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A NOTE ON THE MAXIMAL NUMERICAL RANGE

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Abstract. We show that the maximal numerical range of an operator has a non-empty intersection with the boundary of its numerical range if and only if the operator is normaloid. A description of this intersection is also given.

First, let us set some notation and terminology.

For a subset X of the complex plane $\mathbb{C}$, by $\operatorname{cl} X$, ∂X and $\operatorname{conv} X$ we will denote the closure, boundary and the convex hull of X, respectively.

By an "operator" we throughout the paper understand a bounded linear operator acting on a Hilbert space $\mathcal{H}$. The *numerical range* of such an operator A is defined by the formula

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \|x\| = 1\},$$

where $\langle .,.\rangle$ and $\|.\|$ stand, respectively, for the scalar product on $\mathcal{H}$ and the norm associated with it. Introduced a century ago in the works by Toeplitz [9] and Hausdorrf [7] (and thus also known as the *Toeplitz-Hausdorff set*), it since has been a subject of intensive research. We mention here only [4] as a standard source of references and note the following basic properties:

Due to the Cauchy-Schwarz inequality, the set $W(A)$ is bounded. Namely,

$$w(A) := \sup\{|z| : z \in W(A)\} \leqslant \|A\|; \tag{1}$$

$w(A)$ is called the *numerical radius* of A.

The set $W(A)$ is convex (the Toeplitz-Hausdorff theorem) and if $\dim \mathcal{H} < \infty$ it is also closed.

A (relatively) more recent notion of the *maximal* numerical range $W_0(A)$ was introduced in [8] as the set of all $\lambda \in \mathbb{C}$ for which there exist

$$x_n \in \mathcal{H} \text{ such that } \|x_n\| = 1,\ \|Ax_n\| \to \|A\|, \text{ and } \langle Ax_n, x_n\rangle \to \lambda. \tag{2}$$

It was also shown in [8, Lemma 2] that $W_0(A)$ is convex, closed and is contained in the closure of $W(A)$:

$$W_0(A) \subset \operatorname{cl} W(A). \tag{3}$$

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Observe that in the finite dimensional case $W_0(A) = W(B)$, where B is the compression of A onto the eigenspace of A^*A corresponding to its maximal eigenvalue, so the above mentioned properties of the maximal numerical range are rather straightforward.

Given the inclusion (3), it is natural to try to describe in more detail the positioning of $W_0(A)$ with respect to $W(A)$. In particular, what can be said about the points of $W_0(A)$ which lie on the boundary $\partial W(A)$ of $W(A)$?

We start by describing the intersection of $W_0(A)$ with the circle

$$\mathscr{C}_A := \{z\colon\ |z| = \|A\|\}.$$

LEMMA 1. *For any operator A,*

$$W_0(A) \cap \mathscr{C}_A = \operatorname{cl} W(A) \cap \mathscr{C}_A = \sigma(A) \cap \mathscr{C}_A. \tag{4}$$

Proof. The second equality in (4) is well known [5, Problem 218]. Due to (3) it therefore remains to prove only the inclusion of the middle term in the left hand side. To this end, observe that with any $\lambda \in \operatorname{cl} W(A)$ by definition there is associated a sequence of unit vectors $x_n \in \mathscr{H}$ for which $\langle Ax_n, x_n\rangle \to \lambda$. If, in addition, $|\lambda| = \|A\|$, then the Cauchy-Schwarz inequality implies that $\|Ax_n\| \to \|A\|$. In other words, (2) holds. $\square$

Recall that operators for which the second (and thus, equivalently, the third) term in (4) is non-empty are called *normaloid*. Therefore, Lemma 1 implies the sufficiency in the following

THEOREM 1. *The intersection $W_0(A) \cap \partial W(A)$ is non-empty if and only the operator A is normaloid.*

Proof of necessity. To simplify the notation, without loss of generality suppose that $\|A\| = 1$; this can be arranged by an appropriate scaling not having any effect on the validity of the statement. Then $\mathscr{C}_A$ is simply the unit circle $\mathbb{T}$.

If $W_0(A) \cap \partial W(A) \neq \emptyset$, then (2) holds for some $\lambda \in \partial W(A)$. Choose unit vectors y_n orthogonal to x_n and lying in the span of x_n and Ax_n. Then of course

$$Ax_n = a_n x_n + c_n y_n$$

for some $a_n, c_n \in \mathbb{C}$ such that

$$a_n \to \lambda, \quad |c_n|^2 \to 1 - |\lambda|^2. \tag{5}$$

Consider the compression of A onto the span of x_n, y_n. Its matrix A_n with respect to the basis $\{x_n, y_n\}$ has $[a_n, c_n]^T$ as its first column; denote the second column of A_n as $[b_n, d_n]^T$.

Passing to a subsequence if needed, we may suppose that

$$A_n \to A_0 := \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where due to (5) $a = \lambda$ and $|c|^2 = 1 - |\lambda|^2$.

Since $W(A_n) \subset W(A)$ for all $n = 1, 2, \dots$, we have $W(A_0) \subset \mathrm{cl}\, W(A)$ and so the $(1,1)$-entry a of A_0 lies on the boundary of its numerical range. This is only possible if $|b| = |c|$, as was observed e.g. in [10, Corollary 4], see also [3, Proposition 4.3]. Moreover,

$$A_0^* A_0 = \begin{bmatrix} 1 & \overline{a}b + \overline{c}d \\ \overline{b}a + \overline{d}c & |b|^2 + |d|^2 \end{bmatrix}. \tag{6}$$

Since $\|A_0\| = \lim \|A_n\| \leqslant 1$, the matrix $A_0^* A_0$ must be diagonal. When combined with the already established equality $|b| = |c|$, this implies that either $b = c = 0$, or $|d| = |a|$.

In the former case the normaloidness of A is immediate, because then $\lambda \in \mathbb{T}$. In the latter case (6) simplifies to $A_0^* A_0 = I$, i.e. A_0 is unitary. Then $\mathrm{cl}\, W(A) \cap \mathbb{T} \supset \sigma(A_0) \neq \emptyset$, also implying that A is normaloid. $\square$

It follows from Theorem 1 in particular that an operator A is normaloid if and only if its numerical radius $w(A)$ coincides with $w_0(A) := \max\{|z| : z \in W_0(A)\}$. This result was established in [1, Corollary 1]. Moreover, the paper [1] served as a motivation for the present note and our proof of Theorem 1 is making use of some reasoning from [1]. Note that a simplified version of the proof, adapted to the finite dimensional setting, was included in [6]. The latter paper contains also results on the explicit description of $W_0(A)$ for some classes of matrices A.

A closer look at the proof of Theorem 1 yields an explicit description of the set $W_0(A) \cap \partial W(A)$.

COROLLARY 1. *The intersection of $W_0(A)$ with the boundary of $W(A)$ consists of $\sigma(A) \cap \mathscr{C}_A$ and the set $\mathscr{L}_A$ of all the chords of $\mathscr{C}_A$ lying on $\partial W(A)$:*

$$W_0(A) \cap \partial W(A) = (\sigma(A) \cap \mathscr{C}_A) \cup \mathscr{L}_A.$$

Note that the endpoints of the above mentioned chords belong to $\sigma(A) \cap \mathscr{C}_A$. Considering by convention the remaining points of $\sigma(A) \cap \mathscr{C}_A$ as the endpoints of "degenerate" zero-length chords of $\mathscr{C}_A$, we may say simply that $W_0(A) \cap \partial W(A)$ is exactly the set of all chords of $\mathscr{C}_A$ lying on $\partial W(A)$.

Being convex, along with $\sigma(A) \cap \mathscr{C}_A$ the set $W_0(A)$ must also contain its convex hull $\mathrm{conv}(\sigma(A) \cap \mathscr{C}_A)$. Since $\mathscr{L}_A \subset \mathrm{conv}(\sigma(A) \cap \mathscr{C}_A)$, the equality

$$W_0(A) = \mathrm{conv}(\sigma(A) \cap \mathscr{C}_A) \tag{7}$$

is plausible. It may fail, however, even in finite dimensions.

Example. Let

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\|B\| = 1$ is attained on the 2-dimensional span of the standard basis vectors e_2, e_3. The compression of A onto their span is the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and so $W_0(B)$ is the line segment $[0,1]$. On the other hand, $W(A)$ is the ice-cone shaped convex hull of the circle centered at the origin and having radius $1/2$ and the point 1. In particular, $w(B) = 1$, making A normaloid. In agreement with Corollary 1 we have (taking into consideration that $\mathscr{C}_B = \mathbb{T}$):

$$W_0(B) \cap \partial W(B) = \sigma(B) \cap \mathbb{T} = \{1\},$$

and so $\operatorname{conv}(\sigma(B) \cap \mathbb{T}) = \{1\}$ is a proper subset of $W_0(B)$.

The situation changes if A is normal and not merely normaloid.

THEOREM 2. *Equality* (7) *holds for normal operators* A.

Proof. Due to the inclusion $W_0(A) \supset \sigma(A) \cap \mathscr{C}_A$ and the fact that both sides in (7) are convex and compact, it suffices to show that any open half-plane containing $\sigma(A) \cap \mathscr{C}_A$ also contains $W_0(A)$.

So, consider a half-plane $\Pi \supset \sigma(A) \cap \mathscr{C}_A$. The spectrum of A is disjoint with the arc $\gamma = \mathscr{C}_A \setminus \Pi$, and the distance between γ and $\sigma(A)$ is therefore positive. Denoting it by ε, observe that

$$\sigma_\varepsilon(A) := \{\lambda \in \sigma(A) \colon\ |\lambda| \geqslant \|A\| - \varepsilon\} \subset \Pi.$$

Let A_ε be the restriction of A onto its spectral subset corresponding to $\sigma_\varepsilon(A)$. The definition of $W_0(A)$ implies that $W_0(A) \subset \operatorname{cl} W(A_\varepsilon)$. On the other hand, the operator A_ε is normal along with A and so $\operatorname{cl} W(A_\varepsilon) = \operatorname{conv} \sigma(A_\varepsilon) = \operatorname{conv} \sigma_\varepsilon(A) \subset \Pi$. $\square$

Recall that an operator A acting on a Hilbert space $\mathscr{H}$ is *subnormal* if there exists a Hilbert space $\mathscr{G}$ and operators $B \colon \mathscr{G} \to \mathscr{H}$, $C \colon \mathscr{G} \to \mathscr{G}$ such that the operator

$$N := \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : \mathscr{H} \oplus \mathscr{G} \to \mathscr{H} \oplus \mathscr{G}$$

is normal. As it happens, property (7) extends from normal to subnormal operators.

COROLLARY 2. *Equality* (7) *holds for subnormal operators* A.

Proof. Consider the minimal normal extension N of A, the existence and properties of which are discussed e.g. in [2, 5]. It is true in particular that $\|A\| = \|N\|$. So, whenever a sequence of unit vectors $x_n \in \mathscr{H}$ is such that $\|Ax_n\| \to \|A\|$, it at the same time satisfies $\|Nx_n\| \to \|N\|$. Consequently,

$$W_0(A) \subset W_0(N). \tag{8}$$

Furthermore, $\sigma(A)$ equals $\sigma(N)$ with some holes filled and so

$$\sigma(A) \cap \mathscr{C}_A = \sigma(N) \cap \mathscr{C}_N. \tag{9}$$

Combining (8),(9) with the equality $W_0(N) = \operatorname{conv}(\sigma(N) \cap \mathscr{C}_N)$ which holds due to Theorem 2, we obtain

$$W_0(A) \subset W_0(N) = \operatorname{conv}(\sigma(N) \cap \mathscr{C}_N) = \operatorname{conv}(\sigma(A) \cap \mathscr{C}_A).$$

Since the converse inclusion holds for any A, we are done. $\square$

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