g \in \mathrm{Lip}_0(Y,\rho)$ in $(\mathrm{Lip}_0(Y,\rho), p_{(Y,\rho)}(\cdot))$. By [4, Lemma 2.4], we have $\lim_{n\to\infty} f_n(\varphi(y)) = 0$ and $\lim_{n\to\infty} T(f_n)(y) = g(y)$ for all $y \in Y$. The boundedness of u implies that $\lim_{n\to\infty} u(y)(f_n(\varphi(y))) = 0$ for all $y \in Y$. Therefore, $g = 0$ and by the closed graph theorem T is continuous and so bounded. Hence, (i) holds.

Let $x,y \in Y$ with $\varphi(x) \neq \varphi(y)$. Suppose that x_0 is the basepoint of X. We can assume without lost of generality that $d(\varphi(y),x_0) \leqslant d(\varphi(x),x_0)$ and then $d(\varphi(x),\varphi(y)) \leqslant 2d(\varphi(x),x_0)$. Take $\delta = \min\{d(\varphi(x),x_0), d(\varphi(x),\varphi(y))\}$. Then $\delta > 0$. Define the function $h_{\varphi(x),\delta} : X \to \mathbb{K}$ by

$$h_{\varphi(x),\delta}(z) = \max\{0, 1-\frac{d(\varphi(x),z)}{\delta}\} \qquad (z \in X).$$

Then $h_{\varphi(x),\delta}(x_0) = 0$, $h_{\varphi(x),\delta}$ is a $\mathbb{K}$-valued Lipschitz function on (X,d) and $p_{(X,d)}(h_{\varphi(x),\delta}) \leqslant \frac{1}{\delta}$. Define the function $f_{x,y,\delta} : X \to \mathbb{K}$ by

$$f_{x,y,\delta}(z) = d(\varphi(x),\varphi(y))h_{\varphi(x),\delta}(z) \qquad (z \in X).$$

We can easily show that $f_{x,y,\delta} \in \mathrm{Lip}_0(X,d)$ and $p_{(X,d)}(f_{x,y,\delta}) \leqslant 2$. Since $f_{x,y,\delta}(\varphi(x)) =$

$d(\varphi(x),\varphi(y))$ and $f_{x,y,\delta}(\varphi(y))=0$, we deduce that

$$\begin{aligned}
&|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}\\
=&\frac{|u(x)d(\varphi(x),\varphi(y))h_{\varphi(x),\delta}(\varphi(x))-u(y)d(\varphi(x),\varphi(y))h_{\varphi(x),\delta}(\varphi(y))|}{\rho(x,y)}\\
=&\frac{|u(x)f_{x,y,\delta}(\varphi(x))-u(y)f_{x,y,\delta}(\varphi(y))|}{\rho(x,y)}=\frac{|T(f_{x,y,\delta})(x)-T(f_{x,y,\delta})(y)|}{\rho(x,y)}\leqslant p_{(Y,\rho)}(T(f_{x,y,\delta}))\\
\leqslant& p_{(X,d)}(f_{x,y,\delta})\|T\|\leqslant 2\|T\|.
\end{aligned}$$

The above inequality holds whenever $x,y\in Y$ with $x\neq y$ and $\varphi(x)=\varphi(y)$. Hence, (ii) holds. $\square$

COROLLARY 2.3. *Let (X,d) and (Y,ρ) be pointed metric spaces and let $\varphi: Y\to X$ be a basepoint-preserving map. If $f\circ\varphi\in \mathrm{Lip}_0(Y,\rho)$ for all $f\in\mathrm{Lip}_0(X,d)$, then C_φ is a bounded composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$ and φ is a Lipschitz mapping from (Y,ρ) to (X,d).*

Proof. Suppose that $f\circ\varphi\in\mathrm{Lip}_0(Y,\rho)$ for all $f\in\mathrm{Lip}_0(X,d)$. Then C_φ is a composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. Hence, $T=uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$, whenever $u=1_Y$. By Theorem 2.2, C_φ is bounded and $\sup\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}: x,y\in Y, x\neq y\}\leqslant 2\|C_\varphi\|$, i.e., φ is Lipschitz. $\square$

COROLLARY 2.4. *Let (X,d) and (Y,ρ) be pointed metric spaces and let u be a $\mathbb{K}$-valued bounded function on Y which is continuous on $\mathrm{coz}(u)$. Let $\varphi: Y\to X$ be a basepoint-preserving map and let $T=uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. Then φ is Lipschitz on every nonempty compact subset of $\mathrm{coz}(u)$.*

Proof. Let K be a nonempty compact subset of $\mathrm{coz}(u)$. Take $C=\inf\{|u(y)|: y\in K\}$. The continuity of u on $\mathrm{coz}(u)$ implies that $C>0$. Suppose that $x,y\in K$ with $x\neq y$. By Theorem 2.2, we deduce that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}\leqslant\frac{2\|T\|}{C}$. Hence, φ is a Lipschitz mapping from (K,ρ) to (X,d). $\square$

THEOREM 2.5. *Let (X,d) and (Y,ρ) be pointed metric spaces, let u be a $\mathbb{K}$-valued continuous function on Y, let $\varphi: Y\to X$ be a basepoint-preserving map and let $T=uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. Then φ is continuous on $\mathrm{coz}(u)$.*

Proof. Suppose that there exists $y\in\mathrm{coz}(u)$ such that φ is not continuous at y. Then there exist a positive number ε and a sequence $\{y_n\}_{n=1}^\infty$ in Y such that $\rho(y_n,y)<\frac{1}{n}$ and $d(\varphi(y_n),\varphi(y))\geqslant\varepsilon$ for all $n\in\mathbb{N}$. Let x_0 be the basepoint of X and define the function $f_0: X\to\mathbb{K}$ by

$$f_0(x)=d(x,\varphi(y))-d(x_0,\varphi(y))\qquad(x\in X).$$

Clearly, $f_0 \in \mathrm{Lip}_0(X,d)$. Since $\lim_{n\to\infty} y_n = y$ in (Y,ρ) and $T(f_0) \in \mathrm{Lip}_0(Y,\rho)$, we deduce that

$$\lim_{n\to\infty} T(f_0)(y_n) = T(f_0)(y),$$

that is $\lim_{n\to\infty} u(y_n) f_0(\varphi(y_n)) = u(y) f_0(\varphi(y))$ and so

$$\lim_{n\to\infty} u(y_n)\left[d(\varphi(y_n),\varphi(y)) - d(x_0,\varphi(y))\right] = -u(y)d(x_0,\varphi(y)). \tag{2.1}$$

The continuity of u at y implies that

$$\lim_{n\to\infty} u(y_n) = u(y), \tag{2.2}$$

and so

$$\lim_{n\to\infty} u(y_n)d(x_0,\varphi(y)) = u(y)d(x_0,\varphi(y)). \tag{2.3}$$

Using (2.1) and (2.3), we have

$$\lim_{n\to\infty} u(y_n)d(\varphi(y_n),\varphi(y)) = 0. \tag{2.4}$$

From $u(y) \neq 0$, (2.2) and (2.4), we conclude that there exists $m \in \mathbb{N}$ such that

$$|u(y_m)| > \frac{|u(y)|}{2}, \tag{2.5}$$

and

$$|u(y_m)|d(\varphi(y_m),\varphi(y)) < \frac{\varepsilon|u(y)|}{3}. \tag{2.6}$$

Since $d(\varphi(y_m),\varphi(y)) \geqslant \varepsilon$, by (2.5), we have

$$|u(y_m)|d(\varphi(y_m),\varphi(y)) > \frac{\varepsilon|u(y)|}{2},$$

which is contradicts to (2.6). Therefore, φ is continuous at every $y \in \mathrm{coz}(u)$ and the proof is complete. $\square$

3. Injectivity and surjectivity of weighted composition operators

We first give necessary and sufficient conditions for the injectivity of weighted composition operators between Lipschitz spaces of $\mathbb{K}$-valued functions on pointed metric spaces.

THEOREM 3.1. *Let (X,d) and (Y,ρ) be pointed metric spaces, let y_0 be the basepoint of Y, let u be a $\mathbb{K}$-valued function on Y with $u(y_0) \neq 0$, let $\varphi : Y \to X$ be a basepoint-preserving map and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. Then T is injective if and only if $\varphi(\mathrm{coz}(u))$ is dense in X.*

Proof. Suppose that $\varphi(\mathrm{coz}(u))$ is not dense in X. Choose $x_1 \in X$ such that $\mathrm{dist}(x_1, \varphi(\mathrm{coz}(u))) > 0$ and take $\delta = \mathrm{dist}(x_1, \varphi(\mathrm{coz}(u)))$. Define the function $f : X \to \mathbb{K}$ by

$$f_{x_1,\delta}(x) = \max\{0, 1 - \frac{d(x_1,x)}{\delta}\} \qquad (x \in X).$$

Clearly, $f_{x_1,\delta}$ is a $\mathbb{K}$-valued Lipschitz function on (Y,ρ). Let x_0 be the basepoint of X. Since $y_0 \in \mathrm{coz}(u)$ and $\varphi(y_0) = x_0$, we deduce that $x_0 \in \varphi(\mathrm{coz}(u))$. This implies that $f_{x_1,\delta} \in \mathrm{Lip}_0(X,d)$. On the other hand, $T(f_{x_1,\delta}) = 0$ and $f_{x_1,\delta}(x_1) = 1$. Hence, T is not injective.

Conversely, suppose that $\varphi(\mathrm{coz}(u))$ is dense in X. Let $f \in \mathrm{Lip}_0(X,d)$ with $T(f) = 0$. Assume that $x \in \varphi(\mathrm{coz}(u))$ and choose $y \in \mathrm{coz}(u)$ such that $x = \varphi(y)$. Since $u(y) \neq 0$ and $0 = T(f)(y) = u(y)f(\varphi(y)) = u(y)f(x)$, we deduce that $f(x) = 0$. Hence, the continuous $\mathbb{K}$-valued function f on X vanishes on the dense subset $\varphi(\mathrm{coz}(u))$ of X. This implies that $f = 0$ on X. Therefore, T is injective. $\square$

Here, we give some sufficient conditions for the surjectivity of weighted composition operators between $\mathrm{Lip}_0(X,d)$ and $\mathrm{Lip}_0(Y,\rho)$.

THEOREM 3.2. *Let (X,d) and (Y,ρ) be pointed metric spaces. Suppose that u is a $\mathbb{K}$-valued function on Y such that $u(y) \neq 0$ for all $y \in Y$ and $\frac{g}{u}$ is a Lipschitz function on (Y,ρ) for all $g \in \mathrm{Lip}_0(Y,\rho)$. Let $\varphi : Y \to X$ be a basepoint-preserving map and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. If $\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} > 0$, then T is surjective.*

Proof. Suppose that

$$\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} > 0. \tag{3.1}$$

Let x_0 and y_0 be the basepoints of X and Y, respectively. We can consider $(\varphi(Y),d)$ as a pointed metric space with the basepoint $\varphi(y_0) = x_0$. Define the map $\psi : \varphi(Y) \to Y$ by

$$\psi(\varphi(y)) = y \qquad (y \in Y).$$

Then ψ is well-defined since φ is injective. Moreover, (3.1) implies that ψ is a Lipschitz mapping from $(\varphi(Y),d)$ to (Y,ρ). Let $g \in \mathrm{Lip}_0(Y,\rho)$. Then $\frac{g}{u} \circ \psi$ is a $\mathbb{K}$-valued Lipschitz function on $(\varphi(Y),d)$. On the other hand,

$$(\frac{g}{u} \circ \psi)(x_0) = (\frac{g}{u} \circ \psi)(\varphi(y_0)) = \frac{g(y_0)}{u(y_0)} = 0.$$

Hence, $\frac{g}{u} \circ \psi \in \mathrm{Lip}_0(\varphi(Y),d)$. By [7, Theorem 1.5.6], there exists a function $f \in \mathrm{Lip}_0(X,d)$ such that $f = \frac{g}{u} \circ \psi$ on $\varphi(Y)$. Therefore,

$$T(f)(y) = u(y)f(\varphi(y)) = u(y)(\frac{g}{u} \circ \psi)(\varphi(y)) = g(y),$$

for all $y \in Y$. Hence, $T(f) = g$ and so T is surjective. $\square$

We now obtain some necessary conditions for the surjectivity of weighted composition operators from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. For this purpose we need the following lemma.

LEMMA 3.3. [7, Proposition 1.8.4(a)]. *Let (X,d) and (Y,ρ) be pointed metric spaces and let $\mathrm{diam}(Y) < \infty$ and let $\varphi : Y \to X$ be a basepoint-preserving Lipschitz mapping. Then the composition operator C_φ from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$ is surjective if and only if $\varphi : Y \to \varphi(Y)$ is a Lipschitz homeomorphism.*

THEOREM 3.4. *Let (X,d) and (Y,ρ) be pointed metric spaces and let $\mathrm{diam}(Y) < \infty$. Let y_0 be the basepoint of Y, let $u \in \mathrm{Lip}(Y,\rho)$ with $u(y_0) \neq 0$, let $\varphi : Y \to X$ be a basepoint-preserving Lipschitz mapping and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. If T is surjective then $u(y) \neq 0$ for all $y \in Y$, $\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} > 0$ and $\inf\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x \in K, y \in Y, x \neq y\} > 0$, where K is a nonempty compact subset of Y.*

Proof. We first show that $u(y) \neq 0$ for all $y \in Y \setminus \{y_0\}$. Suppose that $y \in Y \setminus \{y_0\}$. Take $\delta = \frac{1}{2}\rho(y,y_0)$. Define the function $h_{y,\delta} : Y \to \mathbb{K}$ by

$$h_{y,\delta}(z) = \max\{0, 1 - \frac{\rho(y,z)}{\delta}\} \qquad (z \in Y).$$

Clearly, $h_{y,\delta} \in \mathrm{Lip}_0(Y,\rho)$ and $h_{y,\delta}(y) = 1$. The surjectivity of T implies that there exists a function $f \in \mathrm{Lip}_0(X,d)$ such that $h_{y,\delta} = T(f)$. Thus $1 = h_{y,\delta}(y) = T(f)(y) = u(y)f(\varphi(y))$ and so $u(y) \neq 0$.

Since φ is a basepoint-preserving Lipschitz mapping from (Y,ρ) to (X,d), we deduce that $f \circ \varphi \in \mathrm{Lip}_0(Y,\rho)$ for all $f \in \mathrm{Lip}_0(X,d)$ and so C_φ is a composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. We claim that C_φ is surjective. Suppose that $g \in \mathrm{Lip}_0(Y,\rho)$. Then $ug \in \mathrm{Lip}_0(Y,\rho)$ since $\mathrm{diam}(Y) < \infty$. The surjectivity of T implies that $ug = T(f)$ for some $f \in \mathrm{Lip}_0(X,d)$. Therefore, $g = f \circ \varphi = C_\varphi(f)$. Hence, our claim is justified.

By Lemma 3.3, $\varphi : Y \to \varphi(Y)$ is injective and a Lipschitz homeomorphism from (Y,ρ) to $(\varphi(Y),d)$. Hence, there exists $M > 0$ such that $\rho(x,y) \leqslant Md(\varphi(x),\varphi(y))$ for all $x,y \in Y$. This implies that

$$\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} \geqslant M' > 0, \tag{3.2}$$

where $M' = 1/M$.

We now assume that K is a nonempty compact subset of Y. Then $\inf\{|u(y)| : y \in K\} = |u(y_1)|$ for some $y_1 \in K$. This implies that

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \geqslant |u(y_1)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}$$

for all $x \in K$, $y \in Y$ with $x \neq y$. Hence, by (3.2) and $|u(y_1)| > 0$, we have

$$\begin{aligned}&\inf\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x \in K, y \in Y, x \neq y\}\\ \geqslant &|u(y_1)|\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x, y \in Y, x \neq y\} \geqslant M'|u(y_1)|.\end{aligned}$$

Therefore, the proof is complete. $\square$

4. Compactness of weighted composition operators

To obtain a necessary and sufficient condition for the compactness of a weighted composition operator between Lipschitz spaces on pointed metric spaces, we need the following result, which follows from an adaptation of the proof of [4, Lemma 2.5] to the non-separable case.

LEMMA 4.1. [4, Lemma 2.5] *Let (X,d) be a pointed metric space. Then every bounded net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $(\mathrm{Lip}_0(X,d), p_{(X,d)}(\cdot))$ has a subnet that converges pointwise on X to a function $f \in \mathrm{Lip}_0(X,d)$. Moreover, this convergence is uniform on each totally bounded subset of X.*

Having a look to the proof of [4, Proposition 2.3] and making use of Lemma 4.1, the following theorem should be clear.

THEOREM 4.2. *Let (X,d) and (Y,ρ) be pointed metric spaces. Let u be a $\mathbb{K}$-valued bounded function on Y, let $\varphi : Y \to X$ be a basepoint-preserving map and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. Then T is compact if and only if for each bounded net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $(\mathrm{Lip}_0(X,d), p_{(X,d)}(\cdot))$ which converges to the function 0 uniformly on totally bounded subsets of X, there exists a subnet $\{f_{\lambda_\gamma}\}_{\gamma\in\Gamma}$ of $\{f_\lambda\}_{\lambda\in\Lambda}$ such that $\lim_\gamma p_{(Y,\rho)}(T(f_{\lambda_\gamma})) = 0$.*

THEOREM 4.3. *Let (X,d) and (Y,ρ) be pointed metric spaces, let $u \in \mathrm{Lip}(Y,\rho)$, let $\varphi : Y \to X$ be a basepoint-preserving map, let $\varphi(\mathrm{coz}(u))$ be totally bounded in (X,d) and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. Then T is compact if and only if $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0.*

Proof. We first assume that $T = uC_\varphi$ is compact. Suppose that there exist $\varepsilon > 0$ and two sequence $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in Y with $x_n \neq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(\varphi(x_n),\varphi(y_n)) = 0$, but $|u(x_n)|\frac{d(\varphi(x_n),\varphi(y_n))}{\rho(x_n,y_n)} \geqslant \varepsilon$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define the function $f_n : X \to \mathbb{K}$ by

$$f_n(t) = \begin{cases} d(t,\varphi(y_n)) & d(t,\varphi(y_n)) \leqslant d(\varphi(x_n),\varphi(y_n)),\\ d(\varphi(x_n),\varphi(y_n)) & d(t,\varphi(y_n)) \geqslant d(\varphi(x_n),\varphi(y_n)),\end{cases}$$

for all $t \in X$. It is easy to see that f_n is a $\mathbb{K}$-valued Lipschitz function on (X,d), $\|f_n\|_X \leqslant d(\varphi(x_n),\varphi(y_n))$ and $p_{(X,d)}(f_n) \leqslant 1$. Then $\{f_n\}_{n\in\mathbb{N}}$ is a net of $\mathbb{K}$-valued Lipschitz functions on (X,d) which converges to the function 0 uniformly on X and so converges to the function 0 pointwise. Hence,

$$\lim_n f_n(x_0) = 0, \tag{4.1}$$

where x_0 is the basepoint of X. For each $n \in \mathbb{N}$ we define the function $g_n : X \to \mathbb{K}$ by

$$g_n(t) = f_n(t) - f_n(x_0) \qquad (t \in X).$$

It is clear that $\{g_n\}_{n\in\mathbb{N}}$ is a net in $\mathrm{Lip}_0(X,d)$ which converges to the function 0 uniformly on X. Moreover,

$$p_{(X,d)}(g_n) = p_{(X,d)}(f_n) \leqslant 1,$$

for all $n \in \mathbb{N}$. By Theorem 4.2 and the compactness of T, there exists a subnet $\{g_{n_\gamma}\}_{\gamma\in\Gamma}$ of the net $\{g_n\}_{n\in\mathbb{N}}$ such that

$$\lim_\gamma p_{(Y,\rho)}(T(g_{n_\gamma})) = 0. \tag{4.2}$$

From (4.1) and (4.2), we get

$$\lim_\gamma \left[p_{(Y,\rho)}(T(g_{n_\gamma})) + |f_{n_\gamma}(x_0)|p_{(Y,\rho)}(u)\right] = 0.$$

This implies that there exists a $\gamma \in \Gamma$ such that

$$p_{(Y,\rho)}(T(g_{n_\gamma})) + |f_{n_\gamma}(x_0)|p_{(Y,\rho)}(u) < \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned}
&\frac{|u(x_{n_\gamma})d(\varphi(x_{n_\gamma}),\varphi(y_{n_\gamma})) + f_{n_\gamma}(x_0)(u(y_{n_\gamma}) - u(x_{n_\gamma}))|}{\rho(x_{n_\gamma},y_{n_\gamma})} \\
=&\frac{|u(x_{n_\gamma})g_{n_\gamma}(\varphi(x_{n_\gamma})) - u(y_{n_\gamma})g_{n_\gamma}(\varphi(y_{n_\gamma}))|}{\rho(x_{n_\gamma},y_{n_\gamma})} = \frac{|T(g_{n_\gamma})(x_{n_\gamma}) - T(g_{n_\gamma})(y_{n_\gamma})|}{\rho(x_{n_\gamma},y_{n_\gamma})} \\
\leqslant& p_{(Y,\rho)}(T(g_{n_\gamma})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|u(x_{n_\gamma})|\frac{d(\varphi(x_{n_\gamma}),\varphi(y_{n_\gamma}))}{\rho(x_{n_\gamma},y_{n_\gamma})} \leqslant& |u(x_{n_\gamma})\frac{d(\varphi(x_{n_\gamma}),\varphi(y_{n_\gamma}))}{\rho(x_{n_\gamma},y_{n_\gamma})} + f_{n_\gamma}(x_0)\frac{(u(y_{n_\gamma}) - u(x_{n_\gamma}))}{\rho(x_{n_\gamma},y_{n_\gamma})}| \\
&+ |f_{n_\gamma}(x_0)|p_{(Y,\rho)}(u) \\
\leqslant& p_{(Y,\rho)}(T(g_{n_\gamma})) + |f_{n_\gamma}(x_0)|p_{(Y,\rho)}(u) < \frac{\varepsilon}{2},
\end{aligned}$$

which is a contradiction.

Conversely, suppose that $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}=0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a bounded net in $(\mathrm{Lip}_0(X,d),p_{(X,d)}(\cdot))$ that converges uniformly to the function 0 on totally bounded subsets of X. Let $M>0$ with $p_{(X,d)}(f_\lambda)<M$ for all $\lambda\in\Lambda$. Assume that

$$C=\sup\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}:x,y\in Y,x\neq y\}. \tag{4.3}$$

Since u is bounded on Y and $T=uC_\varphi$ is a weighted composition operator, we deduce that T is a bounded linear operator and $C\leqslant 2\|T\|$ by Theorem 2.2. Let $\varepsilon>0$ be given. Then there exists $\delta>0$ such that

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}<\frac{\varepsilon}{2M}, \tag{4.4}$$

whenever $x,y\in Y$ with $0<d(\varphi(x),\varphi(y))<\delta$.

Since $\varphi(\mathrm{coz}(u))$ is totally bounded in (X,d), the net $\{f_\lambda\}_{\lambda\in\Lambda}$ converges uniformly to the function 0 on $\varphi(\mathrm{coz}(u))$. Hence, there exists an $\eta\in\Lambda$ such that for each $\lambda\in\Lambda$ with $\eta\preceq\lambda$, we have $|f_\lambda(\varphi(y))|<\frac{\varepsilon}{A}$ for all $y\in\mathrm{coz}(u)$, where $A=3(\frac{2C}{\delta}+\frac{1}{2}+p_{(Y,\rho)}(u))$. Let $\lambda\in\Lambda$ with $\eta\preceq\lambda$. Then

$$|f_\lambda(\varphi(y))|<\frac{\varepsilon}{A}, \tag{4.5}$$

for all $y\in\mathrm{coz}(u)$.

Let us now prove that

$$\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)}<\frac{5\varepsilon}{6}, \tag{4.6}$$

holds for all $x,y\in X$ with $x\neq y$. To this aim, pick $x,y\in Y$ with $x\neq y$. Let us distinguish the following cases.

Case 1. $x,y\in\mathrm{coz}(u)$ with $\varphi(x)\neq\varphi(y)$. Then we have

$$\begin{aligned}\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)}&=\frac{|u(x)f_\lambda(\varphi(x))-u(y)f_\lambda(\varphi(y))|}{\rho(x,y)}\\&\leqslant\frac{|f_\lambda(\varphi(x))-f_\lambda(\varphi(y))|}{\rho(x,y)}|u(x)|+\frac{|u(x)-u(y)|}{\rho(x,y)}|f_\lambda(\varphi(y))|\\&\leqslant\frac{|f_\lambda(\varphi(x))-f_\lambda(\varphi(y))|}{d(\varphi(x),\varphi(y))}\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}|u(x)|\\&\quad+p_{(Y,\rho)}(u)|f_\lambda(\varphi(y))|.\end{aligned}$$

If $0<d(\varphi(x),\varphi(y))<\delta$, then

$$\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)}<p_{(X,d)}(f_\lambda)\frac{\varepsilon}{2M}+p_{(Y,\rho)}(u)\frac{\varepsilon}{A}\leqslant\frac{\varepsilon}{2}+\frac{\varepsilon}{3}=\frac{5\varepsilon}{6},$$

by (4.4) and (4.5). If $d(\varphi(x),\varphi(y)) \geqslant \delta$, then

$$\begin{aligned}\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)} &\leqslant \frac{|f_\lambda(\varphi(x))|+|f_\lambda(\varphi(y))|}{\delta}C+\frac{\varepsilon}{A}p_{(Y,\rho)}(u)\\ &\leqslant \frac{2C\varepsilon}{\delta A}+\frac{\varepsilon}{A}p_{(Y,\rho)}(u)=(\frac{2C}{\delta}+p_{(Y,\rho)}(u))\frac{\varepsilon}{A}<\frac{\varepsilon}{2},\end{aligned}$$

by (4.3) and (4.5).

Case 2. $x,y\in \mathrm{coz}(u)$ with $x\neq y$ and $\varphi(x)=\varphi(y)$. Then

$$\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)} \leqslant \frac{|u(x)-u(y)|}{\rho(x,y)}|f_\lambda(\varphi(y))| \leqslant p_{(Y,\rho)}(u)\frac{\varepsilon}{A}<\frac{\varepsilon}{2},$$

by (4.5).

Case 3. $x\in \mathrm{coz}(u)$ and $u(y)=0$. Then

$$\begin{aligned}\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)} &= \frac{|u(x)f_\lambda(\varphi(x))|}{\rho(x,y)} = \frac{|u(x)-u(y)|}{\rho(x,y)}|f_\lambda(\varphi(x))|\\ &\leqslant p_{(Y,\rho)}(u)\frac{\varepsilon}{A}<\frac{\varepsilon}{2},\end{aligned}$$

by (4.5).

Case 4. $u(x)=0$ and $y\in \mathrm{coz}(u)$. By similar to the argument in case 3, we have

$$\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)}<\frac{\varepsilon}{2}.$$

Case 5. $x,y\in Y$ with $x\neq y$ and $u(x)=u(y)=0$. Then

$$\frac{|T(f_\lambda)(x)-T(f_\lambda)(y)|}{\rho(x,y)}=0.$$

Summarising, we have proved that (4.6) holds for all $x,y\in Y$ with $x\neq y$ and so $p_{(Y,\rho)}(T(f_\lambda)) \leqslant \frac{5\varepsilon}{6}<\varepsilon$. This implies that

$$\lim_\lambda p_{(Y,\rho)}(T(f_\lambda))=0.$$

Therefore, T is compact by Theorem 4.2. $\square$

Note that in the sufficiency part of Theorem 4.3, we can not remove the total boundedness of $\varphi(\mathrm{coz}(u))$ in (X,d) in general. To show this assertion we need the following lemmas.

LEMMA 4.4. *Let (X,d) and (Y,ρ) be pointed metric spaces and let $\varphi: Y\to X$ be a basepoint-preserving uniformly continuous mapping. Then $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}=0$ when $d(\varphi(x),\varphi(y))$ tends to 0 if and only if φ is supercontractive from (Y,ρ) to (X,d).*

Proof. We first assume that $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$, when $d(\varphi(x),\varphi(y))$ tends to 0. Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$, when $x,y \in Y$ with $0 < d(\varphi(x),\varphi(y)) < \delta_1$. Since φ is a uniformly continuous mapping from (Y,ρ) to (X,d), we deduce that there exists $\delta > 0$ such that $d(\varphi(s),\varphi(t)) < \delta_1$, when $s,t \in Y$ with $\rho(s,t) < \delta$. Suppose that $x,y \in Y$ with $0 < \rho(x,y) < \delta$. Then $d(\varphi(x),\varphi(y)) < \delta_1$. If $\varphi(x) = \varphi(y)$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0 < \varepsilon$. If $0 < d(\varphi(x),\varphi(y)) < \delta_1$, then by the argument above, we have $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$. Therefore, φ is supercontractive from (Y,ρ) to (X,d).

We now assume that φ is supercontractive. Let $\varepsilon > 0$ be given. Then there exists $\delta_0 > 0$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$ when $x,y \in Y$ with $0 < \rho(x,y) < \delta_0$. Take $\delta = \varepsilon\delta_0$ and assume that $0 < d(\varphi(x),\varphi(y)) < \delta$ when $x,y \in Y$. If $0 < \rho(x,y) < \delta_0$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$. If $\rho(x,y) \geqslant \delta_0$ then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leqslant \frac{d(\varphi(x),\varphi(y))}{\delta_0} < \frac{\delta}{\delta_0} = \varepsilon$. Therefore, $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$, when $d(\varphi(x),\varphi(y))$ tends to 0. Hence, the proof is complete. □

LEMMA 4.5. *Let (X,d) and (Y,ρ) be pointed metric spaces, let* $\operatorname{diam}(Y) < \infty$*, let $\varphi : Y \to X$ be a basepoint-preserving Lipschitz mapping and let $u \in \operatorname{Lip}(Y,\rho)$ with $|u(y)| = 1$ for all $y \in Y$. Then $C_\varphi : \operatorname{Lip}_0(X,d) \to \operatorname{Lip}_0(Y,\rho)$ is compact if and only if $uC_\varphi : \operatorname{Lip}_0(X,d) \to \operatorname{Lip}_0(Y,\rho)$ is compact.*

Proof. Since $u \in \operatorname{Lip}(Y,\rho)$ and $|u(y)| = 1$ for all $y \in Y$, we deduce that $\frac{1}{u} \in \operatorname{Lip}(Y,\rho)$ and $|\frac{1}{u}(y)| = 1$ for all $y \in Y$. It is easy to see that if $\{f_\lambda\}_{\lambda\in\Lambda}$ be a net in $\operatorname{Lip}_0(X,d)$, then the net $\{f_\lambda \circ \varphi\}_{\lambda\in\Lambda}$ converges in $(\operatorname{Lip}_0(Y,\rho), p_{(Y,\rho)}(\cdot))$ if and only if $\{u\cdot(f_\lambda \circ \varphi)\}_{\lambda\in\Lambda}$ converges in $(\operatorname{Lip}_0(Y,\rho), p_{(Y,\rho)}(\cdot))$. This implies that C_φ is compact if and only if uC_φ is compact. □

THEOREM 4.6. *Let (X,d) be a bounded pointed metric space and let $\varphi : X \to X$ be a basepoint-preserving supercontractive Lipschitz mapping such that $\varphi(X)$ is not totally bounded in (X,d) and let $u \in \operatorname{Lip}(X,d)$ with $|u(x)| = 1$ for all $x \in X$. Then $T = uC_\varphi$ is a weighted composition operator from $\operatorname{Lip}_0(X,d)$ to $\operatorname{Lip}_0(X,d)$ which is not compact.*

Proof. By Lemma 4.4, $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. This implies that $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0 since $|u(x)| = 1$ for all $x \in X$. Since $\varphi(X)$ is not totally bounded in (X,d), an inspection in the proof of [4, Theorem 1.2] reveals that C_φ is not compact operator from $\operatorname{Lip}_0(X,d)$ to $\operatorname{Lip}_0(X,d)$. Hence, $T = uC_\varphi$ is not compact by Lemma 4.5. □

In the following example we give a metric space (X,d), a supercontractive Lipschitz mapping $\varphi : X \to X$ and a $\mathbb{K}$-valued function u on X satisfying the conditions of Theorem 4.6.

EXAMPLE 3. Let X be an infinite subset of $\mathbb{K} \setminus \{iy : y \in \mathbb{R}\}$ and let d be the discrete metric on X. Choose a point $x_0 \in X$ as the basepoint of X. Define the map

$\varphi : X \to X$ by

$$\varphi(z) = z \qquad (z \in X).$$

Then φ is a basepoint-preserving supercontractive Lipschitz mapping from (X,d) to (X,d) and $\varphi(X)$ is not totally bounded in (X,d). Let $\alpha \in \mathbb{K}$ with $|\alpha| = 1$. Define the function $u_\alpha : X \to \mathbb{K}$ by

$$u_\alpha(z) = \alpha \operatorname{sgn}(\operatorname{Re} z) \qquad (z \in X).$$

Then u_α is a Lipschitz function on (X,d) and $|u_\alpha(z)| = 1$ for all $z \in X$. It is clear that $T_\alpha = u_\alpha C_\varphi$ is a weighted composition operator from $\operatorname{Lip}_0(X,d)$ to $\operatorname{Lip}_0(X,d)$.

Applying Theorem 4.3, we give a sufficient condition for the compactness of a weighted composition operator uC_φ from $\operatorname{Lip}_0(X,d)$ to $\operatorname{Lip}_0(Y,\rho)$.

THEOREM 4.7. *Let (X,d) and (Y,ρ) be pointed metric spaces, let $u \in \operatorname{Lip}(Y,\rho)$, let $\varphi : Y \to X$ be a basepoint-preserving map, let $\varphi(\operatorname{coz}(u))$ be totally bounded in (X,d) and let $T = uC_\varphi$ be a weighted composition operator from $\operatorname{Lip}_0(X,d)$ to $\operatorname{Lip}_0(Y,\rho)$. If φ is supercontractive on $\operatorname{coz}(u)$, then T is compact.*

Proof. Assume that φ is supercontractive on $\operatorname{coz}(u)$. Let $\varepsilon > 0$ be given. Then there exists a positive number δ_0 with $\delta_0 < 1$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \frac{\varepsilon}{1+\|u\|_{\operatorname{Lip}(Y,\rho)}}$ when $x,y \in \operatorname{coz}(u)$ with $0 < \rho(x,y) < \delta_0$. Take $\delta = \frac{\varepsilon\delta_0}{1+\|u\|_{\operatorname{Lip}(Y,\rho)}}$ and assume that $x,y \in Y$ with $0 < d(\varphi(x),\varphi(y)) < \delta$. Let us now prove that

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon. \tag{4.7}$$

To this aim, we distinguish the following cases.

Case 1. $x,y \in \operatorname{coz}(u)$ with $0 < \rho(x,y) < \delta_0$. Then

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leqslant \|u\|_Y \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \|u\|_Y \frac{\varepsilon}{1+\|u\|_{\operatorname{Lip}(Y,\rho)}} < \varepsilon.$$

Case 2. $x,y \in \operatorname{coz}(u)$ with $\rho(x,y) \geqslant \delta_0$. Then

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leqslant \|u\|_Y \frac{d(\varphi(x),\varphi(y))}{\delta_0} < \frac{\|u\|_Y \varepsilon\delta_0}{\delta_0(1+\|u\|_{\operatorname{Lip}(Y,\rho)})} < \varepsilon.$$

Case 3. $x \in \operatorname{coz}(u)$ and $u(y) = 0$. Then

$$\begin{aligned}|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} &= \frac{|u(x)-u(y)|}{\rho(x,y)} d(\varphi(x),\varphi(y)) < p_{(Y,\rho)}(u)\delta = \frac{p_{(Y,\rho)}(u)\varepsilon\delta_0}{1+\|u\|_{\operatorname{Lip}(Y,\rho)}} \\ &< \varepsilon.\end{aligned}$$

Case 4. $u(x) = 0$ and $y \in \mathrm{coz}(u)$. Then

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0 < \varepsilon.$$

Therefore, (4.7) holds and so $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$, when $d(\varphi(x),\varphi(y))$ tends to 0. Hence, T is compact by Theorem 4.3. □

The following example shows that the converse of Theorem 4.7 is not valid.

EXAMPLE 4. Let $X = (-2,2)$, let d be the Euclidean metric on X and let $x_0 = 0$ be the basepoint of X. Define the function $u : X \to \mathbb{C}$ by

$$u(x) = x \qquad (x \in X).$$

Then $u \in \mathrm{Lip}(X,d)$. Define the map $\varphi : X \to X$ by

$$\varphi(x) = \mathrm{sgn}(x) \qquad (x \in X).$$

Then φ is a basepoint-preserving map and it is easy to see that

$$\sup\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in X, x \neq y\} < 2.$$

Hence, $T = uC_\varphi$ is a weighted composition operator on $\mathrm{Lip}_0(X,d)$ by Theorem 2.1. Moreover, it is clear that $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Since $\varphi(\mathrm{coz}(u)) = \{-1,1\}$, we deduce that $\varphi(\mathrm{coz}(u))$ is a totally bounded set in (X,d). Therefore, T is a compact weighted composition operator by Theorem 4.3.

On the other hand,

$$\frac{d(\varphi(\frac{1}{n}),\varphi(\frac{-1}{n}))}{d(\frac{1}{n},\frac{-1}{n})} = \frac{2}{\frac{2}{n}} = n,$$

for all $n \in \mathbb{N}$ with $n \geqslant 2$. Hence, φ is not supercontractive on $\mathrm{coz}(u)$.

In spite of the previous example, the following result reveals φ has to be supercontractive on some subsets of $\mathrm{coz}(u)$. More precisely we have the following theorem.

THEOREM 4.8. *Let (X,d) and (Y,ρ) be pointed metric spaces, let $u \in \mathrm{Lip}(Y,\rho)$, let $\varphi : Y \to X$ be a basepoint-preserving map, let $\varphi(\mathrm{coz}(u))$ be totally bounded in (X,d) and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}_0(X,d)$ to $\mathrm{Lip}_0(Y,\rho)$. If T is compact, then φ is supercontractive on compact subsets of $\mathrm{coz}(u)$.*

Proof. Suppose that T is compact. By Theorem 4.3, $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Let K be a nonempty compact subset of $\mathrm{coz}(u)$. Let $\varepsilon > 0$

be given. Take $C = \inf\{|u(y)| : y \in K\}$. The continuity of u on $\mathrm{coz}(u)$ implies that $C > 0$. By the assumptions there exists $\delta_1 > 0$ such that

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < C\varepsilon, \tag{4.8}$$

when $x,y \in Y$ with $0 < d(\varphi(x),\varphi(y)) < \delta_1$. By Corollary 2.4, φ is a Lipschitz mapping from (K,ρ) to (X,d). Hence, φ is a uniformly continuous mapping from (K,ρ) to (X,d). This implies that there exists $\delta > 0$ such that $d(\varphi(x),\varphi(y)) < \delta_1$ when $x,y \in K$ with $\rho(x,y) < \delta$. Suppose that $x,y \in K$ with $0 < \rho(x,y) < \delta$. Then $d(\varphi(x),\varphi(y)) < \delta_1$. If $\varphi(x) = \varphi(y)$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0 < \varepsilon$. If $0 < d(\varphi(x),\varphi(y)) < \delta_1$, then we have

$$\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leqslant \frac{|u(x)|d(\varphi(x),\varphi(y))}{C\rho(x,y)} < \frac{C\varepsilon}{C} = \varepsilon,$$

by (4.8). Therefore, φ is supercontractive from (K,ρ) to (X,d) and the proof is complete. □

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