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ON DESCARTES' RULE OF SIGNS FOR MATRIX POLYNOMIALS

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Abstract. We present a generalized Descartes' rule of signs for self-adjoint matrix polynomials whose coefficients are either positive or negative definite, or null. In particular, we conjecture that the number of real positive (negative) eigenvalues of a matrix polynomial is bounded above by the product of the size of the matrix coefficients and the number of definite sign alternations (permanences) between consecutive coefficients. Our main result shows that this generalization holds under the additional assumption that the matrix polynomial is hyperbolic. In addition, we prove individual cases where the matrix polynomial is diagonalizable by congruence, or of degree three or less. The full proof of our conjecture is an open problem; we discuss analytic and algebraic approaches for solving this problem and ultimately, what makes this open problem non-trivial. Finally, we prove generalizations of two famous extensions of Descartes' rule: If all eigenvalues are real then the bounds in Descartes' rule are sharp and the number of real positive and negative eigenvalues have the same parity as the associated bounds in Descartes' rule.

1. Introduction

The theory of matrix polynomials has been strongly influenced by its applications to differential equations and vibrating systems. In fact, vibrating systems motivated the first works devoted primarily to matrix polynomials: one by Frazer, Duncan and Collar in 1938 [10] and the other by Lancaster in 1966 [14]. In addition, the theory of matrix polynomials has been influenced by results from matrix theory and complex analysis. The canonical set of Jordan chains defined in [12] are a natural generalization of a cycle of generalized eigenvectors. A generalized Rouché's theorem is presented in [11] and is then used to prove a generalized Pellet's theorem for matrix polynomials in [17]. The generalized Pellet's theorem has been used to give sharp bounds on the spectrum of unitary matrix polynomials [6] and in the development of approximation methods for the eigenvalues of a matrix polynomial [4, 5, 18].

These generalizations have greatly influenced the study of matrix polynomials and their spectra. It is in this spirit that we present a generalization of Descartes' rule of signs for matrix polynomials. Descartes rule was first published in 1637 [8], but the first widely recognized rigorous proof of the rule was not given until 1742 by de Gua [3]. Since then, many proofs of Descartes' rule and its extensions have been presented [2, 3, 7, 20]. Today, we understand Descartes' rule as follows:

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- A real scalar polynomial has no more positive roots than alternations of signs between two consecutive coefficients.

- A real scalar polynomial has no more negative roots than permanences of signs between two consecutive coefficients.

By a matrix polynomial, we mean a polynomial in a complex variable with matrix coefficients. More specifically, a *matrix polynomial* of degree m and size n is given by

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \qquad A_m \neq 0, \tag{1}$$

where the coefficients $A_i \in \mathbb{C}^{n\times n}$ and λ is a complex variable. We assume that the matrix polynomial is regular, that is, $\det P(\lambda)$ is not identically zero. A finite eigenvalue of the matrix polynomial $P(\lambda)$ is any scalar $\mu \in \mathbb{C}$ such that $\det P(\mu) = 0$. For each eigenvalue $\mu \in \mathbb{C}$, a corresponding eigenvector $v \in \mathbb{C}^n$ is any nonzero vector such that $P(\mu)v = 0$. We note that the matrix polynomial $P(\lambda)$ has an infinite eigenvalue if and only if the leading coefficient A_m is singular. Any further discussion of infinite eigenvalues is beyond the scope of this article and hereafter all eigenvalues are assumed to be finite.

Throughout this article, we assume that the coefficients of $P(\lambda)$ are self-adjoint (i.e., Hermitian) and positive or negative definite, or null. We adapt the convention from [3] for counting alternations and permanences of signs of $P(\lambda)$. If consecutive coefficients A_{i+1} and A_i are both positive or negative definite, then the pair (A_{i+1}, A_i) contributes 1 permanence and 0 alternation of signs. If one is positive definite and the other is negative definite, then the pair (A_{i+1}, A_i) contributes 1 alternation and 0 permanence of signs. For null coefficients, we count as follows:

- For alternations, every coefficient $A_i = 0$ is considered to be the same sign as A_{i+1}. Note that this is the same as ignoring the null coefficients.

- For permanences, the sign of the coefficient $A_i = 0$ is considered to be opposite to the sign of A_{i+1}. This is the same as counting the alternations of $P(-\lambda)$.

Henceforth, we denote by $z^+(P)$ the number of positive eigenvalues of $P(\lambda)$ and by $z^-(P)$ the number of negative eigenvalues. We let $\mathbb{S}$ denote the set of all self-adjoint matrix polynomials whose coefficients are either positive or negative definite, or null. Furthermore, for each $P(\lambda) \in \mathbb{S}$, we use $\alpha(P)$ to denote the number of alternations of signs of $P(\lambda)$ and $\pi(P)$ to denote the number of its permanences of signs.

The outline of this article is as follows: In Section 2, we show that a generalized Descartes' rule holds for hyperbolic and diagonalizable matrix polynomials in $\mathbb{S}$. We note that in this section we take an analytic approach using derivatives to build inductive arguments to obtain our results. Whereas, in Section 3, we take an algebraic approach to show that this generalization holds for all matrix polynomials in $\mathbb{S}$ of degree three or less. We conjecture that this generalization holds for all matrix polynomials in $\mathbb{S}$ and discuss what makes this open problem non-trivial. Finally, in Section 4, we generalize two famous extensions of Descartes' rule for matrix polynomials.

2. An analytic approach

The *numerical range* of a matrix polynomial is the set

$$W(P) = \{\mu \in \mathbb{C} : \; x^* P(\mu) x = 0 \text{ for some } x \in S\}, \tag{2}$$

where S denotes the unit sphere in $\mathbb{C}^n$, that is,

$$S = \{x \in \mathbb{C}^n : \; x^* x = 1\}.$$

It is important to note that $W(P)$ contains all the eigenvalues of $P(\lambda)$. In addition, if $P(\lambda) = \lambda I - A$, then $W(P)$ reduces to the *field of values* of the matrix A [13]. Finally, we call the matrix polynomial $P(\lambda)$ *hyperbolic* if $W(P)$ is a bounded subset of $\mathbb{R}$. Note that if $P(\lambda)$ is hyperbolic then its coefficients must be self-adjoint and the leading coefficient is either positive definite or negative definite [15, Theorem 2.3]. We let $\mathscr{H} \subset \mathbb{S}$ denote the set of all hyperbolic matrix polynomials whose coefficients are either positive or negative definite, or null.

Suppose that $P(\lambda)$ is hyperbolic and denote by $\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_m(x)$ the roots of $x^* P(\lambda) x$, where $x \in S$. The set

$$\Lambda_j = \{\lambda_j(x) : \; x \in S\}$$

is called the jth *spectral zone* of $P(\lambda)$. Note that each spectral zone, Λ_j, is a closed bounded interval (possibly degenerate) $[\alpha_j, \beta_j]$ on the real line. In addition, it follows from [16, Theorem 31.5] that $\beta_j \leqslant \alpha_{j+1}$, for $j = 1, 2, \ldots, m-1$ and from [15, Theorem 3.1] that each spectral zone contains exactly n eigenvalues of $P(\lambda)$.

Now, let $\lambda_1'(x) \leqslant \lambda_2'(x) \leqslant \ldots \leqslant \lambda_{m-1}'(x)$ denote the roots of $x^* P'(\lambda) x$, where $x \in S$. Then, by Rolle's theorem, we have $\lambda_j(x) \leqslant \lambda_j'(x) \leqslant \lambda_{j+1}(x)$, which implies that the spectral zones of the derivative $P'(\lambda)$ satisfy

$$\Lambda_j' = \{\lambda_j'(x) : \; x \in S\} \subseteq [\alpha_j, \beta_{j+1}], \qquad j = 1, 2, \ldots, m-1.$$

LEMMA 1. *Let $P(\lambda) \in \mathscr{H}$. Then,*

$$z^+(P') \geqslant z^+(P) - n. \tag{3}$$

Proof. Let $s^+(P)$ denote the number of spectral zones of $P(\lambda)$ that lie in $\mathbb{R}_+$. By the discussion preceding the theorem statement, it follows that

$$s^+(P') \geqslant s^+(P) - 1. \tag{4}$$

If a spectral zone contains the origin, then it must be the degenerate interval $[0]$ as the coefficient matrices cannot be indefinite. In particular, if $A_0 \neq 0$, then we have $0 \notin W(P)$; otherwise, if $A_0 = 0$ and A_k is the trailing coefficient of $P(\lambda)$, then $F(\lambda) = \lambda^{-k} P(\lambda)$ satisfies $0 \notin W(F)$ and $W(P) = W(F) \cup \{0\}$. Therefore, since each spectral zone contains exactly n eigenvalues and no spectral zone contains both positive and negative eigenvalues, the result follows by multiplying both sides of (4) by n. $\square$

Now, we follow the proof of de Gua for Descartes' rule to prove our main result. More precisely, we use the next lemma to construct a matrix polynomial from $P(\lambda)$ that has exactly one less alternation than $P(\lambda)$, which allows us to establish a proof by induction on the number of alternations of signs.

LEMMA 2. *Let $P(\lambda) \in \mathbb{S}$. Suppose that the consecutive coefficients A_k and A_{k-1} are of opposite sign and define $F(\lambda) = \lambda^{-k}P(\lambda)$. Then,*

$$G(\lambda) = \lambda^{k+1}F'(\lambda)$$

has exactly one less alternation than $P(\lambda)$.

Proof. Note that

$$G(\lambda) = -kP(\lambda) + \lambda P'(\lambda) = (m-k)A_m\lambda^m + \cdots + (k-k)A_k\lambda^k + \cdots + (0-k)A_0.$$

It is clear from the above equation that the counting of alternations from $P(\lambda)$ to $G(\lambda)$ can differ only in the $(k+1)$, k and $(k-1)$ indexed coefficients. Furthermore, when counting alternations in $G(\lambda)$, the null coefficient $(k-k)A_k$ is considered of the same sign as $[(k+1)-k]A_{k+1} = A_{k+1}$. The result follows by considering the cases where A_{k+1} and A_k are of opposite sign and the same sign and noting that in either case $G(\lambda)$ has precisely one less alternation than $P(\lambda)$. $\square$

Several remarks are in order: First, Lemma 2 holds for all matrix polynomials in $\mathbb{S}$, whereas Lemma 1 only holds for matrix polynomials in $\mathscr{H} \subset \mathbb{S}$. Second, given the hypothesis of Lemma 2, A_k cannot be a trailing coefficient of $P(\lambda)$ since A_k and A_{k-1} are of opposite sign. Therefore, the numerical range of $F(\lambda)$ satisfies

$$W(F) = W(P) \setminus \{0\}.$$

Here we are extending the definition in (2) in a natural way to hold for the rational matrix function $F(\lambda)$. It follows that if $P(\lambda)$ is hyperbolic, then so too is $G(\lambda)$. Finally,

$$z^+(F) = z^+(P) \text{ and } z^+(G) = z^+(F'),$$

and it follows from Lemma 1 that if $P(\lambda)$ is hyperbolic, then

$$z^+(G) \geqslant z^+(P) - n. \tag{5}$$

We are now ready to prove our main result.

THEOREM 3. *Let $P(\lambda) \in \mathscr{H}$. Then,*

$$z^+(P) \leqslant n \cdot \alpha(P) \text{ and } z^-(P) \leqslant n \cdot \pi(P). \tag{6}$$

Proof. We prove the first assertion and note that the second assertion follows by changing $P(\lambda)$ into $P(-\lambda)$.

The following is a proof by induction on the number of alternations of signs. If $\alpha(P)=0$, then $x^*P(\lambda)x$ is a real scalar polynomial with zero alternations of signs, for all $x \in S$. Therefore, by Descartes' rule, it follows that $W(P)\cap\mathbb{R}_+=\emptyset$. Thus, $z^+(P)=0$ and (6) holds.

Suppose, then, that the result holds for all matrix polynomials in $\mathscr{H}$ with $k \geqslant 0$ alternations. Let $P(\lambda)\in\mathscr{H}$ such that $\alpha(P)=k+1$. Applying Lemma 2, we construct $G(\lambda)\in\mathscr{H}$ such that $\alpha(G)=k$. Hence, by the induction hypothesis, $z^+(G)\leqslant n\cdot\alpha(G)$. Furthermore, by (5), we have

$$z^+(P)\leqslant z^+(G)+n\leqslant n\cdot\alpha(G)+n=n(k+1)=n\cdot\alpha(P).$$

Therefore, by the principle of mathematical induction, (6) holds for all $P(\lambda)\in\mathscr{H}$. □

Informally, we note that the proofs of Descartes' rule for scalar polynomials have fallen into two categories: analytic and algebraic. Typically, analytic proofs rely on the geometry of curves and extrema to obtain information, or derivatives to build inductive arguments. In contrast, algebraic proofs rely on the factorization of the polynomial and the properties of an ordered field [3]. For this reason, we say that our proof of Theorem 3 is analytic in nature.

We are interested in extending the set of matrix polynomials for which our main result (6) holds. To that end, we note that (6) holds for all $P(\lambda)\in\mathbb{S}$ such that $\alpha(P)=0$. Furthermore, given any $P(\lambda)\in\mathbb{S}$ with $\alpha(P)=k+1$ we can use Lemma 2 to form a $G(\lambda)\in\mathbb{S}$ such that $\alpha(G)=k$. It follows from the proof of Theorem 3 that if we can extend the set of matrix polynomials for which (3) holds, then we can extend the set of matrix polynomials for which (6) holds. For instance, (3) clearly holds for all scalar polynomials in $\mathbb{S}$, due to Rolle's theorem. Therefore, (6) holds for all real scalar polynomials and it follows that Descartes' rule is a special case of a much broader theory. Similarly, the following proposition implies that (6) holds for all $P(\lambda)\in\mathbb{S}$ that are diagonalizable by congruence.

PROPOSITION 4. *If $P(\lambda)\in\mathbb{S}$ is diagonalizable by congruence then* (3) *holds.*

Proof. If $P(\lambda)$ is diagonalizable by congruence, then there exists a nonsingular $M\in\mathbb{C}^{n\times n}$ such that $MP(\lambda)M^*=D(\lambda)$, where $D(\lambda)$ is a diagonal matrix polynomial with real coefficients. Note that $D'(\lambda)=MP'(\lambda)M^*$.

It follows that the eigenvalues of $P(\lambda)$ are the roots of the diagonal entries of $D(\lambda)$, denoted by $d_{ii}(\lambda)$, and the eigenvalues of $P'(\lambda)$ are the roots of the diagonal entries of $D'(\lambda)$, denoted by $d'_{ii}(\lambda)$, for $i=1,2,\dots,n$. Moreover, as a consequence of Rolle's theorem, between any two distinct real positive roots of $d_{ii}(\lambda)$ there is a real positive root of $d'_{ii}(\lambda)$.

Finally, for $i=1,2,\dots,n$, let r_i and r'_i denote the number of real positive roots of $d_{ii}(\lambda)$ and $d'_{ii}(\lambda)$, respectively. It follows that $r'_i\geqslant r_i-1$, for $i=1,2,\dots,n$. Hence, $z^+(D')\geqslant z^+(D)-n$ and, therefore, $z^+(P')\geqslant z^+(P)-n$. □

We conjecture that (6), in fact, holds for all matrix polynomials in $\mathbb{S}$. However, as the following example illustrates, (3) does not hold for all matrix polynomials in $\mathbb{S}$. For this reason, we must consider other methods for extending the set of matrix polynomials for which (6) holds.

EXAMPLE 5. Consider the matrix polynomial:

$$P(\lambda) = \begin{bmatrix} \lambda^5 - 6\lambda^4 + 15\lambda^3 - 20\lambda^2 + 14\lambda - 4 & \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 \\ \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 & \lambda^5 - 7\lambda^4 + 19\lambda^3 - 25\lambda^2 + 16\lambda - 4 \end{bmatrix}.$$

Note that $P(\lambda) \in \mathbb{S}$ and $\alpha(P) = 5$. Hence, (6) holds trivially (see Lemma 6). However, $P(\lambda)$ has 8 real positive eigenvalues and $P'(\lambda)$ has only 4 real positive eigenvalues. Therefore, (3) does not hold.

3. An algebraic approach

The concluding remarks made in Section 2 serve as the impetus of this section, where we prove that (6) holds for all matrix polynomials in $\mathbb{S}$ of degree three or less. We begin with a lemma that establishes the result for a specific number of alternations.

LEMMA 6. *Let $P(\lambda) \in \mathbb{S}$. Then, $z^+(P) \leqslant n \cdot \alpha(P)$ provided that $\alpha(P) \in \{0, 1, m\}$.*

Proof. Note that the case $\alpha(P) = m$ is trivial since the total number of eigenvalues of $P(\lambda)$ is equal to mn and the case $\alpha(P) = 0$ was already covered in the proof of Theorem 3 and did not rely on the matrix polynomial being hyperbolic.

Therefore, we only need to consider the case where $\alpha(P) = 1$. In this case, for each $x \in S$, the real scalar polynomial $x^*P(\lambda)x$ has exactly one alternation and, by Descartes' rule, precisely one positive root. We may assume that the leading coefficient of $P(\lambda)$ is positive definite. Otherwise, the matrix polynomial $-P(\lambda)$ has a positive definite leading coefficient and the same number of alternations and positive eigenvalues as $P(\lambda)$. Therefore, there exists $a, b \in \mathbb{R}_+$ such that $a < b$, $P(a)$ is negative definite, $P(b)$ is positive definite and for each $x \in S$, $x^*P(\lambda)x$ has exactly one root in (a,b). It follows from [16, Theorem 30.6] that $P(\lambda)$ admits the factorization

$$P(\lambda) = P_+(\lambda)(\lambda I - Z),$$

where $P_+(\lambda)$ is invertible for all $\lambda \in (a,b)$ and all eigenvalues of Z are contained in (a,b). Thus, $P(\lambda)$ has exactly n positive eigenvalues and the result follows. □

The fact that (6) holds for linear and quadratic matrix polynomials in $\mathbb{S}$ follows readily from the previous lemma. With a little more work, we also prove the cubic case.

THEOREM 7. *Let $P(\lambda) \in \mathbb{S}$. Then,* (6) *holds provided that $P(\lambda)$ has degree $m \in \{1, 2, 3\}$.*

Proof. For degree $m = 1$, or $m = 2$, the only cases to consider are covered by Lemma 6.

If $m = 3$, then in addition to appealing to Lemma 6, we must also show that the result holds for $\alpha(P) = 2$ and $\pi(P) = 2$, where the latter result follows from the first by changing $P(\lambda)$ into $P(-\lambda)$.

Suppose that $\alpha(P) = 2$ and for the sake of contradiction assume that the purely imaginary number it, where $t \in \mathbb{R} \setminus \{0\}$, lies in $W(P)$. Then, there exists a vector $x \in S$ such that

$$-i(x^*A_3x)t^3 - (x^*A_2x)t^2 + i(x^*A_1x)t + (x^*A_0x) = 0.$$

Note that if $A_2 = 0$, then in order to satisfy the above equation it follows that $A_0 = 0$ and, therefore, there is at most 1 alternation, which contradicts $\alpha(P) = 2$. Hence, we may assume that $A_2 \neq 0$ and it follows that

$$\frac{x^*A_0x}{x^*A_2x} = \frac{x^*A_1x}{x^*A_3x} = t^2 > 0.$$

However, this implies that there are 0 or 3 alternations, both of which contradict $\alpha(P) = 2$. Therefore, $W(P)$ does not intersect the imaginary axis, except, possibly, at the origin. That being said, if $A_0 = 0$, then the matrix polynomial $Q(\lambda)$ obtained by dividing $P(\lambda)$ by λ is quadratic with 2 alternations and the result follows from Lemma 6. Therefore, we may assume that $P(\lambda)$ does not have a zero coefficient after its trailing coefficient and $W(P)$ does not intersect the imaginary axis anywhere.

Thus, $W(P)$ consists of two parts: Ω_L in the open left half-plane of $\mathbb{C}$ and Ω_R in the open right half-plane. As a real cubic polynomial with 2 alternations $x^*P(\lambda)x$ has at least one root in Ω_L, but Ω_R may be empty. If Ω_R is empty, then $z^+(P) = 0$ and the result follows.

Suppose, then, that Ω_R is non-empty. By the continuity of the roots of $x^*P(\lambda)x$ with respect to $x \in S$, it follows that the number of roots of the polynomials $x^*P(\lambda)x$ in Ω_R is constant and equal to 2. Therefore, we may construct a simple closed rectifiable curve Γ in the open left half-plane such that for each $x \in S$, $x^*P(\lambda)x$ has no roots on the boundary of Γ and exactly 1 root inside Γ. By [16, Theorem 26.19], it follows that $P(\lambda)$ can be factored in the form

$$P(\lambda) = P_L(\lambda)P_R(\lambda),$$

where $P_L(\lambda)$ has n eigenvalues in Ω_L and $P_R(\lambda)$ has $2n$ eigenvalues in Ω_R. Therefore, $z^+(P) \leqslant n \cdot \alpha(P)$ when $\alpha(P) = 2$ and the result follows. □

The proof of Theorem 7 is algebraic in nature since it relies on the factorization of the matrix polynomial. In this fashion, we could attempt to extend the set of matrix polynomials for which (6) holds one degree at a time. For all possible degrees, this will require an inductive strategy and some analogue of Segner's lemma [3]:

If a scalar polynomial is multiplied by $(\lambda - \gamma)$, *with* $\gamma > 0$, *then its number of alternations of signs increases by at least* 1. *If it is multiplied by* $(\lambda + \gamma)$, *then its number of permanences of signs increases by at least* 1.

Since the coefficients of the matrix polynomials in $\mathbb{S}$ do not form a field we have little hope in applying this strategy.

In summary, both analytic and algebraic strategies for proving Descartes' rule have serious complications in the generalization for matrix polynomials. The strategy for proving Theorem 3 relies on Lemma 1, which does not hold for all matrix polynomials in $\mathbb{S}$ (see Example 5). Furthermore, there is no inductive strategy for working with the factorization results employed in this section. For this reason, we consider the extension of the set of matrix polynomials for which (6) holds to be a non-trivial open problem. Furthermore, in order to significantly extend this set (recall that our conjecture is (6) holds for all matrix polynomials in $\mathbb{S}$), we will likely need to find a completely different approach.

4. Extensions of Descartes' rule

There are two famous extensions of Descartes' rule that, throughout history, have become synonymous with the original rule. The first extension is that if all roots of a real scalar polynomial are real, then the bounds of Descartes' rule are sharp. This statement is an easy to prove corollary of Descartes' rule. In addition, the converse of this statement holds for polynomials that have no null coefficients, except, possibly, after their trailing coefficient.

The second extension states that the parities of the number of positive roots and alternations of signs are equal and the parities of the number of negative roots and the permanence of signs are equal. We refer to this extension as *Fourier's rule* since it was noted and proven in the 1820 dissertation of Joseph Fourier [3]. It is worth noting that Fourier's rule holds independently of Descartes' rule.

In this section, we prove generalizations of both extensions for matrix polynomials in $\mathbb{S}$. When proving both results, we assume, without loss of generality, that $P(\lambda)$ has no null coefficients after its trailing coefficient. Otherwise, the matrix polynomial $F(\lambda)$ obtained by dividing $P(\lambda)$ by a suitable power λ^k has no null coefficients after its trailing coefficient and satisfies

$$z^+(P) = z^+(F),\ z^-(P) = z^-(F),\ \alpha(P) = \alpha(F) \text{ and } \pi(P) = \pi(F).$$

THEOREM 8. *Let $P(\lambda) \in \mathbb{S}$. If* (6) *holds and all eigenvalues of $P(\lambda)$ are real, then*

$$z^+(P) = n \cdot \alpha(P) \text{ and } z^-(P) = n \cdot \pi(P),$$

i.e., the bounds in (6) *are sharp.*

Proof. Note that, for $i = m-1, m-2, \ldots, 0$, each pair of consecutive coefficients (A_{i+1}, A_i) can contribute at most one alternation or permanence. Therefore, we have $\alpha(P) + \pi(P) \leqslant m$. For the sake of contradiction, suppose that all eigenvalues of $P(\lambda)$ are real and $z^+(P) < n \cdot \alpha(P)$ or $z^-(P) < n \cdot \pi(P)$. Since $P(\lambda)$ has no null coefficients after its trailing coefficient, it follows that

$$nm = z^+(P) + z^-(P) < n(\alpha(P) + \pi(P)),$$

which implies that $\alpha(P) + \pi(P) > m$ and contradicts what we know to be true. $\square$

The converse of Theorem 8 holds for all matrix polynomials in $\mathbb{S}$ that have no null coefficients, except, possibly, after their trailing coefficient. Indeed, let $P(\lambda) \in \mathbb{S}$ and let k denote the index of the last null coefficient after the trailing coefficient, where

$k = -1$ if $P(\lambda)$ has no null coefficient. Then, $\alpha(P) + \pi(P) = m - (k+1)$ and assuming the bounds in (6) are sharp we have $z^+(P) + z^-(P) = n(m - (k+1))$. Since $P(\lambda)$ has $n(k+1)$ zero eigenvalues, it follows that all nm eigenvalues of $P(\lambda)$ are real.

THEOREM 9. *Let $P(\lambda) \in \mathbb{S}$. Then $n \cdot \alpha(P)$ and $z^+(P)$ are of the same parity, as are $n \cdot \pi(P)$ and $z^-(P)$.*

Proof. We prove the first assertion and note that the second assertion follows by changing $P(\lambda)$ into $P(-\lambda)$. Recall that we assume, without loss of generality, that $P(\lambda)$ has no null coefficients after its trailing coefficient.

Let $p(\lambda) = \det P(\lambda)$. Then $p(\lambda)$ is a real scalar polynomial, which factors as $p(\lambda) = cp_1(\lambda)p_2(\lambda)$, where $p_1(\lambda)$ is monic polynomial with no positive roots and $p_2(\lambda)$ is a monic polynomial with only positive roots. Clearly the degree of $p_2(\lambda)$ is equal to $z^+(P)$ and $c = \det A_m$. Since the complex roots of a real scalar polynomial come in conjugate pairs, it follows that $p_1(0) > 0$. In addition, note that $p(0) = \det A_0$ and $p_2(0) > 0$ if and only if $z^+(P)$ is even. It follows that the signs of $\det A_0$ and $\det A_m$ are equal if and only if $z^+(P)$ is even.

Since a sequence of signs contains an even number of alternations if and only if its extremities are equal, it follows that $\alpha(P)$ is even if and only if both matrices A_0 and A_m are of the same sign. If $\alpha(P)$ is odd, then the signs of $\det A_0$ and $\det A_m$ are equal if and only if n is even. Thus, $z^+(P)$ is even if and only if $n \cdot \alpha(P)$ is even and the result follows. □

5. Conclusion

The set $\mathbb{S}$ of self-adjoint matrix polynomials whose coefficients are positive or negative definite, or null, provides a framework for the generalization of Descartes' rule of signs and its extensions. In Section 2, we prove our generalized Descartes' rule, which is stated formally in (6), holds for all hyperbolic and diagonalizable matrix polynomials in $\mathbb{S}$. Then, in Section 3, we prove that (6) holds for all matrix polynomials in $\mathbb{S}$ of degree three or less. Also, two generalized extensions of Descartes' rule are presented in Theorem 8 and Theorem 9.

We conjecture that (6) holds for all matrix polynomials in $\mathbb{S}$ and, in Sections 2 and 3, we discuss the difficulties surrounding a complete solution to this problem. In particular, we conclude that a significant extension of the set of matrix polynomials for which (6) holds will require a completely different approach.

A natural first step is to attempt to generalize Lemma 1 to quasi-hyperbolic matrix polynomials [1]. However, even this extension appears to be non-trivial. For instance, even if $P(\lambda)$ is strictly isospectral to a diagonal matrix polynomial $D(\lambda)$, we are not aware of a relationship between $P'(\lambda)$ and $D'(\lambda)$, nor are we aware of a relationship between the definiteness of $P(\lambda)$ and the definiteness of $D(\lambda)$. Moreover, much of the current results for quasi-hyperbolic matrix polynomials deal with the positive or negative type of the eigenvalues and not their sign.

That being said, this is one of many possible directions to consider for extending the set of matrix polynomials in $\mathbb{S}$ for which (6) holds. We note that the set $\mathbb{S}$ consists of matrix polynomials that arise naturally in the study of differential equations and vibrating systems [9, 10, 12, 14, 19]. Hence, future investigation of our generalized Descartes' rule and its extensions, such as those noted in [7], may lead to additional interesting results and useful applications.

REFERENCES

[1] AL-AMMARI, M. AND TISSEUR, F., *Hermitian matrix polynomials with real eigenvalues of definite type. Part I: Classification*, Linear Algebra Appl. 436, 10 (2012), 3954–3973.

[2] ANDERSON, B., JACKSON, J. AND SITHARAM, M., *Descartes' rule of signs revisited*, Amer. Math. Monthly 105, 5 (1998), 447–451.

[3] BENSIMHOUN, M., *Historical account and ultra-simple proofs of Descartes' rule of signs, De Gua, Fourier and Budan's rules*, arXiv:1309.6664v5 [math.HO], 2016.

[4] BINI, D. A., NOFERINI, V. AND SHARIFY, M., *Locating the eigenvalues of matrix polynomials*, SIAM J. Matrix Anal. Appl. 34, 4 (2013), 1708–1727.

[5] BINI, D. A. AND NOFERINI, V., *Solving polynomial eigenvalue problems by means of the Ehrlich-Aberth method*, Linear Algebra Appl. 439, 4 (2013), 1130–1149.

[6] CAMERON, T. R., *Spectral bounds for matrix polynomials with unitary coefficients*, Electron. J. Linear Algebra 30, (2015), 585–591.

[7] CURTISS, D. R., *Recent extensions of Descartes' rule of signs*, Ann. of Math., (1918), 251–278.

[8] DESCARTES, R., *La géometrie (Discours de la méthode, third part)*, Ed. of Leyde, 1637.

[9] DUFFIN, R. J., *A minmax theory for overdamped networks*, J. Rat. Mech. Anal. 4, (1955), 221–233.

[10] FRAZER, R. A., DUNCAN, W. J. AND COLLAR, A. R., *Elementary matrices, 2nd ed.*, Cambridge Univ. Press, London and New York, 1955.

[11] GOHBERG, I., GOLDBERG, S. AND KAASHOEK, M., *Classes of Linear Operators, vol. I*, Birkhäuser Verlag, Basel, Switzerland, 1990.

[12] GOHBERG, I., LANCASTER, P. AND RODMAN, L., *Matrix Polynomials*, SIAM, Philadelphia, PA, 2009.

[13] HORN, R. A. AND JOHNSON, C. R., *Topics in Matrix Analysis*, Cambridge University Press, New York, NY, 1991.

[14] LANCASTER, P., *Lambda-matrices and Vibrating Systems*, Pergamon Press, Oxford, United Kingdom, 1966.

[15] LI, C.-K. AND RODMAN, L., *Numerical range of matrix polynomials*, SIAM J. Matrix Anal. Appl. 15, 4 (1994), 1256–1265.

[16] MARKUS, A. S., *Introduction to the Spectral Theory of Polynomial Operator Pencils*, AMS, Providence, RI, 1988.

[17] MELMAN, A., *Generalization and variations of Pellet's theorem for matrix polynomials*, Linear Algebra Appl. 439, 5 (2013), 1550–1567.

[18] NOFERINI, V., SHARIFY, M. AND TISSEUR, F., *Tropical roots as approximations to eigenvalues of matrix polynomials*, SIAM J. Matrix Anal. Appl. 36, 1 (2015), 138–157.

[19] TISSEUR, F. AND MEERBERGEN, K., *The quadratic eigenvalue problem*, SIAM Review 43, 2 (2001), 235–286.

[20] WANG, X. *A simple proof of Descartes' rule of signs*, Amer. Math. Monthly 111, 6 (2004), 525–526.

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