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MAPS PRESERVING THE PERIPHERAL LOCAL SPECTRUM OF SOME PRODUCT OF OPERATORS

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Abstract. Let $\mathscr{H}$ and $\mathscr{K}$ be two infinite-dimensional complex Hilbert spaces. Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If T is an operator in $\mathscr{B}(\mathscr{H})$ and x a vector in $\mathscr{H}$ then $\gamma_T(x)$ denotes the peripheral local spectrum of T at x. In this paper we characterize all surjective maps φ from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{K})$ satisfying

$$\gamma_{(\mu ST^*S+\nu T^*S)}(h_0)=\gamma_{(\mu\varphi(S)\varphi(T)^*\varphi(S)+\nu\varphi(T)^*\varphi(S))}(k_0),\ \ (S,\,T\in\mathscr{B}(\mathscr{H})),$$

for a given couple of complex scalars $(\mu,\nu)\neq(0,0)$ and nonzero vectors $h_0\in\mathscr{H}$ and $k_0\in\mathscr{K}$. This result provides a complete description of all surjective maps from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{K})$ preserving the peripheral local spectrum of the skew double product "T^*S" and the skew triple product "TS^*T" of operators. It also unifies and extends several known results on local spectrum preservers.

1. Introduction

The study of linear and nonlinear local spectra preserver problems has attracted the attention of a number of authors. Mainly, several authors have described maps on matrices or operators that preserve local spectrum, local spectral radius, and inner local spectral radius, see for instance [14, 15, 17, 21]. In [11, 12], nonlinear surjective maps on Banach space operators preserving the local spectrum of the product and the triple product of operators have been investigated. In [5, 6], maps preserving the local spectrum of the product and the triple product of matrices have been characterized. In [9], maps on $\mathscr{M}_n(\mathbb{C})$, the algebra of all $n\times n$ complex matrices, preserving the local spectrum of Jordan product of matrices have been described. In [1], maps preserving the local spectrum of skew triple and double product of operators are described. Many recent results on this research area can be found on [13].

Recently, A. Bourhim, T. Jari and J. Mashreghi described in [8] surjective maps on $\mathscr{B}(X)$, the algebra of all bounded operators on a complex Banach space X, preserving the peripheral local spectrum at a nonzero fixed vector of double and triple product of operators. In [2] maps on $\mathscr{M}_n(\mathbb{C})$ preserving the local spectrum of the matrix product $\mu AB^*A+\nu B^*A$ were characterized. In this paper, we provide an infinite dimensional

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variant of [2] with the refinement of using the peripheral spectrum instead of local spectrum. Our aim in this paper is to characterize surjective maps ϕ on $\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on a complex Hilbert space $\mathscr{H}$, preserving the peripheral local spectrum at a nonzero fixed vector of a specific product of operators. This provides, in particular, a complete description of all surjective maps ϕ from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{K})$ preserving the peripheral spectrum of the skew double product "TS^*" and the skew triple product "TS^*T" of operators. This is a new result that extends the main results of [1].

2. Main result

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are two infinite-dimensional complex Hilbert spaces. As usual $\mathscr{B}(\mathscr{H},\mathscr{K})$ denotes the space of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$. When $\mathscr{H}=\mathscr{K}$ we simply write $\mathscr{B}(\mathscr{H})$ instead of $\mathscr{B}(\mathscr{H},\mathscr{H})$. The inner product of $\mathscr{H}$ or $\mathscr{K}$ will be denoted by $\langle\ ,\ \rangle$ if there is no confusion. Let $\mathscr{F}(\mathscr{H})$ denote the ideal of all finite rank operators on $\mathscr{H}$. For a positive integer n, let $\mathscr{F}_n(\mathscr{H})$ be the set of all operators of $\mathscr{B}(\mathscr{H})$ of rank at most n. For an operator $T\in\mathscr{B}(\mathscr{H})$, let T^* denote as usual its adjoint. The *local resolvent* set, $\rho_T(x)$, of an operator $T\in\mathscr{B}(\mathscr{H})$ at a point $x\in\mathscr{H}$ is the union of all open subsets U of the complex plane $\mathbb{C}$ for which there is an analytic function $\phi: U\to\mathscr{H}$ such that $(T-\lambda)\phi(\lambda)=x,\ \ (\lambda\in U)$. Clearly $\rho_T(x)$ contains the resolvent set $\rho(T)$ of T, but this containment could be proper. The *local spectrum* of T at x is defined by

$$\sigma_T(x):=\mathbb{C}\backslash\rho_T(x),$$

and thus it is a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. In fact, $\sigma_T(x)\neq\emptyset$ for all nonzero vectors x in $\mathscr{H}$ precisely when T has the single-valued extension property (SVEP). Recall that T is said to have SVEP provided that for every open subset U of $\mathbb{C}$, the equation $(T-\lambda)\phi(\lambda)=0,\ \ (\lambda\in U)$, has no nontrivial analytic solution ϕ. Every operator $T\in\mathscr{F}(\mathscr{H})$ enjoys this property. The local spectral radius of T at x is defined by

$$r_T(x):=\limsup_{n\to+\infty}\|T^n(x)\|^{\frac{1}{n}}.$$

The set

$$\gamma_T(x):=\{\lambda\in\sigma_T(x):|\lambda|=r_T(x)\}$$

is called the peripheral local spectrum of T at x. Note that $\gamma_T(x)=\emptyset$ provided that $\max\{|\lambda|:\lambda\in\sigma_T(x)\}<r_T(x)$. The books [3] by P. Aiena and [26] by K. B. Laursen, M. M. Neumann provide an excellent exposition as well as a rich bibliography of the local spectral theory.

For two scalars μ and ν for which $(\mu,\nu)\neq(0,0)$, define a map θ from $\mathscr{B}(\mathscr{H})\times\mathscr{B}(\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$ by

$$\theta(S,T):=\mu STS+\nu TS,\ (S,\ T\in\mathscr{B}(\mathscr{H})).$$

The following theorem is our main result, It describes all surjective maps on $\mathscr{B}(\mathscr{H})$ preserving peripheral local spectrum of $\theta(S,T)$ at a nonzero fixed vector $h_0 \in \mathscr{H}$. Its proof is given in section 5 and uses some ideas influenced by arguments quoted from [1, 11, 12]. It also uses new results and lemmas presented in section 4.

THEOREM 2.1. *Let $h_0 \in \mathscr{H}$ and $k_0 \in \mathscr{K}$ be two fixed nonzero vectors. A map φ from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{K})$ satisfies*

$$\gamma_{\theta(S,T^*)}(h_0) = \gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0), \quad (T,\, S \in \mathscr{B}(\mathscr{H})), \tag{2.1}$$

if and only if there exist two unitary operators U and V in $\mathscr{B}(\mathscr{H},\mathscr{K})$ and a nonzero scalars α and β such that for every $T \in \mathscr{B}(\mathscr{H})$,

$$\varphi(T) = UTV^* \quad \textit{and} \quad Vh_0 = \beta k_0 \quad \textit{if } \mu = 0, \tag{2.2}$$

and

$$\varphi(T) = UTU^* \quad \textit{and} \quad Uh_0 = \alpha\ k_0 \quad \textit{if } \mu \neq 0. \tag{2.3}$$

Note that if $\mathscr{H}$ and $\mathscr{K}$ are isomorphic, then they are isomotrically isomorphic. Thus the statments of our theorem can be reduced to the case when $\mathscr{H} = \mathscr{K}$ and $h_0 = k_0$. But the fact that "$\mathscr{H}$ and $\mathscr{K}$ are isomorphic " is one of the conclusions of the main result.

3. Preliminaries

In this section, we fix some notions and exhibit some tools on the local spectral theory and some essential results needed for the proof of our main result. The first lemma summarizes some basic properties of the local spectrum that will be used in the sequel.

LEMMA 3.1. *For an invertible operator $A \in \mathscr{B}(\mathscr{H})$, a vector $x \in \mathscr{H}$ and a nonzero scalar $\alpha \in \mathbb{C}$, the following statements hold.*

(a) $\sigma_T(\alpha x) = \sigma_T(x)$ *and* $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$ *for all* $T \in \mathscr{B}(\mathscr{H})$.

(b) *If T has the SVEP, $x \neq 0$ and $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) = \{\lambda\}$.*

(c) $\sigma_T(x+y) \subset \sigma_T(x) \cup \sigma_T(y)$. *The equality holds if* $\sigma_T(x) \cap \sigma_T(y) = \emptyset$.

(d) $\sigma_{ATA^{-1}}(Ax) = \sigma_T(x)$ *for all* $T \in \mathscr{B}(\mathscr{H})$.

(e) $\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ *for all* $x \in \mathscr{H}$ *and* $n \geqslant 1$.

In [11, 12], the authors gave some essential lemmas and theorems which are useful tools to establish our main results. For our purpose, we state these results only in the case of Hilbert spaces. Let x and y be two nonzero vectors in $\mathscr{H}$, the rank one operator

$x \otimes y$ is defined by $(x \otimes y)z = \langle z, y \rangle x$, for all $z \in \mathscr{H}$. The peripheral local spectrum of an operator of rank one at a vector $h_0 \in \mathscr{H}$ is given by

$$\gamma_{x \otimes y}(h_0) = \begin{cases} \{0\} & \text{if } \langle h_0, y \rangle = 0 \\ \\ \{\langle x, y \rangle\} & \text{if } \langle h_0, y \rangle \neq 0, \end{cases} \tag{3.1}$$

LEMMA 3.2. *Let h_0 be a nonzero vector in $\mathscr{H}$. For every rank one operator $R \in \mathscr{B}(\mathscr{H})$, we have*

$$\gamma_{\theta(R,(T+S))}(h_0) = \gamma_{\theta(R,T)}(h_0) + \gamma_{\theta(R,S)}(h_0),$$

for all $T, S \in \mathscr{B}(\mathscr{H})$.

Proof. Let $R \in \mathscr{B}(\mathscr{H})$ be a rank one operator and write $R = x \otimes y$. For every $T, S \in \mathscr{B}(\mathscr{H})$ we have

$$\theta(x \otimes y, T) = [\mu \langle Tx, y \rangle x + \nu Tx] \otimes y,$$

$$\theta(x \otimes y, S) = [\mu \langle Sx, y \rangle x + \nu Sx] \otimes y,$$

and

$$\theta(x \otimes y, T+S) = [\mu \langle (T+S)x, y \rangle x + \nu (T+S)x] \otimes y.$$

Therefore, if $\langle h_0, y \rangle = 0$ then we have

$$\begin{aligned} \gamma_{\theta(R,T+S)}(h_0) &= \{0\} \\ \gamma_{\theta(R,T)}(h_0) &= \{0\} \\ \gamma_{\theta(R,S)}(h_0) &= \{0\}. \end{aligned}$$

Hence,

$$\gamma_{\theta(R,T+S)}(h_0) = \gamma_{\theta(R,T)}(h_0) + \gamma_{\theta(R,S)}(h_0)$$

Now, if $\langle h_0, y \rangle \neq 0$, then

$$\begin{aligned} \gamma_{\theta(R,T+S)}(h_0) &= \{\langle (T+S)x, y \rangle [\mu \langle x, y \rangle + \nu]\} \\ \gamma_{\theta(R,T)}(h_0) &= \{\langle Tx, y \rangle [\mu \langle x, y \rangle + \nu]\} \\ \gamma_{\theta(R,S)}(h_0) &= \{\langle Sx, y \rangle [\mu \langle x, y \rangle + \nu]\}. \end{aligned}$$

Again we get

$$\gamma_{\theta(R,T+S)}(h_0) = \gamma_{\theta(R,T)}(h_0) + \gamma_{\theta(R,S)}(h_0).$$

The proof is therefore complete. $\square$

4. Auxilary results

In this section we first etablish a local spectral idendity principle that provides necessary and sufficient conditions for two operators to be equal in term of the peripheral local spectrum of $\theta(S,T)$.

THEOREM 4.1. *For a nonzero vector h_0 in $\mathscr{H}$ and two operators A and B in $\mathscr{B}(\mathscr{H})$, the following statements are equivalent.*

(a) $A = B$.

(b) $\gamma_{\theta(S,A)}(h_0) = \gamma_{\theta(S,B)}(h_0)$ *for all* $S \in \mathscr{B}(\mathscr{H})$.

(c) $\gamma_{\theta(R,A)}(h_0) = \gamma_{\theta(R,B)}(h_0)$ *for all* $R \in \mathscr{F}_1(\mathscr{H})$.

Proof. We only need to prove that the implication $(c) \Longrightarrow (a)$ holds. So assume that

$$\gamma_{\theta(R,A)}(h_0) = \gamma_{\theta(R,B)}(h_0) \tag{4.1}$$

for all $R \in \mathscr{F}_1(\mathscr{H})$, and fix a nonzero vector $x \in \mathscr{H}$. If $\langle h_0, x\rangle \neq 0$, then (3.1) implies that

$$\{\langle Ax,x\rangle[\mu\|x\|^2+\nu]\} = \gamma_{\theta(R,A)}(h_0) = \gamma_{\theta(R,B)}(h_0) = \{\langle Bx,x\rangle[\mu\|x\|^2+\nu]\}$$

If necessary, replace x by tx for wich $t^2\mu\|x\|^2+\nu \neq 0$ to deduce that

$$\langle Ax,x\rangle = \langle Bx,x\rangle.$$

If however, $\langle h_0,x\rangle = 0$, then $\langle h_0, x+th_0\rangle = t\|h_0\|^2 \neq 0$ for all nonzero real scalars t and

$$\langle A(x+th_0),(x+th_0)\rangle = \langle B(x+th_0),(x+th_0)\rangle.$$

Now take the limit as t goes to 0 to get that $\langle Ax,x\rangle = \langle Bx,x\rangle$ in this case too. Since x is an arbitrary vector in $\mathscr{H}$, we clearly have $A = B$. $\square$

The following theorem gives a local spectral characterization of rank one operators in term of the peripheral local spectrum of $\theta(S,T)$.

THEOREM 4.2. *Let h_0 be a nonzero vector of $\mathscr{H}$. For a nonzero operator $R \in \mathscr{B}(\mathscr{H})$, the following statements are equivalent.*

(a) *R has rank one.*

(b) $\gamma_{\theta(T,R)}(h_0)$ *is a singleton for all* $T \in \mathscr{B}(\mathscr{H})$.

Proof. Obviously, if R has rank one and $T \in \mathscr{B}(\mathscr{H})$ is an arbitrary operator, then, $\theta(T,R)$ has rank one too and thus $\gamma_{\theta(T,R)}(h_0)$ is a singleton.

Conversely, assume that R has rank at least two, and let us show that there exists $T \in \mathscr{B}(\mathscr{H})$ such that $\gamma_{\theta(T,R)}(h_0)$ contains at least two elements. We may and shall assume that $\mu \neq 0$ as the case when $\mu = 0$ is given in [8]. We shall discuss two situations.

Case 1. If there exist two vectors $h_1, h_2 \in \mathscr{H}$ such that h_0, Rh_1 and Rh_2 are linearly independent, then there also exists $h \in \mathscr{H}$ such that h, h_0, Rh_1 and Rh_2 are linearly independent. Hence, there exists an operator $T \in \mathscr{B}(\mathscr{H})$ of a finite rank such that $Th_0 = h_1$, $Th = h_2$, $\mu TRh_2 = -h - \nu Rh_2$ and $\mu TRh_1 = h_0 - 2h - \nu Rh_1$. Then we have $\theta(T,R)(h) = -h$ and $\theta(T,R)(h_0) = h_0 - 2h$. Thus $\theta(T,R)(h_0 - h) = h_0 - h$, and consequently

$$\sigma_{\theta(T,R)}(h_0) = \sigma_{\theta(T,R)}(h) \cup \sigma_{\theta(T,R)}(h_0 - h) = \{-1, 1\},$$

and then, $\gamma_{\theta(T,R)}(h_0) = \{-1, 1\}$ contains two different scalars.

Case 2. If h_0, Rh_1 and Rh_2 are linearly dependent for all $h_1, h_2 \in \mathscr{H}$, then R has rank 2 and its image contains h_0. So, $R := h_1 \otimes y_1 + h_2 \otimes y_2$ and $h_0 = \alpha_1 h_1 + \alpha_2 h_2$ for some linearly independent vectors $h_1, h_2 \in \mathscr{H}$, linearly independent vectors $y_1, y_2 \in \mathscr{H}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. If both α_1 and α_2 are nonzero scalars, then take z_1 and $z_2 \in \mathscr{H}$ linearly independent of h_1 and h_2 such that $\langle z_1, y_1 \rangle = \alpha_1^{-1}\mu + \nu \langle h_1, y_1 \rangle$, $\langle z_2, y_1 \rangle = \nu \langle h_2, y_1 \rangle$, $\langle z_1, y_2 \rangle = \nu \langle h_1, y_2 \rangle$ and $\langle z_2, y_2 \rangle = -\alpha_2^{-1}\mu + \nu \langle h_2, y_2 \rangle$.

Now, let $h := h_0 - z_1 - z_2 \neq 0$ and define $\mu T h_i = z_i - \nu h_i$ and $\mu . T z_i = \alpha_i z_i$. We infer that $\theta(T,R)h = 0$, $\theta(T,R)z_1 = z_1$ and $\theta(T,R)z_2 = -z_2$. It follows that

$$\begin{aligned} \sigma_{\theta(T,R)}(h_0) &= \sigma_{\theta(T,R)}(h + z_1 + z_2) \\ &= \sigma_{\theta(T,R)}(h) \cup \sigma_{\theta(T,R)}(z_1) \cup \sigma_{\theta(T,R)}(z_2) \\ &= \{-1, 0, 1\}. \end{aligned}$$

Then, $\gamma_{\theta(T,R)}(h_0) = \{-1, 1\}$ contains two different scalars.

If $\alpha_2 = 0$, then $h_0 = \alpha_1(h_1 - h_2) + \alpha_1 h_2$ and $R = (h_1 - h_2) \otimes y_1 + h_2 \otimes (y_1 + y_2)$. By what has shown above, there is $T \in \mathscr{B}(\mathscr{H})$ such that $\gamma_{\theta(T,R)}(h_0) = \{-1, 1\}$ contains two different scalars. The case when $\alpha_1 = 0$ is similar, and thus the implication $(b) \Rightarrow (a)$ is established. $\square$

For the proof of theorem 2.1, we also need the following essential lemmas.

LEMMA 4.3. *Let $h_0 \in \mathscr{H}$ and $k_0 \in \mathscr{K}$ be two nonzero vectors and A, B be two bijective linear operators from $\mathscr{H}$ into $\mathscr{K}$, and $\varphi : \mathscr{F}_1(\mathscr{H}) \to \mathscr{F}_1(\mathscr{K})$ defined by $\varphi(x \otimes y) := Ax \otimes By$ for all x, $y \in \mathscr{H}$. If φ satisfies*

$$\gamma_{\theta(S,T^*)}(h_0) = \gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0) \quad (T,\, S \in \mathscr{F}_1(\mathscr{H})), \tag{4.2}$$

then there exist two positives scalars ξ and η such that $A^*A = \eta I$ and $B^*B = \xi I$. Moreover if $\mu \neq 0$ then, $B^*A = I$.

Proof. Let x, y, l and h be four vectors in $\mathscr{H}$, and let us first show that

$$\langle x, l \rangle \langle h, y \rangle [\mu \langle x, y \rangle + \nu] = \langle Ax, Al \rangle \langle Bh, By \rangle [\mu \langle Ax, By \rangle + \nu]. \tag{4.3}$$

Note that (4.2) applied to $x \otimes y$ and $l \otimes h$ entails that

$$\langle x,l\rangle \gamma_{[\mu\langle h,y\rangle x+\nu h]\otimes y}(h_0)=\langle Ax,Al\rangle \gamma_{[\mu\langle Bh,By\rangle Ax+\nu Bh]\otimes By}(k_0), \tag{4.4}$$

and let us show that

$$\langle h_0,y\rangle \neq 0 \iff \langle k_0,By\rangle \neq 0. \tag{4.5}$$

Indeed, if $\langle h_0,y\rangle \neq 0$ and $\langle k_0,By\rangle = 0$, then (4.4), applied when $x=h=l=ty/\|y\|^2$, for some scalar t such that $t\mu+\nu\neq 0$, yields to

$$\left\{\|y\|^{-2}[t\mu+\nu]\right\}=\{0\}.$$

Which is a contradiction and shows that if $\langle k_0,By\rangle = 0$ then $\langle h_0,y\rangle = 0$. Conversely, if $\langle h_0,y\rangle = 0$ and $\langle k_0,By\rangle \neq 0$, apply (4.4) when $x=l=A^{-1}By/\|By\|^2$ and $h=y$ so that

$$\{0\}=\{t\mu+\nu\}.$$

This contradiction shows that if $\langle h_0,y\rangle = 0$ then $\langle k_0,By\rangle = 0$. Therefore, (4.5) is established.

By (4.5) and (3.1) we see that (4.3) holds provided that $\langle h_0,y\rangle \neq 0$. Now, if $\langle h_0,y\rangle = 0$, note that for every $\lambda > 0$, $\langle h_0,y+\lambda h_0\rangle \neq 0$ and apply (4.3). We get

$$\langle x,l\rangle\langle h,y+\lambda h_0\rangle[\mu\langle x,y+\lambda h_0\rangle+\nu]=\langle Ax,Al\rangle\langle Bh,By+\lambda Bh_0\rangle[\mu\langle Ax,By+\lambda Bh_0\rangle+\nu].$$

By expanding this identity and getting λ to 0, we deduce that (4.3) holds in this case too. Hence, (4.3) is true for all $x,y,l,h\in\mathscr{H}$.

Now, we show that the mappings A and B are continuous. Take x such that $\|x\|=1$ and set $\delta_x=\frac{t\mu+\nu}{\|Bx\|^2(t\mu\langle Ax,Bx\rangle+\nu)}$. From (4.3), we get

$$\langle Ax,Al\rangle=\delta_x\langle x,l\rangle,$$

for all $l\in\mathscr{H}$. This obviously shows that $u\mapsto\langle Ax,Au\rangle$ is continuous, and thus, since x is an arbitrary vector in $\mathscr{H}$ and A is bijective, the closed graph theorem implies that A itself is continuous. Similarly, we can show that B is continuous, and we therefore omit the details here.

Now, let us show at first that A^*Ax and x are linearly dependent for every $x\in\mathscr{H}$. To do this, we rewrite (4.3) as follows

$$\langle x,l\rangle\langle h,y\rangle[\mu\langle x,y\rangle+\nu]=\langle A^*Ax,l\rangle\langle B^*Bh,y\rangle[\mu\langle B^*Ax,y\rangle+\nu]. \tag{4.6}$$

for all x, y, l, $h\in\mathscr{H}$. Indeed, assume by the way of contradiction that there exists a nonzero vector $x_1\in\mathscr{H}$ such that A^*Ax_1 and x_1 are linearly independent, and let l_1 be a nonzero vector in $\mathscr{H}$ such that $\langle x_1,l_1\rangle=1$ and $\langle A^*Ax_1,l_1\rangle=0$. From (4.6), we get that for all $h,y\in\mathscr{H}$,

$$\langle h,y\rangle[\mu\langle x_1,y\rangle+\nu]=0.$$

Which is not possible, this contradiction shows that $A^*A=\eta I_{\mathscr{H}}$ for some positive scalar η. By a similar way, we show that $B^*B=\xi I_{\mathscr{H}}$ for some positive scalar ξ. Now, assume that $\mu\neq 0$ and let us show that $B^*A=I_{\mathscr{H}}$. Note that, since A^*A and B^*B are scalar operators, (4.3) implies that

$$\mu\langle x,y\rangle+\nu=\alpha\beta[\mu\langle B^*Ax,y\rangle+\nu].$$

If B^*A is not scalar, then there is a nonzero vector $x_2\in\mathscr{H}$ such that B^*Ax_2 and x_2 are linearly independant. Therefore, there exists a nonzero vector $y_2\in\mathscr{H}$ such that $\langle x_2,y_2\rangle:=-\frac{\nu}{\mu}$ and $\langle B^*Ax_2,y_2\rangle:=t$. Back to the previous formula we get $t\mu+\nu=0$ for all scalars t, which is a contradiction.

Hence, in the case when $\mu\neq 0$, we have $B^*A=\gamma I_{\mathscr{H}}$ for some nonzero scalar γ. Moreover, observe that (4.6) implies that $\alpha\beta\gamma=1$. Moreover, such a scalar γ must be 1. Indeed, since A and B are invertibles and $AB^*=\gamma I_{\mathscr{K}}$, $AA^*=\eta I_{\mathscr{K}}$, $BB^*=\xi I_{\mathscr{K}}$, thus

$$\eta\xi I_{\mathscr{H}}=A^*AB^*B=\gamma\bar{\gamma}I_{\mathscr{H}}.$$

This shows that $\eta\xi=\gamma\bar{\gamma}$, and $\gamma^2\bar{\gamma}=\eta\xi\gamma=1$. Therefore $\gamma=1$ and $B^*A=I_{\mathscr{H}}$, and the proof of the lemma is complete. $\square$

LEMMA 4.4. *Let $h_0\in\mathscr{H}$ and $k_0\in\mathscr{K}$ be two nonzero fixed vectors, and let C and D be two bijective linear operators from $\mathscr{H}$ into $\mathscr{K}$, and $\varphi:\mathscr{F}_1(\mathscr{H})\to\mathscr{F}_1(\mathscr{K})$ defined by*

$$\varphi(x\otimes y)=Cy\otimes Dx,\quad (x,y\ \in\mathscr{H})$$

Then, there are rank one operators T and $S\in\mathscr{F}_1(\mathscr{H})$ such that

$$\gamma_{\theta(S,T^*)}(h_0)\neq\gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0).$$

Proof. Assume by the way of contradiction that $\gamma_{\theta(S,T^*)}(h_0)=\gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0)$ for all rank one operators $T,S\in\mathscr{F}_1(\mathscr{H})$ and choose a nonzero vector $y_1\in\mathscr{H}$ such that $\langle k_0,y_1\rangle=0$ and $x=D^{-1}y_1$. Since x and h_0 are nonzero vectors, there exists $y\in\mathscr{H}$ such that $\langle h_0,y\rangle\neq 0$ and $\langle x,y\rangle=1$. We therefore have

$$\begin{aligned}
\{(\mu+\nu)\}&=\gamma_{\{\mu+\nu\}(x\otimes y)}(h_0)\\
&=\gamma_{\mu(x\otimes y)(x\otimes y)(x\otimes y)+\nu(x\otimes y)(x\otimes y)}(h_0)\\
&=\gamma_{\mu(x\otimes y)(y\otimes x)^*(x\otimes y)+\nu(y\otimes x)^*(x\otimes y)}(h_0)\\
&=\gamma_{\mu(Cy\otimes Dx)(Cx\otimes Dy)^*(Cy\otimes Dx)+\nu(Cx\otimes Dy)^*(Cy\otimes Dx)}(k_0)\\
&=\gamma_{\mu(Cy\otimes Dx)(Dy\otimes Cx)(Cy\otimes Dx)+\nu(Dy\otimes Cx)(Cy\otimes Dx)}(k_0)\\
&=\gamma_{\langle Cy,Cx\rangle[\mu\langle Dy,Dx\rangle Cy+\nu Dy]\otimes Dx)}(k_0)\\
&=\gamma_{\langle Cy,Cx\rangle[\mu\langle Dy,Dx\rangle Cy+\nu Dy]\otimes y_1)}(k_0)\\
&=\{0\}.
\end{aligned}$$

Thus, $\mu+\nu=0$. Now, using $(x\otimes -y)$ instead of $(x\otimes y)$ we get $-\mu+\nu=0$. Hence, $\mu=\nu=0$. This leads to a contradiction and the lemma is therefore proved. $\square$
In the next section we establish the proof of our main result.

5. Proof of theorem 2.1

Proof of theorem 2.1. For the proof of the if part of theorem 2.1, let us assume that there exist two unitary operators U and V in $\mathscr{B}(\mathscr{H},\mathscr{K})$ and a nonzero scalars α and β such that φ satisfies (2.2) or (2.3). Assume that φ satisfies (2.2), then, for every $T,S\in\mathscr{B}(\mathscr{H})$ we have

$$\begin{aligned}\gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0) &= \gamma_{\nu\varphi(T)^*\varphi(S)}(k_0)\\ &= \gamma_{\nu VT^*U^*USV^*}(k_0)\\ &= \gamma_{\nu VT^*SV^*}(k_0)\\ &= \gamma_{\nu T^*S}(V^*k_0)\\ &= \gamma_{\nu T^*S}(\beta^{-1}h_0)\\ &= \gamma_{\nu T^*S}(h_0)\\ &= \gamma_{\theta(S,T^*)}(h_0).\end{aligned}$$

Now, if φ satisfies (2.3) we have

$$\begin{aligned}\gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0) &= \gamma_{\mu\varphi(S)\varphi(T)^*\varphi(S)+\nu\varphi(T)^*\varphi(S)}(k_0)\\ &= \gamma_{\mu USU^*UT^*U^*USU^*+\nu UT^*U^*USU^*}(k_0)\\ &= \gamma_{\mu UST^*SU^*+\nu UT^*U^*USU^*}(k_0)\\ &= \gamma_{U(\mu ST^*S+\nu T^*S)U^*}(k_0)\\ &= \gamma_{(\mu ST^*S+\nu T^*S)}(U^*k_0)\\ &= \gamma_{(\mu ST^*S+\nu T^*S)}(h_0)\\ &= \gamma_{\theta(S,T^*)}(h_0),\end{aligned}$$

for all $T,\ S\in\mathscr{B}(\mathscr{H})$, and therefore (2.1) is established.

Conversely, assume that φ satisfies (2.1) and let us show that φ takes the desired form. The proof breaks down into three steps.

Step 1. φ is a one to one map preserving rank one operators in both directions.

We first show φ is a one to one map and $\varphi(0)=0$. Take two operators $A,\ B\in\mathscr{B}(\mathscr{H})$ such that $\varphi(A)=\varphi(B)$, and note that

$$\begin{aligned}\gamma_{\theta(S,A^*)}(h_0) &= \gamma_{\theta(\varphi(S),\varphi(A)^*)}(k_0)\\ &= \gamma_{\theta(\varphi(S),\varphi(B)^*)}(k_0)\\ &= \gamma_{\theta(S,B^*)}(h_0)\end{aligned}$$

for all $S\in\mathscr{B}(\mathscr{H})$. Theorem 4.1 tells us that $A=B$, and thus φ is a one to one. In a similar way, we show that $\varphi(0)=0$. For every $S\in\mathscr{B}(\mathscr{H})$, we have

$$\gamma_{\theta(\varphi(S),\varphi(0)^*)}(k_0)=\gamma_{\theta(S,0)}(h_0)=\{0\}=\gamma_{\theta(\varphi(S),0)}(k_0)$$

Again, by theorem 4.1 and the bijectivity of φ we see that $\varphi(0)=0$.

Next, we show that φ preserves rank one operators in both direction. Let R be a rank one operator, and note that $\varphi(R) \neq 0$ and that $\gamma_{\theta(S,R^*)}(h_0)$ has at most one element for all $S \in \mathscr{B}(\mathscr{H})$, and so is $\gamma_{\theta(\varphi(S),\varphi(R)^*)}(k_0)$. By theorem 4.2 and the bijectivity of φ we see that $\varphi(R)^*$ is rank one operator and so does $\varphi(R)$.

Conversely, assume that $\varphi(R)$ is rank one for some operator $R \in \mathscr{B}(\mathscr{H})$, and note that $R \neq 0$ and that $\gamma_{\theta(\varphi(S),\varphi(R)^*)}(k_0)$ has at most one element for all $S \in \mathscr{B}(\mathscr{H})$. Therefore, $\gamma_{\theta(S,R^*)}(h_0)$ has at most one element for all $S \in \mathscr{B}(\mathscr{H})$. Again theorem 4.2 tells us that R^* is rank one operator and so does R.

Step 2. φ is linear.

First we show that φ is additive. Let R be a rank one operator, and let T and S two operators in $\mathscr{B}(\mathscr{H})$, then by Lemma 3.2, we have

$$\begin{aligned}
\gamma_{\theta(\varphi(R),\varphi(T+S)^*)}(k_0) &= \gamma_{\theta(R,(T+S)^*)}(h_0) \\
&= \gamma_{\theta(R,T^*)}(h_0) + \gamma_{\theta(R,S^*)}(h_0) \\
&= \gamma_{\theta(\varphi(R),\varphi(T)^*)}(k_0) + \gamma_{\theta(\varphi(R),\varphi(S)^*)}(k_0) \\
&= \gamma_{\theta(\varphi(R),(\varphi(T)+\varphi(S))^*)}(k_0)
\end{aligned}$$

for all rank one operators $R \in \mathscr{B}(\mathscr{H})$. By theorem 4.1, we conclude that

$$\varphi(T+S) = \varphi(T) + \varphi(S)$$

for all T, $S \in \mathscr{B}(\mathscr{H})$, and φ is additive; as desired.

Now, let us show that φ is homogeneous. Indeed, take a nonzero $\lambda \in \mathbb{C}$ and an operator $T \in \mathscr{B}(\mathscr{H})$, and note that

$$\begin{aligned}
\gamma_{\theta(\varphi(S),(\lambda\varphi(T))^*)}(k_0) &= \overline{\lambda}\gamma_{\theta(\varphi(S),\varphi(T)^*)}(k_0) \\
&= \overline{\lambda}\gamma_{\theta(S,T^*)}(h_0) \\
&= \gamma_{\theta(S,(\lambda T)^*)}(k_0) \\
&= \gamma_{\theta(\varphi(S),\varphi(\lambda T)^*)}(k_0)
\end{aligned}$$

for all rank one operators $R \in \mathscr{B}(\mathscr{H})$. By theorem 4.1, we see that

$$\varphi(\lambda T) = \lambda\varphi(T)$$

for all $T \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \mathbb{C}$. Hence, φ is linear.

Step 3. φ takes the desired forms "(2.3)" and "(2.2)".

By the previous steps, φ is a bijective linear map preserving rank one operators in both directions. By [25, theorem 3.3], either there are two bijective linear mappings $A, B : \mathscr{H} \to \mathscr{K}$ such that

$$\varphi(x \otimes y) = Ax \otimes By, \quad (x, y \ \in \mathscr{H}), \tag{5.1}$$

or there are two bijective linear mappings $C,\ D: \mathscr{H} \to \mathscr{K}$ such that

$$\varphi(x\otimes y) = Cy\otimes Dx, \quad (x,y \ \in \mathscr{H}). \tag{5.2}$$

By Lemma 4.4, φ cannot take the second form, and thus φ takes the form (5.1) when it is restricted on $\mathscr{F}(\mathscr{H})$. Since φ satisfies (5.1), Lemma 4.3 shows that $AA^* = \eta I$ and $BB^* = \xi I$ for somes positives scalars η and ξ. Take $U = \frac{1}{\delta}A$ where $\delta = \sqrt{\eta}$ and $V = \frac{1}{\lambda}B$ where $\lambda = \sqrt{\xi}$. Note that U and V are unitary operators on $\mathscr{H}$. Next we discuss the cases $\mu = 0$ and $\mu \neq 0$. Now if $\mu = 0$, by (4.3) we see that $\xi\eta = 1$. In this case we have $\varphi(x\otimes y) = \frac{1}{\delta}A(x\otimes y)\frac{1}{\lambda}B^* = U(x\otimes y)V$ for all $x,y\in\mathscr{H}$. We continue by showing that $Vh_0 = \alpha k_0$ for some nonzero scalar $\alpha\in\mathbb{C}$. Assume by the way of contradiction that they are linearly independent, and let u be a vector in $\mathscr{H}$ such that $\langle h_0,u\rangle = 1$ and $\langle V^{-1}k_0,u\rangle = 0$. We have

$$\begin{aligned}
\{\nu\} &= \gamma_{\nu(h_0\otimes u)}(h_0)\\
&= \gamma_{\nu(h_0\otimes u)(h_0\otimes u)}(h_0)\\
&= \gamma_{\nu(u\otimes h_0)^*(h_0\otimes u)}(h_0)\\
&= \gamma_{\nu\varphi(u\otimes h_0)^*\varphi(h_0\otimes u)}(k_0)\\
&= \gamma_{\nu(Uu\otimes h_0V^*)^*(Uh_0\otimes uV^*)}(k_0)\\
&= \gamma_{\nu V(h_0\otimes u)(h_0\otimes u)V^*}(k_0)\\
&= \gamma_{\nu V(h_0\otimes u)V^*}(k_0)\\
&= \nu\gamma_{(h_0\otimes u)}(U^*k_0)\\
&= \{0\}.
\end{aligned}$$

This arises a contradiction, and shows that Vh_0 and k_0 are linearly dependent. Hence, for every rank one operator $R\in\mathscr{B}(\mathscr{H})$ and every operator $T\in\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned}
\gamma_{\theta(\varphi(R),\varphi(T)^*)}(k_0) &= \gamma_{\nu\varphi(T)^*\varphi(R)}(k_0)\\
&= \gamma_{\nu T^*R}(h_0)\\
&= \gamma_{\nu T^*R}(\alpha h_0)\\
&= \gamma_{\nu T^*R}(V^*k_0)\\
&= \gamma_{\nu VT^*RV^*}(k_0)\\
&= \gamma_{\nu VT^*U^*URV^*}(k_0)\\
&= \gamma_{\nu VT^*U^*\varphi(R)}(k_0)\\
&= \gamma_{\theta(\varphi(R),(UTV^*)^*)}(h_0).
\end{aligned}$$

Hence, by theorem 4.1, we see that

$$\varphi(T)^* = (UTV^*)^*.$$

And therefore

$$\varphi(T) = UTV^*,$$

for all $T \in \mathscr{B}(\mathscr{H})$. The proof of 2.3 is then complete.
Now if $\mu \neq 0$. Lemma 4.3 shows that $B^*A = I$. In this case we have $B = \frac{1}{\delta}U$. It follows that $\varphi(x \otimes y) = \frac{1}{\delta}A(x \otimes y)\delta B^* = U(x \otimes y)U^*$ for all $x, y \in \mathscr{H}$. We continue by showing that $Uh_0 = \beta k_0$ for some nonzero scalar $\beta \in \mathbb{C}$. Assume by the way of contradiction that they are linearly independent, and let z be a vector in $\mathscr{H}$ such that $\langle h_0, z\rangle = 1$ and $\langle U^{-1}k_0, z\rangle = 0$. We have

$$\begin{aligned}
\{\mu + \nu\} &= \gamma_{\mu(h_0\otimes z)+\nu(h_0\otimes z)}(h_0) \\
&= \gamma_{\mu(h_0\otimes z)(h_0\otimes z)(h_0\otimes z)+\nu(h_0\otimes z)(h_0\otimes z)}(h_0) \\
&= \gamma_{\mu(h_0\otimes z)(z\otimes h_0)^*(h_0\otimes z)+\nu(z\otimes h_0)^*(h_0\otimes z)}(h_0) \\
&= \gamma_{\mu\varphi(h_0\otimes z)\varphi(z\otimes h_0)^*\varphi(h_0\otimes z)+\nu\varphi(z\otimes h_0)^*\varphi(h_0\otimes z)}(k_0) \\
&= \gamma_{\mu(Uh_0\otimes zU^*)(Uz\otimes h_0U^*)^*(Uh_0\otimes zU^*)+\nu(Uz\otimes h_0U^*)^*(Uh_0\otimes zU^*)}(k_0) \\
&= \gamma_{\mu U(h_0\otimes z)(h_0\otimes z)(h_0\otimes z)U^*+\nu U(h_0\otimes z)(h_0\otimes z)U^*}(k_0) \\
&= \gamma_{(\mu+\nu)U(h_0\otimes z)U^*}(k_0) \\
&= (\mu+\nu)\gamma_{(h_0\otimes z)}(U^*k_0) \\
&= \{0\}.
\end{aligned}$$

Therefore, $\mu + \nu = 0$. We get also $-\mu + \nu = 0$ by using $h_0 \otimes -z$ instead of $h_0 \otimes z$. That means $\mu = \nu = 0$ wich is not possible. Hence, $Uh_0 = \beta k_0$ for some nonzero scalar $\beta \in \mathbb{C}$. To finish the proof note that for every rank one operator $R \in \mathscr{B}(\mathscr{H})$ and every $T \in \mathscr{B}(\mathscr{H})$ we have

$$\begin{aligned}
\gamma_{\theta(\varphi(R),UT^*U^*)}(k_0) &= \gamma_{\mu\varphi(R)UT^*U^*\varphi(R)+\nu UT^*U^*\varphi(R)}(k_0) \\
&= \gamma_{\mu URU^*UT^*U^*URU^*+\nu UT^*U^*URU^*}(k_0) \\
&= \gamma_{\mu URT^*RU^*+\nu UT^*RU^*}(k_0) \\
&= \gamma_{U(\mu RT^*R+\nu T^*R)U^*}(k_0) \\
&= \gamma_{(\mu RT^*R+\nu T^*R)}(U^*k_0) \\
&= \gamma_{(\mu RT^*R+\nu T^*R)}(h_0) \\
&= \gamma_{(\mu\varphi(R)\varphi(T)^*\varphi(R)+\nu\varphi(T)^*\varphi(R)}(k_0) \\
&= \gamma_{\theta(\varphi(R),\varphi(T)^*)}(k_0).
\end{aligned}$$

Hence by theorem 4.1 we see that

$$\varphi(T)^* = UT^*U^*$$

And therefore,

$$\varphi(T) = UTU^*$$

for all $T \in \mathscr{B}(\mathscr{H})$. The proof is then complete. □

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