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A NEW CLASS OF HYPERFINITE KADISON–SINGER FACTORS

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Abstract. In this paper, we construct a new class of hyperfinite Kadison-Singer factors on separable Hilbert spaces, and we show that each of these Kadison-Singer factors is isomorphic to a subalgebra of CSL algebra. Moreover, a sufficient and necessary condition for two of these Kadison-Singer factors being isometrically isomorphic is given. Finally, we obtain that every norm preserving automorphism on these Kadison-Singer algebras is inner.

1. Introduction

In 1960, Kadison and Singer (see [12]) introduced and studied a class of non-selfadjoint operator algebras which they called triangular (operator) algebras. Suppose $\mathscr{H}$ is a separable Hilbert space and $\mathscr{B}(\mathscr{H})$ is the algebra of all bounded linear operators on $\mathscr{H}$, and $\mathscr{M}$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$. A triangular algebra $\mathscr{T}$ is a subalgebra of $\mathscr{M}$ such that $\mathscr{T} \cap \mathscr{T}^*$ is a maximal abelian selfadjoint subalgebra of $\mathscr{M}$. One of the interesting cases is $\mathscr{M} = \mathscr{B}(\mathscr{H})$. Nest algebras introduced by Ringrose (see [7, 8]) are the most well understood non-selfadjoint algebras, it is a class of maximal triangular algebras. Let $\mathscr{L}$ be a set of projections in $\mathscr{B}(\mathscr{H})$, and $\mathrm{Alg}(\mathscr{L})$ denote the set of bounded operators that leave the range of every element of $\mathscr{L}$ invariant, i.e.,

$$\mathrm{Alg}(\mathscr{L}) = \{T \in \mathscr{B}(\mathscr{H}) : (I-P)TP = 0, \forall P \in \mathscr{L}\}.$$

Dually, let $\mathscr{A}$ be a set of operators in $\mathscr{B}(\mathscr{H})$, $\mathrm{Lat}(\mathscr{A})$ denote the collection of projections whose ranges are left invariant by every element of $\mathscr{A}$, i.e.,

$$\mathrm{Lat}(\mathscr{A}) = \{P \in \mathscr{B}(\mathscr{H}) : P^* = P, P^2 = P, (I-P)TP = 0, \forall T \in \mathscr{A}\}.$$

Recall that a subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is called *reflexive* if $\mathscr{A} = \mathrm{Alg}(\mathrm{Lat}(\mathscr{A}))$. Every nest algebra is a reflexive algebra, and reflexive algebras are completely determined by their lattices of invariant subspace projections.

In 2009, Ge and Yuan (see [10]) combined triangularity, reflexivity and von Neumann algebra properties in a single class of algebras and introduced Kadison-Singer (KS) algebras.

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DEFINITION 1.1. (See [10], Definition 1.) Let $\mathscr{H}$ be a separable Hilbert space. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is called a *Kadison-Singer algebra* (or *KS-algebra*) if $\mathscr{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathscr{A}\bigcap\mathscr{A}^*$ of $\mathscr{A}$, in the sense that if there is another reflexive subalgebra $\mathscr{B}$ of $\mathscr{B}(\mathscr{H})$ such that $\mathscr{A}\subseteq\mathscr{B}$ and $\mathscr{B}\bigcap\mathscr{B}^*=\mathscr{A}\bigcap\mathscr{A}^*$, then $\mathscr{A}=\mathscr{B}$. When the diagonal of a KS-algebra $\mathscr{A}$ is a factor, we say $\mathscr{A}$ is a *Kadison-Singer factor* (or *KS-factor*). A lattice $\mathscr{L}$ of projections in $\mathscr{B}(\mathscr{H})$ is called a *Kadison-Singer lattice* (or *KS-lattice*) if $\mathscr{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathscr{L}''$, equivalently, $\mathscr{L}$ is reflexive and $\mathrm{Alg}(\mathscr{L})$ is a Kadison-Singer algebra.

In [10], Ge and Yuan gave a class of algebras with hyperfinite diagonals. Later, in [11] they constructed three free projections with trace $\frac{1}{2}$, and then proved that the reflexive lattices generated by these three projections are homeomorphic to the sphere S^2 plus two points. In [6], Hou and Yuan generalized this result and proved the same holds true for reflexive lattice generated by any double triangle lattice of projections in a finite von Neumann algebra. Ren and Wu in [17] constructed a new kind of KS lattices in separable Hilbert spaces. Dong and Hou in [1] studied the automorphisms of some KS algebras. Wu and Yuan in [15] proved that if an abelian KS algebra $\mathscr{A}$ is a subalgebra of matrix algebra $M_n(\mathbb{C})(n\geqslant 3)$, then $\mathscr{A}$ cannot be generated by a single element. Similar results can be found in [2, 3, 4, 5, 9, 16]. KS-algebras bring connections between selfadjoint and non-selfadjoint theories, so many techniques and tools in von Neumann algebras can be used to study these non-selfadjoint algebras.

In this paper, based on the hyperfinite KS-factors in [10], we construct a class of lattices and a class of unbounded operators in separable Hilbert spaces, then we prove that this lattice algebra is isomorphism to a subalgebra of CSL algebra. Moreover, we show that $\mathrm{Alg}(\mathscr{L}(n_1,n_2,\cdots,n_s,\cdots))$ is isometrically isomorphic to $\mathrm{Alg}(\mathscr{L}(m_1,m_2,\cdots,m_s,\cdots))$ if and only if $n_i=m_i$, for all $i=1,2,\cdots$. Furthermore, in Section 3, we show that if $n_i=2$ for each i, then every norm preserving automorphism on $\mathrm{Alg}(\mathscr{L}_\infty)$ is an inner automorphism.

2. Hyperfinite KS-factors

In this section, we shall construct a new hyperfinite KS-Factor. Similar to [10], let $M_{n_\lambda}(\mathbb{C})$ $(n_\lambda>1)$ be the algebra of $n_\lambda\times n_\lambda$ matrices and $\mathscr{A}$ obtained by taking the completion (with respect to operator norm) of $\otimes_{\lambda=1}^{\infty}M_{n_\lambda}(\mathbb{C})$. Then we may write $\mathscr{A}$ for $\overline{M_{n_1}\otimes M_{n_2}\otimes\cdots}$. We denote by $E_{ij}^{(k)},i,j=1,\ldots,n_k$, the standard matrix unit system for $M_{n_k}(\mathbb{C})$ $(k=1,2\ldots)$, and for all $m=1,2\cdots$, let

$$E_i^{(m)}=\sum_{t=1}^{i}E_{tt}^{(m)}\qquad i=1,2,\cdots,n_m$$

be projections of $M_{n_m}(\mathbb{C})$. Let

$$\mathscr{N}_m(n_1,n_2,\cdots,n_m)=M_{n_1}(\mathbb{C})\otimes M_{n_2}(\mathbb{C})\otimes\cdots\otimes M_{n_m}(\mathbb{C}).\tag{2.1}$$

Then $\mathscr{A}=\overline{\cup_{m=1}^{\infty}\mathscr{N}_m(n_1,n_2,\cdots,n_m)}$. Now, we construct (inductively) a family of projections in $\mathscr{N}_m(n_1,n_2,\cdots,n_m)$.

If $m=1$, define $P_{1,j_1}=\sum_{i=1}^{j_1}E_{ii}^{(1)}$, $j_1=1,\ldots,n_1-1$, and $P_{1,n_1}=\frac{1}{n_1}\sum_{s,t=1}^{n_1}E_{st}^{(1)}$. Suppose when $k\leqslant m-1$, for each $j_k=1,\ldots,n_k$, $P_{k,j_k}\in\mathcal{N}_k(n_1,n_2,\cdots,n_k)$ is defined. Now when $k=m$, we define

$$P_{m,j_m}=P_{m-1,n_{m-1}-1}+(I-P_{m-1,n_{m-1}-1})\sum_{i=1}^{j_m}E_{ii}^{(m)},\qquad j_m=1,\ldots,n_m-1,\tag{2.2}$$

$$P_{m,n_m}=P_{m-1,n_{m-1}-1}+(I-P_{m-1,n_{m-1}-1})\left(\frac{1}{n_m}\sum_{s,t=1}^{n_m}E_{st}^{(m)}\right).\tag{2.3}$$

Denote by $\mathcal{L}_m(n_1,n_2,\cdots,n_m)$ the lattice generated by $\{P_{k,j_k}:1\leqslant k\leqslant m,1\leqslant j_k\leqslant n_m\}$ and $\mathcal{L}_\infty(n_1,n_2,\cdots,n_m,\cdots)=\cup_m\mathcal{L}_m(n_1,n_2,\cdots,n_m)$, the lattice generated by $\{P_{k,j_k}:k\geqslant 1,1\leqslant j_k\leqslant n_k\}$. If the sequence $\{n_k\}_{k=1}^{\infty}$ is clear, without causing confusion, we may write $\mathcal{N}_m,\mathcal{L}_m,\mathcal{L}_\infty$ instead of $\mathcal{N}_m(n_1,n_2,\cdots,n_m),\mathcal{L}_m(n_1,n_2,\cdots,n_m)$, $\mathcal{L}_\infty(n_1,n_2,\cdots,n_m,\cdots)$. We can easily show that $\mathcal{N}_m$ is generated by $\mathcal{L}_m$ (as a finite-dimensional von Neumann algebra).

Let ρ_λ be a faithful state on $M_{n_\lambda}(\mathbb{C})$, and $\rho=\rho_1\otimes\rho_2\otimes\cdots$. Clearly, ρ is a state on $\mathcal{A}$. Let $\mathcal{H}$ and $\mathcal{H}_\lambda$ be the Hilbert space obtained by GNS construction on $(\mathcal{A},\rho)$ and $(M_{n_\lambda}(\mathbb{C}),\rho_\lambda)$. It is well-known (see Chapter 11.4 in [13]) that the weak operator closure of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ is a hyperfinite factor $\mathcal{R}$ (In particular, the factor $\mathcal{R}$ is type II_1 if ρ is a trace). Then $\mathcal{L}_m$ and $\mathcal{L}_\infty$ become lattices of projections in $\mathcal{R}$. It is similar to [10] we can prove that $\mathrm{Alg}(\mathcal{L}_\infty)$ is KS-factor containing the hyperfinite factor $\mathcal{R}'$ as its diagonal and the following lemma.

LEMMA 2.1. [10] *With $\mathcal{L}_1\subset\mathcal{N}_1$ defined above, we have*

$$\begin{aligned}Alg(\mathcal{L}_1)=\{\ &T\in\mathcal{B}(\mathcal{H}):E_{ii}^{(1)}TE_{jj}^{(1)}=0,\ \ 1\leqslant j<i\leqslant n_1;\\&\sum_{j=1}^{n_1}E_{11}^{(1)}TE_{j1}^{(1)}=\sum_{j=2}^{n_1}E_{12}^{(1)}TE_{j1}^{(1)}=\cdots=E_{1n_1}^{(1)}TE_{n_11}^{(1)}\},\end{aligned}$$

where $E_{ij}^{(1)}$ $(i,j=1,\ldots,n_1)$ are the matrix units for $\mathcal{N}_1$.

Let $F_1=\sum_{i=1}^{n_1}E_{i,n_1}^{(1)}$, and

$$F_m=(I-P_{m-1,n_{m-1}-1})\sum_{i=1}^{n_m}E_{i,n_m}^{(m)}.$$

Now, we construct a class of operators $\{V_m\}$. Define $V_m:\mathcal{H}\to\mathcal{H}$ with

$$V_1=\sum_{i=1}^{n_1-1}E_{ii}^{(1)}+\sum_{i=1}^{n_1}E_{in_1}^{(1)}=P_{1,n_1-1}+F_1,$$

and when $m\geqslant 2$,

$$V_m=P_{m-1,n_{m-1}-1}+(I-P_{m-1,n_{m-1}-1})(E_{n_m-1}^{(m)}+F_m).$$

By the definition of V_m we have the following fact.

LEMMA 2.2. *If $k > m$, then $V_k P_{m,n_m-1} = P_{m,n_m-1} = P_{m,n_m-1} V_k$.*

Proof. From the definition of V_m, it is easy to see that when $k > m$,

$$\begin{aligned}
&V_k - P_{m,n_m-1} \\
&= (I - P_{m,n_m-1})(P_{k-1,n_{k-1}-1} - P_{m,n_m}) + (I - P_{k-1,n_{k-1}-1})(E^{(m)}_{n_m-1} + F_m) \\
&= (I - P_{m,n_m-1})(V_k - P_{m,n_m-1}).
\end{aligned}$$

When $k \geqslant m$, $P_{k-1,n_{k-1}-1} \geqslant P_{m-1,n_{m-1}-1}$. For all $k > m$, we have

$$(I - P_{m,n_m-1}) V_k P_{m,n_m-1} = (I - P_{m,n_m-1})(P_{m,n_m-1} + (V_k - P_{m,n_m-1})) P_{m,n_m-1} = 0,$$

and

$$\begin{aligned}
V_k P_{m,n_m-1} &= (P_{1,n_1-1} + (V_k - P_{1,n_1-1}))(P_{1,n_1-1} + (I - P_{1,n_1-1})(P_{m,n_m-1} - P_{1,n_1-1})) \\
&= P_{1,n_1-1} + (I - P_{1,n_1-1})(V_k - P_{1,n_1-1})(P_{m,j_m} - P_{1,n_1-1})) \\
&= \cdots\cdots \\
&= P_{m,n_m-1} + (I - P_{m,n_m-1})(V_k - P_{m,n_m-1}) P_{m,n_m-1} \\
&= P_{m,n_m-1}.
\end{aligned}$$

Similarly, we also have that $P_{m,n_m-1} V_k = P_{m,n_m-1}$ for all $k > m$. $\square$

Since $\forall x \in \cup_{m=1}^{\infty} P_{m,n_m-1}\mathscr{H}$, there exists a smallest integer $k = k(x)$ such that $x \in P_{k,n_k-1}\mathscr{H}$. Then we define an operator V_0 on $\mathscr{D}(V_0) = \cup_{m=1}^{\infty} P_{m,n_m-1}\mathscr{H}$ with

$$V_0 x = (\prod_{i=1}^{\infty} V_i) x = (\prod_{i=1}^{k} V_i) x.$$

By the definition of P_{m,j_m} and using Lemma 2.2, we are able to get $\lim\limits_{m\to\infty} P_{m,n_m-1} = I$, and

$$V_0 P_{m,j_m} = V_1 V_2 \cdots V_m P_{m,j_m} \in P_{m,n_m-1}\mathscr{H},$$

for all $j_m = 1, 2, \ldots, n_m - 1$. Thus, V_0 is densely defined on $\mathscr{H}$.

REMARK 2.1. Note that the V_0 defined above is unbounded. In fact, for any m, choose a unit vector a_m in $\mathscr{H}_m$ such that $\xi_m = (0, 0, \cdots, 0, a_m)^\top \in P_{m,n_m-1}\mathscr{H} \subseteq \mathscr{D}(V_0)$, then we have

$$\begin{aligned}
\|\prod_{i=1}^{\infty} V_i \xi_m\| &= \|\prod_{i=1}^{m} V_i \xi_m\| = \|\prod_{i=1}^{m-1} V_i (0, \cdots, 0, a_m, \cdots, a_m)^\top\| \\
&= \cdots = \|(a_m, a_m, \cdots, a_m, a_m)^\top\| \to \infty
\end{aligned}$$

as $m \to \infty$, and then V_0 is unbounded.

By Lemma 2.2, we know that for each $x \in P_{m,n_m-1}\mathscr{H}$, $V_0 P_{m,n_m-1} = \prod_{i=1}^{m} V_i P_{m,n_m-1}$ and $(P_{m,n_m-1} V_0) x = P_{m,n_m-1} (\prod_{i=1}^{\infty} V_i) x = P_{m,n_m-1} (\prod_{i=1}^{m} V_i) x$, it is clear that $V_0 P_{m,n_m-1}$ is bounded. It is not hard to see that

$$V_1^{-1} = I - \sum_{i=1}^{n_1-1} E^{(1)}_{i,n_1},$$

and when $m \geqslant 2$,

$$V_m^{-1} = P_{m-1,n_{m-1}-1} + (I - P_{m-1,n_{m-1}-1})(I - \sum_{i=1}^{n_m-1} E_{i,n_m}^{(m)}).$$

LEMMA 2.3. *V_0^* is a densely defined closed operator on $\mathscr{H}$.*

Proof. We claim that $\bigcup_{m=1}^{\infty}((V_1^*)^{-1}(V_2^*)^{-1}\cdots(V_m^*)^{-1}P_{m,n_m-1}\mathscr{H}) \subseteq \mathscr{D}(V_0^*)$.

Let $k > m$ and $\xi \in (V_1^*)^{-1}(V_2^*)^{-1}\cdots(V_m^*)^{-1}P_{m,n_m-1}\mathscr{H}$. Then for every $\eta \in P_{k,n_k-1}\mathscr{H}$, we have

$$< \xi, \prod_{i=1}^{k} V_i\eta > = < V_m^*V_{m-1}^*\cdots V_1^*\xi, \prod_{i=m+1}^{k} V_i\eta > .$$

Note that $V_m^*V_{m-1}^*\cdots V_1^* \in P_{m,n_m-1}\mathscr{H}$, by Lemma 2.2,

$$< V_m^*V_{m-1}^*\cdots V_1^*\xi, \prod_{i=m+1}^{k} V_i\eta > = < V_m^*V_{m-1}^*\cdots V_1^*\xi, P_{m,n_m-1}\eta > = < V_m^*V_{m-1}^*\cdots V_1^*\xi, \eta > .$$

This implies that $\eta \in \bigcup_{m=1}^{\infty} P_{m,n_m-1}\mathscr{H} \subset D(V_0^*)$. Therefore V_0^* is densely defined. $\square$

Since V_0^* is densely defined and $\mathscr{D}(V_0) = \cup_{m=1}^{\infty} P_{m,n_m-1}\mathscr{H}$, we know that V_0 is preclosed and refer to $\overline{V_0}$ as the closure of V_0. Now let $V = \overline{V_0} = V_0^{**}$. Then $\mathscr{D}(V_0) \subseteq \mathscr{D}(V)$ and $V\mid_{\mathscr{D}(V_0)} = V_0$. In this case, we say that $\mathscr{D}(V_0)$ is a *core* for V. From Lemma 2.3, V is densely defined and closed on $\mathscr{H}$. By the definition of V, we have that $VP_{m,j_m} = V_1V_2\cdots V_mP_{m,j_m}$, indeed, we have the result as follows.

LEMMA 2.4. *For $j_m = 1,2,\ldots,n_m$, $VP_{m,j_m} = V_1V_2\cdots V_mP_{m,j_m} \in \mathscr{L}_\infty''$.*

Proof. When $m = 1$, clearly, $E_{ii}^{(1)} \in \mathscr{L}_1'' \subseteq \mathscr{L}_\infty''$ and when $j_1 < n_1$, we have

$$VP_{1,j_1} = V_1P_{1,j_1} = P_{1,j_1} \in \mathscr{L}_\infty'',$$

and if $j_1 = n_1$,

$$n_1E_{ii}^{(1)}P_{1,n_1}E_{jj}^{(1)} = E_{ij}^{(1)} \in \mathscr{L}_1'' \subseteq \mathscr{L}_\infty''.$$

Therefore, the lemma holds when $m = 1$.

Now, we assume the lemma holds for all $m \leqslant k$, that is, $VP_{k,j_k} \in \mathscr{L}_\infty''$ and $E_{ij}^{(k)} \in \mathscr{L}_k'' \subseteq \mathscr{L}_\infty''$. Since $\mathscr{L}_k \subseteq \mathscr{L}_{k+1}$, we have

$$\mathscr{L}_k \subseteq \mathscr{L}_k'' \subseteq \mathscr{L}_{k+1}'' \subseteq \mathscr{L}_\infty''.$$

Thus, we conclude that both $\sum_{i=1}^{j} E_{ii}^{(k+1)}$ $(j = 1,\ldots,n_{k+1}-1)$ and $\sum_{s,t=1}^{n_{k+1}} E_{st}^{(k+1)}$ are in $\mathscr{L}_{k+1}''$. By the definition of $P_{k+1,n_{k+1}}$, we see that

$$n_{k+1}E_{ii}^{(k+1)}P_{k+1,j_{k+1}}E_{jj}^{(k+1)} \in \mathscr{L}_{k+1}'' \subseteq \mathscr{L}_\infty''.$$

Hence $VP_{k+1,j_{k+1}} \in \mathscr{L}_\infty''$. $\square$

LEMMA 2.5. *V is affiliated with $\mathscr{L}_\infty''$.*

Proof. Let W be a unitary in $\mathscr{L}_\infty'$. It follows from $W^*P_{m,n_m-1}W = P_{m,n_m-1}$ $(\forall m)$ that $W\mathscr{D}(V_0) = \mathscr{D}(V_0)$. Moreover, since V is the closure of V_0, $W\mathscr{D}(V) = \mathscr{D}(V)$.

Let $\xi \in P_{m,n_m-1}\mathscr{H}$. Since

$$V_0W\xi = V_1V_2\cdots V_mW\xi = WV_1V_2\cdots V_m\xi,$$

for each $\beta \in \mathscr{D}(V)$, there exists a sequence $\{\beta_n\}_{n=1}^\infty \subseteq \bigcup_{n=1}^\infty P_{n,j_n}\mathscr{H}$ such that $\beta_n \to \beta$. Note that $\mathscr{D}(V_0)$ is a core for V, we have

$$V\beta_n = V_0\beta_n \to V\beta.$$

Clearly, $W\beta_n \to W\beta$. By Lemma 2.4, we get

$$V_0W\beta_n = (V_0P_{n,j_n})W\beta_n = W(V_0P_{n,j_n})\beta_n = WV_0\beta_n \to WV\beta.$$

On the other hand,

$$V_0W\beta_n = VW\beta_n \to VW\beta.$$

Then $VW\beta = WV\beta$. This proves that if W commutes with V_0, then W commutes with V. Therefore V is affiliated with $\mathscr{L}_\infty''$. $\square$

By the proof of Lemma 2.3, we know that $\operatorname{Ker}(V_0) = \{0\}$. Observe that $V^* = \overline{V_0}^* = V_0^*$, for every $x \in \operatorname{Ker}(V)$,

$$0 =< Vx, y >=< x, V^*y >=< x, V_0^*y >, \quad \forall y \in V_0^*.$$

This implies that $x \perp \operatorname{ran}(V_0^*)$. Since the range of V_0^* is closed densely defined on $\mathscr{H}$, $x = 0$ and hence $\operatorname{Ker}(V) = \{0\}$. Thus V is ono-to-one, the inverse V^{-1} of V exists, and

$$V^{-1} = V_0^{-1} = \left(\prod_{i=1}^\infty V_i\right)^{-1} = \cdots V_m^{-1}V_{m-1}^{-1}\cdots V_1^{-1}.$$

LEMMA 2.6. *For every $A \in Alg(\mathscr{L}_\infty)$, $\|V^{-1}AV\| \leqslant \|A\|$.*

Proof. Let $A \in \operatorname{Alg}(\mathscr{L}_\infty)$. Since for each m, $VP_{m,n_m-1}\mathscr{H} = P_{m,n_m-1}\mathscr{H}$ and $V^{-1}P_{m,n_m-1}\mathscr{H} = P_{m,n_m-1}\mathscr{H}$, we only need to show that for every m, $\|V^{-1}AVP_{m,n_m-1}\| \leqslant \|A\|$.

Assume that

$$A = \begin{pmatrix} A_{1,1}^{(1)} & A_{1,2}^{(1)} & \cdots & A_{1,n_1}^{(1)} \\ 0 & A_{2,2}^{(1)} & \cdots & A_{2,n_1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_1,n_1}^{(1)} \end{pmatrix} \in \operatorname{Alg}(\mathscr{L}_\infty).$$

By Lemma 2.1, we know $\sum_{i=1}^{n} A_{1,i}^{(1)} = \sum_{i=2}^{n} A_{2,i}^{(1)} = \cdots = A_{n_1,n_1}^{(1)}$, and then

$$V^{-1}AV = \cdots V_2^{-1}V_1^{-1}\begin{pmatrix} A_{1,1}^{(1)} & A_{1,2}^{(1)} & \cdots & A_{1,n_1}^{(1)} \\ 0 & A_{2,2}^{(1)} & \cdots & A_{2,n_1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_1,n_1}^{(1)} \end{pmatrix} V_1V_2\cdots$$

$$= \cdots V_2^{-1}\begin{pmatrix} A_{1,1}^{(1)} & A_{1,2}^{(1)} & \cdots & A_{1,n_1-1}^{(1)} & 0 & \cdots & 0 \\ 0 & A_{2,2}^{(1)} & \cdots & A_{2,n_1-1}^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{n_1-1,n_1-1}^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & A_{1,1}^{(2)} & \cdots & A_{1,n_2}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & A_{n_2,n_2}^{(2)} \end{pmatrix} V_2\cdots$$

$$= \cdots = \begin{pmatrix} M_{n_1} & 0 & \cdots & 0 & \cdots \\ 0 & M_{n_2} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & \cdots & M_{n_k} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix},$$

where for $i = 1, 2, \cdots$,

$$M_{n_i} = \begin{pmatrix} A_{1,1}^{(i)} & A_{1,2}^{(i)} & \cdots & A_{1,n_i-1}^{(i)} \\ 0 & A_{2,2}^{(i)} & \cdots & A_{2,n_i-1}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_i-1,n_i-1}^{(i)} \end{pmatrix}.$$

Therefore, we have that $\|V^{-1}AVP_{m,j_m}\| \leqslant \|A\|$, and hence for every $A \in \mathrm{Alg}(\mathscr{L}_\infty)$, $|V^{-1}AV\| \leqslant \|A\|$. $\square$

A family of projections in $\mathscr{N}_m$ was given by (2.2) and (2.3), we now construct a new family of projections in $\mathscr{N}_m$. Let

$$\widetilde{P}_{m,j_m} = P_{m,j_m}, \qquad j_m = 1, \ldots, n_m - 1; \tag{2.4}$$

$$\widetilde{P}_{m,n_m} = I - P_{m,n_m-1}. \tag{2.5}$$

Denote by $\widetilde{\mathscr{L}_m}$ and $\widetilde{\mathscr{L}_\infty} = \cup_m \widetilde{\mathscr{L}_m}$ the lattice generated by $\{\widetilde{P}_{k,j_m} : 1 \leqslant k \leqslant m, 1 \leqslant j_m \leqslant n_m\}$, and $\{\widetilde{P}_{k,j_k} : k \geqslant 1, 1 \leqslant j_k \leqslant n_k\}$, respectively. Then they are commutative subspace lattices (CSL), and hence $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$ is a commutative subspace lattices algebra. Moreover, the following theorem follows directly from the preceding lemma.

THEOREM 2.1. *With $\mathscr{L}_\infty$ and $\widetilde{\mathscr{L}_\infty}$ defined above, there exists an unbounded operator V and a strong operator topology (SOT) dense subalgebra $\mathscr{A}$ of the CSL-algebra $Alg(\widetilde{\mathscr{L}_\infty})$ such that*

$$V^{-1}Alg(\mathscr{L}_\infty)V \cong \mathscr{A} \subset Alg(\widetilde{\mathscr{L}_\infty}).$$

Proof. By the definition of P_{k,j_k} and $\widetilde{P}_{k,j_k}$, we know that $P_{k,j_k} = \widetilde{P}_{k,j_k}$ for all k and $j_k = 1,2,\cdots n_k - 1$. Since $(I - V^{-1}P_{k,j_k}V)A(V^{-1}P_{k,j_k}V) = V^{-1}((I-P_{k,j_k})VAV^{-1}P_{k,j_k})V = 0$, $(I-\widetilde{P}_{k,j_k})A\widetilde{P}_{k,j_k} = 0$ for all $k \geqslant 1$ and $A \in \mathrm{Alg}(\mathscr{L}_\infty)$. Thus

$$\mathrm{Ran}(V^{-1}P_{k,j_k}V) = \mathrm{Ran}(\widetilde{P}_{k,j_k}),$$

and by the proof of Lemma 2.6, we know $V^{-1}\mathrm{Alg}(\mathscr{L}_\infty)V$ is a dense subalgebra in $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$, which implies that there exists a SOT-dense subalgebra $\mathscr{A}$ of the CSL-algebra $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$, satisfying

$$V^{-1}\mathrm{Alg}(\mathscr{L}_\infty)V \cong \mathscr{A}. \quad \square$$

COROLLARY 2.1. *If $T \in Alg(\mathscr{L}_\infty)$ is in the center of $Alg(\mathscr{L}_\infty)$, then*

$$T = V(\lambda_1 P_{1,n_1-1} + \sum_{i=2}^{\infty} \lambda_i (P_{i,n_i-1} - P_{i-1,n_{i-1}-1}))V^{-1},$$

where $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$.

Proof. By Theorem 2.1, we know that $V^{-1}\mathrm{Alg}(\mathscr{L}_\infty)V$ is a SOT-dense subalgebra of the CSL-algebra $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$, then $V^{-1}TV$ is in the center of $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$. Since an element in the center of CSL-algebra $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$ is of the form $\lambda_1 P_{1,n_1-1} + \sum\limits_{i=2}^{\infty} \lambda_i (P_{i,n_i-1} - P_{i-1,n_{i-1}-1})$ for some $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$, we have that

$$T = V(\lambda_1 P_{1,n_1-1} + \sum_{i=2}^{\infty} \lambda_i (P_{i,n_i-1} - P_{i-1,n_{i-1}-1}))V^{-1}. \quad \square$$

Let $T_{n_1}^{(1)} = P_{1,n_1-1}$, $T_{n_m}^{(m)} = P_{m,n_m-1} - P_{m-1,n_{m-1}-1}$, and $W_m = VT_{n_m}^{(m)}V^{-1}$. Then

$$T = V(\sum_{i=1}^{\infty} \lambda_i T_{n_i}^{(i)})V^{-1} = \sum_{i=1}^{\infty} \lambda_i W_i.$$

REMARK 2.2. It is not hard to see that for all $m \neq k \geqslant 1$, $W_m W_k = W_k W_m = 0$. Since $T_{n_m}^{(m)}$ and $T_{n_k}^{(k)}$ are the minimal idempotents in the center of $\mathrm{Alg}(\widetilde{\mathscr{L}_\infty})$, by Corollary 2.1, we know W_m and W_k are the minimal idempotents in the center of $\mathrm{Alg}(\mathscr{L}_\infty)$.

The following result shows that in the sense of isometrical isomorphism, $\mathrm{Alg}(\mathscr{L}_\infty(n_1,n_2,\cdots,n_s,\cdots))$ is unique.

THEOREM 2.2. *If $Alg(\mathscr{L}_\infty(n_1,n_2,\cdots,n_s,\cdots))$ is isometrically isomorphic to $Alg(\mathscr{L}_\infty(m_1,m_2,\cdots,m_s,\cdots))$, then $n_i = m_i$, for all $i = 1,2,\cdots$.*

Proof. Let $W_m' = E_{n_m}^{(m)} - F_m$. By the definition of W_i, we know that $W_1 = W_1' = E_{n_1}^{(1)} - F_1$. Thus

$$\| W_1 \| = \| W_1^* W_1 \|^{\frac{1}{2}} = \sqrt{n_1},$$

and

$$\begin{aligned}\| W_2 \| &= \| W_2^* W_2 \|^{\frac{1}{2}} \\ &= \left\| \begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ (W_2')^* & \cdots & (W_2')^* \end{pmatrix}_{n_1} \begin{pmatrix} 0 & \cdots & W_2' \\ \cdots & \cdots & \cdots \\ 0 & \cdots & W_2' \end{pmatrix}_{n_1} \right\|^{\frac{1}{2}} = \| n_1 (W_2')^* W_2' \| = \sqrt{n_1 n_2}.\end{aligned}$$

Similarly, we can show $\| W_s \| = \sqrt{n_1 n_2 \cdots n_s}$ for each $s \geqslant 2$.

By Corollary 2.1, we know that $\sum_{i=1}^{\infty} \lambda_i W_i^{(n_i)}$ is in the centralizer of $\mathrm{Alg}(\mathscr{L}_\infty(n_1, n_2, \cdots, n_s, \cdots))$, and $\sum_{i=1}^{\infty} \lambda_i W_i^{(m_i)}$ is in the centralizer of $\mathrm{Alg}(\mathscr{L}_\infty(m_1, m_2, \cdots, m_s, \cdots))$. If $\mathrm{Alg}(\mathscr{L}_\infty(n_1, n_2, \cdots, n_s, \cdots))$ is isometrically isomorphic to $\mathrm{Alg}(\mathscr{L}_\infty(m_1, m_2, \cdots, m_s, \cdots))$, then we have $n_i = m_i$ for all i. Otherwise, we may assume that there exists an integer k such that for $1 \leqslant i < k$, $n_i = m_i$, and $n_k \neq m_k$,

$$\| W_k^{(n_k)} \| = \sqrt{n_1 n_2 \cdots n_k} \neq \sqrt{m_1 m_2 \cdots m_k} = \| W_k^{(m_k)} \| .$$

Since $\mathrm{Alg}(\mathscr{L}_\infty(n_1, n_2, \cdots, n_s, \cdots))$ is isometrically isomorphic to $\mathrm{Alg}_\infty(\mathscr{L}(m_1, m_2, \cdots, m_s, \cdots))$, it must be norm preserving. Note that W_{n_i} and W_{m_i} are minimal idempotent elements of $\mathrm{Alg}(\mathscr{L}_\infty(n_1, n_2, \cdots, n_s, \cdots))$ and $\mathrm{Alg}(\mathscr{L}_\infty(m_1, m_2, \cdots, m_s, \cdots))$, they must have the same norm, which is a contradiction and therefore $n_i = m_i$ for all i. $\square$

3. Automorphisms on Alg($\mathscr{L}_\infty$)

Algebraic automorphisms of reflexive operator algebras acting on separable Hilbert spaces have been investigated by many mathematicians. Recall that a automorphism φ on an algebra $\mathscr{A}$ is *inner* if there exists a unitary $u \in \mathscr{A}$ such that $\varphi(A) = u^* A u, \forall A \in \mathscr{A}$. Moreover, if φ is an isometric isomorphism, it follows from Theorem 2.2 that $\mathrm{Alg}(\mathscr{L}_\infty(n_1, n_2, \cdots, n_s, \cdots))$ has only one structure. In this section, we let all $n_i = 2$ in (2.1), and $\mathscr{L}_\infty = \cup_m \mathscr{L}_m$, we will study automorphism on $\mathrm{Alg}(\mathscr{L}_\infty)$.

THEOREM 3.1. *If an automorphism $\varphi : Alg(\mathscr{L}_\infty) \to Alg(\mathscr{L}_\infty)$ is norm preserving, then φ is an inner automorphism.*

Proof. Let $\varphi : \mathrm{Alg}(\mathscr{L}_\infty) \to \mathrm{Alg}(\mathscr{L}_\infty)$ be an automorphism. By Theorem 2.2, we know that W_i's are minimal idempotent elements of $\mathrm{Alg}(\mathscr{L}_\infty)$, then $\varphi(W_i) = W_i$. Particularly, we have $\varphi\left(\begin{pmatrix} I & -I \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} I & -I \\ 0 & 0 \end{pmatrix}$ and $\varphi\left(\begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix}\right) = \begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix}$. Then

$$\begin{aligned}\varphi\left(\begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix} W_1\right) = \varphi\left(\begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix}\right) W_1 \\ &= \begin{pmatrix} \varphi_1(A) & B - \varphi_1(A) \\ 0 & B \end{pmatrix} W_1 = \begin{pmatrix} \varphi_1(A) & -\varphi_1(A) \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Similarly, we obtain

$$\varphi\left(\begin{pmatrix} 0 & A \\ 0 & A \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & A \\ 0 & A \end{pmatrix}\right)\begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & \varphi_2(A) \\ 0 & \varphi_2(A) \end{pmatrix}.$$

Let P be a projection in $B(P_{1,1}\mathscr{H})$. Then

$$\left\|\begin{pmatrix} P & -P \\ 0 & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} P & -P \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ -P & 0 \end{pmatrix}\right\|^{\frac{1}{2}} = \left\|\begin{pmatrix} 2P & 0 \\ 0 & 0 \end{pmatrix}\right\|^{\frac{1}{2}} = \sqrt{2}.$$

Since $\left\|\begin{pmatrix} \varphi_1(P) & -\varphi_1(P) \\ 0 & 0 \end{pmatrix}\right\| = \sqrt{2} \parallel \varphi_1(P) \parallel$, which implies $\parallel \varphi_1(P) \parallel = 1$, therefore $\varphi_1(P)$ is also a projection in $\mathrm{Alg}(\mathscr{L}_\infty)$. Since $\mathrm{Alg}(\mathscr{L}_\infty) \subset \mathscr{B}(\mathscr{H})$, we have φ_1 is an isometric automorphism. Thus, for all $A \in \mathrm{Alg}(\mathscr{L}_\infty)$, $\varphi(A^*) = \varphi(A)^*$. Then for all $A \in \mathrm{Alg}(\mathscr{L}_\infty)$, there exists a unitary operator u_1 such that $\varphi_1(A) = u_1^* A u_1$.

Now we claim that

$$u_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_1, \ \ u_1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} u_1$$

and

$$\varphi_2\left(\begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix}\right) = u_1^* \begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix} u_1.$$

Indeed, since

$$\varphi\left(\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

there exists a unitary u_2 such that

$$\varphi\begin{pmatrix} 0 & 0 & A & -A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A & -A \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & u_2^* A u_2 & -u_2^* A u_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2^* A u_2 & -u_2^* A u_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let P be a projection and $P_1 = \begin{pmatrix} P & -P \\ 0 & 0 \end{pmatrix}$. It's easy to see that

$$\left\|\begin{pmatrix} P_1 & -P_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & P_1 \\ 0 & P_1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} P_1 & 0 \\ 0 & P_1 \end{pmatrix}\right\| = \sqrt{2}.$$

Since φ is norm preserving, we know

$$\sqrt{2} = \left\|\varphi\left(\begin{pmatrix} P_1 & 0 \\ 0 & P_1 \end{pmatrix}\right)\right\| = \left\|\begin{pmatrix} u_1^* P_1 u_1 & -u_1^* P_1 u_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & u_2^* P u_2 & -u_2^* P u_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2^* P u_2 & -u_2^* P u_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\|.$$

Let $Q = u_1^* P_1 u_1 = u_1^* \begin{pmatrix} P & -P \\ 0 & 0 \end{pmatrix} u_1$ and $E = \begin{pmatrix} u_2^* P u_2 & -u_2^* P u_2 \\ 0 & 0 \end{pmatrix}$. Then

$$\varphi\left(\begin{pmatrix} P_1 & 0 \\ 0 & P_1 \end{pmatrix}\right) = \begin{pmatrix} Q & -Q \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & E \\ 0 & E \end{pmatrix} = \begin{pmatrix} Q & E-Q \\ 0 & E \end{pmatrix}.$$

Hence

$$\begin{aligned} 2 = \left\|\begin{pmatrix} Q & E-Q \\ 0 & E \end{pmatrix}\right\|^2 &= \left\|\begin{pmatrix} Q & E-Q \\ 0 & E \end{pmatrix}\begin{pmatrix} Q^* & 0 \\ E^*-Q^* & E^* \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} QQ^*+(E-Q)(E^*-Q^*) & (E-Q)E^* \\ E(E^*-Q^*) & EE^* \end{pmatrix}\right\|. \end{aligned}$$

This implies that $\|QQ^*+(E-Q)(E^*-Q^*)\| \leqslant 2$.

Since $QQ^* = u_1^* \begin{pmatrix} 2P & 0 \\ 0 & 0 \end{pmatrix} u_1$, we have $\|QQ^*\| = 2$. Note that $QQ^* = 2u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1$, we obtain $u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1 E = Q$, that is

$$u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1 \begin{pmatrix} u_2^* P u_2 & -u_2^* P u_2 \\ 0 & 0 \end{pmatrix} = u_1^* \begin{pmatrix} P & -P \\ 0 & 0 \end{pmatrix} u_1. \tag{3.1}$$

It is easy to check that

$$u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1 \begin{pmatrix} u_2^* P u_2 & 0 \\ 0 & 0 \end{pmatrix} u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1 = u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1.$$

Note that $u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1$ and $\begin{pmatrix} u_2^* P u_2 & 0 \\ 0 & 0 \end{pmatrix}$ are the projections in $B(P_{1,1}\mathscr{H})$, then $u_1^* \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1$ is a subprojection of $\begin{pmatrix} u_2^* P u_2 & 0 \\ 0 & 0 \end{pmatrix}$.

Similarly, we have $u_1 \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} u_1^*$ is a subprojection of $\begin{pmatrix} u_2 P u_2^* & 0 \\ 0 & 0 \end{pmatrix}$. Let $u_1 = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$. In particular, u_1^* commute with $\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$, and therefore $a_2 = 0$.

Since

$$\begin{pmatrix} a_1^* P a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_2^* P u_2 & -u_2^* P u_2 \\ 0 & 0 \end{pmatrix} = u_1^* \begin{pmatrix} P & P \\ 0 & 0 \end{pmatrix} u_1 = \begin{pmatrix} a_1^* & 0 \\ 0 & a_3^* \end{pmatrix}\begin{pmatrix} P & P \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_3 \end{pmatrix},$$

$a_1^* P a_1 = a_1^* P a_3$, and $Pa_1 = Pa_3$ for all projection P, therefore we have $a_1 = a_3$. So $u_1 \in (\mathrm{Alg}(\mathscr{L}_\infty))'$.

From (3.1), we know that for every projection P, $a_1^* P a_1 u_2^* P u_2 = a_1^* P a_1$. Multiplying the above equation by a_1 on left and u_2^* on right, we have $Pa_1 u_2^* P = Pa_1 u_2^*$. We also have $Pa_1 u_2^* P = a_1 u_2^* P$ by $(a_1^* P a_1)^* = a_1^* P a_1$. The claim is proved. Then

$$\varphi\left(\begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \begin{pmatrix} u_2^* & 0 \\ 0 & u_2^* \end{pmatrix} A \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix} & -\begin{pmatrix} u_2^* & 0 \\ 0 & u_2^* \end{pmatrix} A \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix} \\ 0 & 0 \end{pmatrix},$$

and

$$\varphi\left(\begin{pmatrix} 0 & B \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} 0 & \begin{pmatrix} u_2^* & 0 \\ 0 & u_2^* \end{pmatrix} B \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix} \\ 0 & \begin{pmatrix} u_2^* & 0 \\ 0 & u_2^* \end{pmatrix} B \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix} \end{pmatrix},$$

where $B = \begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix}$. Similarly, $u_2 = \begin{pmatrix} u_3 & 0 \\ 0 & u_3 \end{pmatrix}$, and $u_3 = \begin{pmatrix} u_4 & 0 \\ 0 & u_4 \end{pmatrix}$, $\cdots$. This implies that $\begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} \in (\mathscr{L}_\infty)'$. Therefore, we get

$$\varphi\left(\begin{pmatrix} A & B-A \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} u_1^* & 0 \\ 0 & u_1^* \end{pmatrix} \begin{pmatrix} A & B-A \\ 0 & B \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix}.$$

Since the commutator of a von Neumann Algebra is self-adjoint, φ is an inner automorphism. $\square$

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