SELF–ADJOINT OPERATORS AND THE GENERAL GKN–EM THEOREM

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Abstract. We construct self-adjoint operators in the direct sum of a complex Hilbert space H and a finite dimensional complex inner product space W. The operator theory developed in this paper for the Hilbert space $H \oplus W$ is originally motivated by some fourth-order differential operators, studied by Everitt and others, having orthogonal polynomial eigenfunctions. Generated by a closed symmetric operator T_0 in H with equal and finite deficiency indices and its adjoint T_1, we define *families* of minimal operators $\{\widehat{T_0}\}$ and maximal operators $\{\widehat{T_1}\}$ in the extended space $H \oplus W$ and establish, using a recent theory of complex symplectic geometry, developed by Everitt and Markus, a characterization of self-adjoint extensions of $\{\widehat{T_0}\}$ when the dimension of the extension space W is not greater than the deficiency index of T_0. A generalization of the classical Glazman-Krein-Naimark (GKN) theorem - called the GKN-EM theorem to acknowledge the work of Everitt and Markus - is key to finding these self-adjoint extensions in $H \oplus W$. We consider several examples to illustrate our results.

1. Introduction

In [6, p. 105], the authors list ten open problems related to orthogonal polynomial eigenfunctions of differential equations. The one pertinent to this present paper is the following (paraphrased to simplify the original notation):

The GKN theory provides a recipe, in theory, for determining all self-adjoint extensions in the Hilbert space $L^2(I;w)$ of formally symmetric differential expressions of the form

$$\ell_{2r}[y](u) = \frac{1}{w(u)} \sum_{j=0}^{r} (-1)^j (q_j(u) y^{(j)}(u))^{(j)} \quad (u \in I) \qquad (1.1)$$

on some open interval $I = (a,b)$; we assume here that $w > 0$ and each coefficient q_j is sufficiently differentiable on I. This theory works well in developing the spectral theory for the second-order classical differential equations of Jacobi, Laguerre, and Hermite.[1] However, for nonclassical symmetric differential equations (1.1) with orthogonal polynomial

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[1] The GKN theory is applicable as well to developing the self-adjoint operator theory associated with exceptional orthogonal polynomials.

solutions, the appropriate right-definite setting is a Hilbert-Sobolev space
S with orthogonalizing Sobolev inner product

$$\langle f, g \rangle = \int_a^b f(u)\overline{g}(u)w(u)du + \sum_{j=0}^p \left(\alpha_j f^{(j)}(a)\overline{g}^{(j)}(a) + \beta_j f^{(j)}(b)\overline{g}^{(j)}(b) \right).$$
(1.2)

The Sobolev space S has the form $L^2(I;w) \oplus \mathbb{C}^k$ for some $k \leqslant 2p$. De-
velop a general GKN-type theory for this setting; in particular, provide
a 'recipe' for determining the self-adjoint operator having the orthogonal
polynomials as eigenfunctions.

In this paper, we answer this question. In fact, we will see that we can provide
a recipe for *all* self-adjoint operators, generated by $\ell_{2r}[\cdot]$, in this Sobolev setting. Our
result is a generalization of the Glazman-Krein-Naimark (GKN) theory of self-adjoint
extensions of Lagrangian symmetric ordinary differential expressions in a weighted
Hilbert space $L^2(I;w)$, where I is an interval of the real line $\mathbb{R}$.

This work is originally motivated by fourth-order differential equations having
non-classical orthogonal polynomials as eigenfunctions. In each of these fourth-order
examples, the orthogonalizing inner product has the form

$$\langle f, g \rangle = Af(a)\overline{g}(a) + \int_a^b f(u)\overline{g}(u)w(u)du + Bf(b)\overline{g}(b),$$

where $A, B \geqslant 0$. Indeed, H. L. Krall [13, 14] classified, up to a complex linear change of
variable, these orthogonal polynomials which were subsequently named the Legendre
type, Laguerre type and Jacobi type polynomials and studied extensively by A. M. Krall
in [12]. Following the work of the two Kralls, other contributions connecting orthogo-
nal polynomial eigenfunctions to higher-order differential equations have emerged; all
known examples have polynomial eigenfunctions orthogonal with respect to an inner
product of the form (1.2). These various contributions are far too numerous to list in
this manuscript but we refer to the Erice and Patras reports [5, 6] for further details and
the references therein contained.

In [3, 4], the authors construct the Legendre type self-adjoint operator, generated
by the fourth-order Legendre type differential expression

$$\ell_{LT}[y](u) := (u^2 - 1)^2 y^{(4)}(u) + 8u(u^2 - 1)y^{(3)}(u) + (4A + 12)(u^2 - 1)y''(u)$$
$$+ 8Auy'(u),$$
(1.3)

in the Hilbert space $L_\mu^2[-1, 1]$. Here

$$L_\mu^2[-1, 1] = \{f : [-1, 1] \to \mathbb{C} \mid f \text{ is Lebesgue measurable}$$
$$\text{and } \langle f, f \rangle_\mu = \int_{[-1,1]} |f|^2 d\mu < \infty\},$$

where

$$\langle f, g \rangle_\mu = Af(-1)\overline{g}(-1) + \int_{-1}^1 f(u)\overline{g}(u)du + Af(1)\overline{g}(1).$$

We note that $L^2_\mu[-1,1]$ is isometrically isomorphic to $L^2(-1,1) \oplus \mathbb{C}^2$. The classical Glazman-Krein-Naimark (GKN) theory of self-adjoint extensions of Lagrangian symmetric differential expressions is not (immediately) applicable in this situation. To develop the appropriate operator theory in $L^2_\mu[-1,1]$, Everitt and Littlejohn studied properties of functions in the maximal domain Δ of $\ell_{LT}[\cdot]$ in the base space $L^2(-1,1)$. They prove the surprising smoothness condition

$$f \in \Delta \Rightarrow f'' \in L^2(-1,1)$$

from which it follows that $f, f' \in AC[-1,1]$. Using standard operator-theoretic methods, they then prove that the operator $T : \mathscr{D}(T) \subset L^2_\mu[-1,1] \to L^2_\mu[-1,1]$ defined by

$$(Tf)(u) = \begin{cases} -8Af'(-1) & u = -1 \\ \ell_{LT}[f](u) & -1 < u < 1 \\ 8Af'(1) & u = 1 \end{cases}$$

$$\mathscr{D}(T) = \Delta$$

is self-adjoint. It is remarkable that Δ is the domain of the self-adjoint operator T in $L^2_\mu[-1,1]$. Indeed, the expression $\ell_{LT}[\cdot]$ is in the limit-3 case at both singular endpoints $x = \pm 1$ in $L^2(-1,1)$ so every self-adjoint operator in $L^2(-1,1)$, generated by $\ell_{LT}[\cdot]$, is necessarily determined by two appropriate boundary restrictions on the space Δ. We will re-examine this Legendre type example in Section 5.1 as an application of the results developed in this paper.

Let $(W, \langle \cdot, \cdot \rangle_W)$ be a *finite-dimensional* complex inner product space and assume $(H, \langle \cdot, \cdot \rangle_H)$ is a complex Hilbert space. Then $H \oplus W$, the direct sum of H and W, is the Hilbert space defined by

$$H \oplus W = \{(x,a) \mid x \in H, a \in W\} \tag{1.4}$$

with inner product

$$\langle (x,a),(y,b) \rangle_{H \oplus W} := \langle x, y \rangle_H + \langle a, b \rangle_W \tag{1.5}$$

and associated norm

$$\|(x,a)\|^2_{H \oplus W} = \|x\|^2_H + \|a\|^2_W.$$

Throughout this paper, we refer to $H \oplus W$ as an *extended Hilbert space* and call H the *base space* and W the *extension space*.

Our starting point in this paper - assumptions we keep throughout this article - is a closed, symmetric operator T_0 in H having equal and finite deficiency indices, denoted by their common value $\mathrm{def}(T_0)$, and adjoint operator T_1 satisfying the inclusions

$$T_1^* = T_0 \subseteq T_0^* = T_1.$$

We call T_0 the *minimal* operator and T_1 the *maximal* operator in H. Then, under the essential assumption that

$$\dim W \leqslant \mathrm{def}(T_0),$$

we construct one-parameter families $\{\widehat{T_0}\}$ of *minimal* operators and associated *maximal* operators $\{\widehat{T_1}\}$ in $H \oplus W$, generated by T_0 and T_1 in H, satisfying the properties

$$\operatorname{def}(\widehat{T_0}) = \operatorname{def}(T_0) \quad (\widehat{T_0} \in \{\widehat{T_0}\})$$

and

$$\left(\widehat{T_1}\right)^* = \widehat{T_0} \subseteq \left(\widehat{T_0}\right)^* = \widehat{T_1} \quad (\widehat{T_0} \in \{\widehat{T_0}\}, \ \widehat{T_1} \in \{\widehat{T_1}\}).$$

Both families $\{\widehat{T_0}\}$ and $\{\widehat{T_1}\}$ are parametrized by an arbitrary, fixed self-adjoint operator $B : W \to W$.

With the constructions of $\{\widehat{T_0}\}$ and $\{\widehat{T_1}\}$ in place, we then appeal to a general theory of complex symplectic algebra, with important applications and implications to boundary value problems in ordinary and partial differential equations, which was developed by Everitt and Markus in a series of remarkable papers [7, 8, 9, 10]. An important consequence of their theory is a generalized GKN theory - which we call GKN-EM theory after the contributions of Everitt and Markus - that we apply to characterize all self-adjoint extensions (respectively, restrictions) of $\widehat{T_0} \in \{\widehat{T_0}\}$ (respectively, of $\widehat{T_1} \in \{\widehat{T_1}\}$).

The contents of this paper are as follows. In Section 2, we briefly discuss the Stone-von Neumann theory of self-adjoint extensions of symmetric operators in a Hilbert space as well as the now classic GKN theory, including a statement of the GKN theorem (Theorem 2.3). Section 3 deals with key complex symplectic geometric results developed by Everitt and Markus and culminates in the GKN-EM theorem (Theorem 3.2). The families $\{\widehat{T_0}\}$ and $\{\widehat{T_1}\}$ of minimal and maximal operators in $H \oplus W$, generated by T_0 and T_1 in the base space H, are developed in Section 4. Another key notion, the symplectic form $[\cdot, \cdot]_{H \oplus W}$ in the extended space $H \oplus W$, essential to our application of the GKN-EM theory, is defined in Section 4. Also, in this section, we apply Theorem 3.2 to characterize all self-adjoint extensions $\{\widehat{T}\}$ of $\widehat{T_0} \in \{\widehat{T_0}\}$; see the summary theorem given in Theorem 4.4. Lastly, Section 5 deals with several examples to illustrate our results. These examples include another look at the Legendre type example where further light is shed on this particular example. Indeed, we show that, remarkably, *continuity* is a GKN-EM boundary condition.

Notation: $\mathbb{R}, \mathbb{C}$ and $\mathbb{N}$ will denote, respectively, the sets of real numbers, the complex numbers and the positive integers. All inner products in this paper will be denoted by $\langle \cdot, \cdot \rangle$, properly subscripted indicating the particular underlying vector space. Ordered pairs in $H \oplus W$ will be written as $(\cdot, \cdot)$; if $\dim W > 1$, then an ordered pair in $H \oplus W$ will have the form $(\cdot, (\cdot, \cdot))$. Our *base* space will be a complex Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, our *extension* space will be finite-dimensional complex Hilbert space $(W, \langle \cdot, \cdot \rangle_W)$ and the *extended* space will be the direct sum space $(H \oplus W, \langle \cdot, \cdot \rangle_{H \oplus W})$. Linear operators in the base space H will be denoted by T_0, T_1, T, etc. while operators in the extended space $H \oplus W$ will be hatted: $\widehat{T_0}$, $\widehat{T_1}$, $\widehat{T}$, etc. The notation

$$x \text{ has property } P \quad (x \in A)$$

means that property P holds for all x in the set A. Lastly, the cardinality of a set A is denoted by $\operatorname{card}(A)$ whereas the dimension of a subspace W of some vector space will be written $\dim W$.

2. The von-Neumann formulas and the GKN theorem

Standard references for topics discussed in this section are [1, 2, 15, 16, 17, 18, 19, 21].

Throughout this paper, the linear operator $T_0 : \mathscr{D}(T_0) \subseteq H \to H$ will be an arbitrary closed, symmetric operator in H while $T_1 : \mathscr{D}(T_1) \subseteq H \to H$ is a linear operator satisfying the operator inclusions

$$T_1^* = T_0 \subseteq T_0^* := T_1; \tag{2.1}$$

in particular, we see that T_0 and T_1 are adjoints of each other. Because of the inclusions in (2.1), we call T_0 the *minimal* operator and T_1 the *maximal* operator. Specific reasons for this notation will be discussed below in this section (see also Remark 2.1). Notice that if T_0 has a self-adjoint extension T in H, then

$$T_0 \subseteq T = T^* \subseteq T_0^* = T_1,$$

so T necessarily has the same form as T_1; that is,

$$Tx = T_1x \quad (x \in \mathscr{D}(T)).$$

The general theory of self-adjoint extensions of the minimal operator T_0 (equivalently, self-adjoint restrictions of the maximal operator T_1) in a Hilbert space - called the Stone-von Neumann theory - is discussed in depth in [2, Chapter XII, Section 4]. Of central importance in this theory are two particular subspaces $X_\pm$ of $\mathscr{D}(T_1)$, defined by

$$X_\pm := \{x \in \mathscr{D}(T_1) \mid T_1x = \pm ix\},$$

where $i = \sqrt{-1}$. These spaces are called the *positive* and *negative deficiency spaces* of T_0. The *first von Neumann formula* decomposes the maximal domain $\mathscr{D}(T_1)$ into linearly independent submanifolds:

THEOREM 2.1. (The first von Neumann formula) $\mathscr{D}(T_1) = \mathscr{D}(T_0) + X_+ + X_-$.

In fact, the sum in this formula is actually an orthogonal direct sum. Indeed, under the graph inner product

$$\langle x, y \rangle_H^* := \langle x, y \rangle_H + \langle T_1x, T_1y \rangle_H \quad (x, y \in \mathscr{D}(T_1)) \tag{2.2}$$

and associated norm

$$(\|x\|_H^*)^2 = \|x\|_H^2 + \|T_1x\|_H^2 \geqslant \|x\|_H^2, \tag{2.3}$$

$\mathscr{D}(T_1)$ is a Hilbert space and, with this inner product, $\mathscr{D}(T_0)$, X_+ and X_- are closed, orthogonal subspaces of $\mathscr{D}(T_1)$; see [2, Chapter XII]. Notice that if $x \in \mathscr{D}(T_1)$ and

$$x = x^0 + x^+ + x^-,$$

where $x^0 \in \mathscr{D}(T_0)$ and $x^\pm \in X_\pm$, then

$$(\|x\|_H^*)^2 = \left(\|x^0\|_H^*\right)^2 + \left(\|x^+\|_H^*\right)^2 + \left(\|x^-\|_H^*\right)^2. \tag{2.4}$$

The dimensions of $X_\pm$, denoted by $\dim(X_\pm)$, are called the *positive and negative deficiency indices* of T_0. A key result in the Stone-von Neumann theory is that the equality of these deficiency indices is equivalent to the existence of self-adjoint extensions T of T_0 in H. Moreover, if $\dim(X_+) = \dim(X_-) = 0$, $T_0 = T_1$ is self-adjoint and is, in fact, the only self-adjoint extension of T_0 in H. In the case that $\dim(X_+) = \dim(X_-)$, we refer to this common value as the *deficiency index* and denote it by $\mathrm{def}(T_0)$. In addition to requiring the equality of these deficiency indices for the entirety of this paper, we assume the deficiency indices are also finite. Thus, another key assumption in this paper is:

CONDITION 2.1. $1 \leqslant \mathrm{def}(T_0) := \dim(X_+) = \dim(X_-) < \infty$.

The *second von Neumann formula* gives a description of the domain of any self-adjoint extension T of T_0 in H:

THEOREM 2.2. (The second von Neumann formula) *Let* $T : \mathscr{D}(T) \subseteq H \to H$ *be a self-adjoint extension of* T_0. *Then there exists an isometric isomorphism* $V : X_+ \to X_-$ *from the positive deficiency space* X_+ *onto the negative deficiency space* X_- *such that*

$$Tx = T_1 x \tag{2.5}$$
$$\mathscr{D}(T) = \{x + x_+ + V x_+ \mid x \in \mathscr{D}(T_0), x_+ \in X_+\}. \tag{2.6}$$

Conversely, if T *and its domain* $\mathscr{D}(T)$ *are defined through* (2.5) *and* (2.6) *for some isometric isomorphism* $V : X_+ \to X_-$ *, then* T *is a self-adjoint extension of* T_0.

The Glazman-Krein-Naimark (GKN) theory is both a refinement and an application of the Stone-von Neumann theory to self-adjoint operator extensions of ordinary differential expressions. Excellent expositions of this theory can be found in Akhiezer and Glazman [1, volume II, Chapter 8] and Naimark [15, part II, Chapter V]. To describe this theory we assume, for the sake of simplicity, that $\ell[\cdot]$ is a real, $2n$-th order Lagrangian symmetrizable differential expression of the form

$$\ell[y](u) = \frac{1}{w(u)} \sum_{j=0}^{n} (-1)^j \left(q_j(u) y^{(j)}(u) \right)^{(j)} \quad (u \in I), \tag{2.7}$$

where each coefficient $q_j : I \to \mathbb{R}$ in (2.7) is j-times continuously differentiable on I (noting, however, that general 'quasi-differentiable' conditions can be placed on these coefficients; see also [20]). The setting for the study of $\ell[\cdot]$ is the Hilbert space

$$L^2(I; w) = \{f : I \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_I |f|^2 \, w \, du < \infty\}$$

endowed with the standard inner product

$$\langle f,g \rangle_w = \int_a^b f(u)\overline{g}(u)w(u)du \quad (f,g \in L^2(I;w)).$$

Here $I \subseteq \mathbb{R}$ is an open interval and w is a positive (a.e.) Lebesgue measurable function on I. The *maximal operator* $L_1 : \mathscr{D}(L_1) \subseteq L^2(I;w) \to L^2(I;w)$, generated by $\ell[\cdot]$, is defined to be

$$L_1 f = \ell[f]$$
$$f \in \mathscr{D}(L_1) = \{f : I \to \mathbb{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(I) \ (j = 0,1,\ldots,2n-1);$$
$$f, \ell[f] \in L^2(I;w)\}.$$

In this setting, the term 'maximal' is appropriate; indeed, $\mathscr{D}(L_1)$ - which is called the *maximal domain* - is the largest subspace of $L^2(I;w)$ for which the expression $\ell[\cdot]$ acts on and maps into $L^2(I;w)$. It is clear that L_1 is a densely defined operator. We denote the adjoint of L_1 by L_0; it is natural then to call L_0 the *minimal operator* generated by $\ell[\cdot]$. The GKN theory shows that, in fact, L_1 and L_0 are adjoint to each other and L_0 is a closed symmetric operator in $L^2(I;w)$. More explicitly, $L_1^* = L_0$ and

$$L_0 = \overline{L_0} \subseteq L_0^* = L_1.$$

REMARK 2.1. The operators T_0 and T_1, defined earlier, are analogous to the minimal operator L_0 and maximal operator L_1, respectively. Because of this, we call T_0 and T_1, respectively, the minimal and maximal operators even though, in the general situation, the terms maximal and minimal may not seem as appropriate as they do in the GKN theory. Likewise, we shall call their respective domains the minimal domain $\mathscr{D}(T_0)$ and the maximal domain $\mathscr{D}(T_1)$.

The domain $\mathscr{D}(L_0)$ of the minimal operator is given explicitly by

$$\mathscr{D}(L_0) = \{f \in \mathscr{D}(L_1) \mid [f,g]_w|_a^b = 0 \text{ for all } g \in \mathscr{D}(L_1)\}, \tag{2.8}$$

where $[\cdot,\cdot]_w|_a^b$ is the skew-symmetric bilinear form obtained from the classic Green's formula

$$\langle L_1 f, g \rangle_w - \langle f, L_1 g \rangle_w = [f,g]_w|_a^b \quad (f,g \in \mathscr{D}(L_1)). \tag{2.9}$$

Moreover, we note that condition 2.1 is automatically satisfied in this setting. Indeed, the deficiency indices of L_0 are equal since ℓ has real coefficients and thus

$$\ell[f] = if \iff \overline{\ell[f]} = \ell[\overline{f}] = -i\overline{f}.$$

Moreover, in this case,

$$0 \leqslant \mathrm{def}(L_0) \leqslant 2n.$$

We are now in position to state the GKN theorem. Notice that this theorem provides a 'recipe' for constructing all self-adjoint extensions of the minimal operator L_0 in

$L^2(I;w)$ by specifying certain restrictions (boundary conditions), using the bilinear form $[\cdot,\cdot]_w$, on the maximal domain $\mathscr{D}(L_1)$. We emphasize, however, that the original GKN theorem is valid *only* for the minimal operator L_0 associated with a real Lagrangian symmetrizable differential expressions of even order in the specific Hilbert space $L^2(I;w)$. Compare the statement of the GKN theorem below with that of the GKN-EM theorem (Theorem 3.2) at the end of the next section.

THEOREM 2.3. (The GKN theorem) *Suppose L_0 and L_1 are, respectively, the minimal and maximal operators in $L^2(I;w)$, generated by the differential expression $\ell[\cdot]$, given in (2.7). In addition, let $m = \operatorname{def}(L_0)$ so $0 \leqslant m \leqslant 2n$.*

(i) *Suppose the set $\{g_j \mid j = 1,\ldots,m\} \subseteq \mathscr{D}(L_1)$ satisfies the two conditions*

(a)

$$\sum_{j=1}^{m} \alpha_j g_j \in \mathscr{D}(L_0) \Longrightarrow \alpha_j = 0 \ (j = 1,\ldots,m) \ and \tag{2.10}$$

(b)

$$[g_i, g_j]_w \big|_a^b = 0 \quad (i,j = 1,\ldots,\operatorname{def}(L_0)). \tag{2.11}$$

Define the operator $S : \mathscr{D}(S) \subseteq L^2(I;w) \to L^2(I;w)$ by

$$Sf = L_1 f \tag{2.12}$$

$$f \in \mathscr{D}(S) = \{f \in \mathscr{D}(L_1) \mid [f,g_j]_w \big|_a^b = 0 \quad (j = 1,\ldots,\operatorname{def}(L_0))\}. \tag{2.13}$$

Then S is a self-adjoint extension of the minimal operator L_0 in $L^2(I;w)$.

(ii) *Conversely, if $S : \mathscr{D}(S) \subseteq L^2(I;w) \to L^2(I;w)$ is a self-adjoint extension of the minimal operator L_0 in $L^2(I;w)$, then there exists a set $\{g_j \mid j = 1,\ldots,m\} \subseteq \mathscr{D}(L_1)$ satisfying the conditions (2.10) and (2.11) such that S is given explicitly by (2.12) and (2.13).*

REMARK 2.2. A collection of vectors $\{g_j \mid j = 1,\ldots,m\} \subseteq \mathscr{D}(L_1)$ satisfying condition (2.10) are said to be linearly independent modulo $\mathscr{D}(T_0)$ while those that satisfy (2.11) are said to satisfy Glazman symmetry conditions. Further light, as well as a generalization, into these concepts will be made in the next section.

REMARK 2.3. Each of the conditions

$$[f,g_j]_w \big|_a^b = 0 \quad (j = 1,\ldots,\operatorname{def}(L_0)),$$

given in (2.13), is called a 'boundary condition'. In the case that $\operatorname{def}(L_0) = 0$, then L_1 $(= L_0)$ is the only self-adjoint extension of L_0 and, in this case, there are no boundary conditions.

The GKN-EM theorem, which we discuss in the next section in Theorem 3.2, is a generalization of the GKN theorem but, remarkably, is valid in an *arbitrary* Hilbert space for an *arbitrary* closed symmetric operator with equal and *finite* deficiency indices. This theorem is a highlight application of the general complex symplectic theory developed by Everitt and Markus.

3. Complex symplectic geometry and a generalization of the GKN theorem

In a series of papers [7, 8, 9, 10], Everitt and Markus developed an extensive theory of complex symplectic geometry with applications to linear ordinary and partial differential equations. Their work was motivated by their interest in boundary value problems. In this section, we report on their results that pertain to this manuscript. A highlight application of their investigations is an important, and remarkable, generalization of Theorem 2.3; see Theorems 3.1 and 3.2 below. This generalization is key to the results we establish in the next section.

DEFINITION 3.1. A complex symplectic space S is a complex vector space together with a conjugate bilinear (sesquilinear) complex-valued function $[\cdot : \cdot] : S \times S \to \mathbb{C}$ satisfying the properties

(i) $[c_1 x_1 + c_2 x_2 : y] = c_1 [x_1 : y] + c_2 [x_2 : y],$

(ii) $[x : y] = -\overline{[y : x]},$

(iii) $[x : S] = 0 \implies x = 0$ (non-degenerate condition).

We call $[\cdot : \cdot]$ a (non-degenerate) symplectic form.

Complex symplectic spaces are non-trivial generalizations (not merely complexifications) of classical real symplectic spaces of Lagrangian and Hamiltonian mechanics (see [11]). Indeed, complex symplectic spaces have a much wider scope and admit new applications. For example, whereas real symplectic spaces cannot be odd dimensional, it is the case that, for every $n \in \mathbb{N}$, there exists complex symplectic spaces of dimension n.

Along with their real symplectic counterparts, complex symplectic spaces support the notion of *Lagrangian* subspaces (see [10] equation (1.10)).

DEFINITION 3.2. A subspace L of a complex symplectic space S is called Lagrangian if $[L : L] = 0$; that is to say, when

$$[x : y] = 0 \quad (x, y \in L).$$

A Lagrangian $L \subseteq S$ is called a complete Lagrangian when

$$x \in S \text{ and } [x : L] = 0 \implies x \in L.$$

We can characterize complete Lagrangian subspaces as follows. This characterization is key for later results.

LEMMA 3.1. *A Lagrangian subspace $L \subseteq S$ is a complete Lagrangian if and only if*

$$L = \{x \in S \mid [x : y] = 0 \ (y \in L)\}. \tag{3.1}$$

Proof. Suppose L is a complete Lagrangian subspace of S. By definition of complete, it is clear that $\{x \in S \mid [x : y] = 0 \ (y \in L)\} \subseteq L$. On the other hand, if $x \in L$ then $[x : y] = 0$ for all $y \in L$ since L is Lagrangian. Hence $L \subseteq \{x \in S \mid [x : y] = 0 \ (y \in L)\}$. Conversely, if L is Lagrangian and given by (3.1), then it is clear that L is a complete. $\square$

An essential step in the work of Everitt and Markus is a natural generalization of the skew-symmetric bilinear form $[\cdot, \cdot]_w |_a^b$ given by Green's formula (2.9).

DEFINITION 3.3. $[x,y]_H := \langle T_1 x, y \rangle_H - \langle x, T_1 y \rangle_H$ for $x, y \in \mathscr{D}(T_1)$.

Following the work of Everitt and Markus, we will see below that $[\cdot, \cdot]_H$ can be identified with a degenerate symplectic form. We also note that $[\cdot, \cdot]_H$ coincides with $[\cdot, \cdot]_w |_a^b$ in the case T_1 is the maximal differential operator, generated by $\ell[\cdot]$ (see (2.7)), in the weighted Hilbert space $L^2(I; w)$.

As shown in [10], the quotient space

$$S' := \mathscr{D}(T_1)/\mathscr{D}(T_0), \tag{3.2}$$

with zero element $0 = \mathscr{D}(T_0)$, is a complex symplectic space when endowed with the form $[\cdot, \cdot]_H$; we outline the specific details below.

Notice that, from Theorem 2.1 and Condition 2.1, S' has dimension $2 \operatorname{def}(T_0)$. Indeed one may view S' as an isomorphic copy of the orthogonal sum of the deficiency spaces $X_\pm$ of T_0. Everitt and Markus call the space S' the *boundary space* of T_0. The elements of S' are, of course, cosets $x = \{x + \mathscr{D}(T_0)\} \ (x \in \mathscr{D}(T_1))$. In this case, we call the vector x a *representative vector* of the coset $\{x + \mathscr{D}(T_0)\}$.

We now consider the natural projection $\phi : \mathscr{D}(T_1) \to S'$ defined by

$$\phi x = \{x + \mathscr{D}(T_0)\} \quad (x \in \mathscr{D}(T_1)) .$$

The following proposition makes clear the connection between a basis of a subspace of S' and the notion of linear independence modulo $\mathscr{D}(T_0)$ which we first encountered in Theorem 2.3 and Remark 2.2.

LEMMA 3.2. *A collection of cosets* $\{\phi t_j\}_{j=1}^d$, *where* $\{t_j\}_{j=1}^d \subseteq \mathscr{D}(T_1)$, *is a basis for a subspace of dimension d of the boundary space S' if and only if the representative vectors* $\{t_j\}_{j=1}^d$ *satisfy*

$$\sum_{j=1}^d \alpha_j t_j \in \mathscr{D}(T_0) \Longrightarrow \alpha_j = 0 \quad (j = 1, 2, \ldots, d);$$

that is to say, $\{t_j\}_{j=1}^d$ *is linearly independent modulo* $\mathscr{D}(T_0)$.

Proof. The equation $\sum_{j=1}^d \alpha_j \phi t_j = 0$ is equivalent to $\sum_{j=1}^d \alpha_j t_j \in \mathscr{D}(T_0)$. $\square$

The following lemma generalizes the characterization of the domain of the minimal operator; see (2.8).

LEMMA 3.3. $\mathscr{D}(T_0) = \{x \in \mathscr{D}(T_1) \mid [x,y]_H = 0 \ (y \in \mathscr{D}(T_1))\}$.

Proof. Fix $x \in \mathscr{D}(T_1)$ and suppose

$$[x,y]_H = 0 \quad (y \in \mathscr{D}(T_1)).$$

Since $[x,y]_H = -\overline{[y,x]}_H$, we see that $\langle T_1 y, x \rangle_H = \langle y, T_1 x \rangle_H$ so $x \in \mathscr{D}(T_1^*) = \mathscr{D}(T_0)$. Conversely, let $x \in \mathscr{D}(T_0)$. Since $T_0^* = T_1$ and $T_0 x = T_1 x$, we see that

$$\langle T_1 x, y \rangle_H = \langle T_0 x, y \rangle_H = \langle x, T_1 y \rangle_H \quad (y \in \mathscr{D}(T_1));$$

that is, for each $y \in \mathscr{D}(T_1)$,

$$[x,y]_H = \langle T_1 x, y \rangle_H - \langle x, T_1 y \rangle_H = 0. \quad \square$$

This result allows the boundary space S' to be equipped with a complex symplectic form.

DEFINITION 3.4. (Boundary Space Symplectic Form)

$$[\phi x : \phi y]_{S'} := [x,y]_H \quad (x,y \in \mathscr{D}(T_1)). \tag{3.3}$$

Lemma 3.3 assures Definition 3.4 above is independent of the choice of representative vectors. Moreover, Lemma 3.3 establishes the non-degeneracy property of Definition 3.1. From the definition of a Lagrangian subspace, the following extension of Lemma 3.2 is clear.

PROPOSITION 3.1. *A collection of cosets* $\{\phi t_j\}_{j=1}^d$ *forms a basis for a d-dimensional Lagrangian subspace of the boundary space* S' *if and only if the representative vectors* $\{t_j\}_{j=1}^d$ *satisfy*

(a)

$$\sum_{j=1}^d \alpha_j t_j \in \mathscr{D}(T_0) \implies \alpha_j = 0 \ (j = 1,\ldots,d); \tag{3.4}$$

and

(b)

$$[t_i, t_j]_H = 0 \quad (i, j = 1,\ldots,d). \tag{3.5}$$

Notice that the properties (3.4) and (3.5) are identical to those conditions discussed in Theorem 2.3. Because of their importance in the special case when $d = \text{def}(T_0)$, we incorporate these two properties into the following definition.

DEFINITION 3.5. A collection of vectors $\{t_j \mid j = 1,\ldots,\text{def}(T_0)\} \subseteq \mathscr{D}(T_1)$ is called a GKN set for T_0 if

(i) the set $\{t_j \mid j = 1, \ldots, \mathrm{def}(T_0)\}$ is linearly independent modulo the minimal domain $\mathscr{D}(T_0)$; that is to say

$$\text{if } \sum_{j=1}^{\mathrm{def}(T_0)} \alpha_j t_j \in \mathscr{D}(T_0) \text{ then } \alpha_j = 0 \text{ for } j = 1, \ldots, \mathrm{def}(T_0); \qquad (3.6)$$

and

(ii) the set $\{t_j \mid j = 1, \ldots, \mathrm{def}(T_0)\}$ satisfies the symmetry conditions

$$[t_i, t_j]_H = 0 \qquad (i, j = 1, \ldots, \mathrm{def}(T_0)). \qquad (3.7)$$

REMARK 3.1. Observe that if $G \subseteq \mathscr{D}(T_1)$ is a GKN set for T_0, then any nonempty, proper subset $P \subseteq G$ is linearly independent modulo $\mathscr{D}(T_0)$ and satisfies the symmetry conditions in (3.7). We refer to P as a partial GKN set. However, we note that the only partial GKN sets P that we use in this manuscript are those which satisfy $\mathrm{card}(P) = \dim(W) \leqslant \mathrm{def}(T_0)$, where W is a complex finite-dimensional extension space; see Condition 4.1 in Section 4.

We now turn our attention to characterizing complete Lagrangians. A key result of Everitt and Markus in this setting is that not only do complete Lagrangians L exist (see [10, Equations (1.54) and (1.61)]) but their dimensions are precisely that of the deficiency index; that is,

$$\dim \mathsf{L} = \mathrm{def}(T_0) \qquad (3.8)$$

(see [10, Equation (3.9)]). Moreover,

LEMMA 3.4. *With* $\mathrm{def}(T_0) < \infty$, *a Lagrangian subspace* $\mathsf{L} \subseteq S'$ *is complete if and only if each of the two conditions hold:*

(i) $\dim \mathsf{L} = \mathrm{def}(T_0)$;

(ii) $\mathsf{L} = \{\phi x \mid [\phi x : \phi t_j]_{S'} = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\}$ *for some GKN set*

$$\{t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}.$$

Moreover, in this case,

$$\phi^{-1}\mathsf{L} = \{x \in \mathscr{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\}. \qquad (3.9)$$

Proof. Suppose $\mathsf{L} \subseteq S'$ *is complete. Then, by* (3.8), $\dim \mathsf{L} = \mathrm{def}(T_0)$ *establishing* (i). *By Lemma* 3.1,

$$\mathsf{L} = \{\phi x \mid [\phi x : \phi y]_{S'} = 0 \ (\phi y \in \mathsf{L})\}. \qquad (3.10)$$

Let $\{\phi t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}$ *be a basis for* L. *Then, by Proposition* 3.1, $\{t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}$ *is a GKN set for* T_0. *It follows from* (3.10), *that*

$$\mathsf{L} = \{\phi x \mid [\phi x : \phi t_j]_{S'} = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\}, \qquad (3.11)$$

proving (*ii*). *Lastly, using the identification in* (3.3) *along with the identity in* (3.11), (3.9) *is clear.*

Conversely, suppose (*i*) *and* (*ii*) *hold. It is straightforward to show that* L *is a subspace of* S'. *Clearly* (3.9) *follows from* (*ii*). *Moreover, since* $\{t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}$ *is a GKN set for* T_0, *we see that*

$$[\phi t_i : \phi t_j]_{S'} = [t_i, t_j]_H = 0.$$

It follows by taking linear combinations that L *is Lagrangian. Finally, from* (3.8), *we see that* L *is complete.* $\square$

The authors in [10, Theorem 1.14 and Remark 1.15] establish the following characterization of self-adjoint extensions of T_0 in terms of complete Lagrangian subspaces L of S'.

THEOREM 3.1. (The finite-dimensional GKN-EM theorem) *Let* T_0 *and* T_1 *be, respectively, the minimal and maximal operators as defined in Section* 2 *and let* S' *be given by* (3.2). *There exists a one-to-one correspondence between the set* $\{T\}$ *of all self-adjoint extensions of* T_0 *and the set* $\{\mathsf{L}\}$ *of all complete Lagrangians* $\mathsf{L} \subseteq \mathsf{S}'$. *More specifically,*

(a) *if* T *is a self-adjoint operator with* $T_0 \subseteq T \subseteq T_1$, *then*

$$\mathsf{L} := \{\phi x \in \mathsf{S}' \mid x \in \mathscr{D}(T)\} \tag{3.12}$$

   is a complete Lagrangian subspace of S' *of dimension* $\mathrm{def}(T_0)$. *Moreover,* $\phi^{-1}\mathsf{L} = \mathscr{D}(T)$.

(b) *If* L *is a complete Lagrangian subspace of* S', *then* L *has dimension* $\mathrm{def}(T_0)$. *Define*
$$\mathscr{D}(T) = \{x \in \mathscr{D}(T_1) \mid \phi x \in \mathsf{L}\}.$$
   Then $T : \mathscr{D}(T) \subset H \to H$ *given by*
$$Tx = T_1 x$$
$$x \in \mathscr{D}(T)$$
   is a self-adjoint operator satisfying $T_0 \subseteq T \subseteq T_1$. *Moreover,* $\phi^{-1}\mathsf{L} = \mathscr{D}(T)$.

Combining Theorem 3.1 with Lemmas 3.1 and 3.4, we are now in position to state and prove an important consequence of Theorem 3.1 which, for our purposes, is key to the results developed in the next section and in the examples of Section 5. We note that the next theorem is an *exact* generalization of the GKN theorem stated in Theorem 2.3.

THEOREM 3.2. (The finite-dimensional symplectic GKN-EM theorem) *Suppose* T_0 *and* T_1 *are linear operators satisfying the conditions set forth in Section* 2 *and* $[\cdot, \cdot]_H$ *is the symplectic form defined in Definition* 3.3. *In particular, we assume* T_0 *has equal and finite deficiency indices denoted by* $\mathrm{def}(T_0)$.

(i) If the operator $T : \mathscr{D}(T) \subseteq H \to H$ is self-adjoint and satisfies

$$T_0 \subseteq T \subseteq T_1$$

then there exists a GKN set $\{t_j \mid j = 1, \ldots, \mathrm{def}(T_0)\} \subseteq \mathscr{D}(T_1)$ for T_0 such that

$$\mathscr{D}(T) = \{x \in \mathscr{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, \ldots, \mathrm{def}(T_0))\}. \qquad (3.13)$$

(ii) If $\{t_j \mid j = 1, \ldots, \mathrm{def}(T_0)\} \subseteq \mathscr{D}(T_1)$ is a GKN set for T_0 then the operator $T : \mathscr{D}(T) \subseteq H \to H$ given by

$$Tx = T_1 x \qquad (3.14)$$
$$x \in \mathscr{D}(T) = \{x \in \mathscr{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, \ldots, \mathrm{def}(T_0))\} \qquad (3.15)$$

is self-adjoint and satisfies

$$T_0 \subseteq T \subseteq T_1.$$

Proof. (i) Suppose $T : \mathscr{D}(T) \subseteq H \to H$ is self-adjoint and satisfies $T_0 \subseteq T \subseteq T_1$. By Theorem 3.1,

$$L = \{\phi x \in S' \mid x \in \mathscr{D}(T)\} \qquad (3.16)$$

is a complete Lagrangian subspace of S' of dimension $\mathrm{def}(T_0)$ from which it follows that

$$\phi^{-1} L = \mathscr{D}(T). \qquad (3.17)$$

Moreover, by Lemma 3.4, there exists a GKN set $\{t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}$ for T_0 such that

$$L = \{\phi x \mid [\phi x : \phi t_j]_{S'} = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\}$$

and

$$\phi^{-1} L = \{x \in \mathscr{D}(T_1) \mid [x, t_j]_H = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\}. \qquad (3.18)$$

Comparing (3.17) with (3.18), we obtain (3.13).

(ii) Suppose $\{t_j \mid j = 1, 2, \ldots, \mathrm{def}(T_0)\}$ is a GKN set for T_0. Let

$$L = \{\phi x \mid [\phi x : \phi t_j]_{S'} = 0 \ (j = 1, 2, \ldots, \mathrm{def}(T_0))\}. \qquad (3.19)$$

By Lemma 3.4, L is a complete Lagrangian subspace of S' of dimension $\mathrm{def}(T_0)$. Define T as in (3.14) and (3.15). Then, from (3.15) and (3.19), we see that

$$L = \{\phi x \mid x \in \mathscr{D}(T)\}$$

so that

$$\mathscr{D}(T) = \phi^{-1} L = \{x \in \mathscr{D}(T_1) \mid \phi x \in L\}.$$

By Theorem 3.1, T is self-adjoint and $T_0 \subseteq T \subseteq T_1$. $\quad \square$

REMARK 3.2. In the case that $H = L^2(I; w)$ and T_0 and T_1 are, respectively, the minimal and maximal operators L_0 and L_1, generated by the ordinary differential expression (2.7), Theorem 3.2 is <u>identical</u> to the classical GKN theorem given in Theorem 2.3. Again, it is remarkable that the GKN theorem extends verbatim to a general Hilbert space with an arbitrary closed symmetric operator having equal deficiency indices. As in the classical GKN setting, we also call the conditions

$$[x, t_j]_H = 0 \text{ for } j = 1, \ldots, \text{def}(T_0)$$

'boundary conditions'. Lastly, we note that, as in Remark 2.3, if $\text{def}(T_0) = 0$, there are no such boundary conditions and, in this case, the only self-adjoint extension of T_0 is the maximal operator $T_1 \ (= T_0)$.

REMARK 3.3. Everitt and Markus discuss other important applications of their results to ordinary and partial differential operators. We refer the reader to Sections 2.1, 2.2 and 4.2 in [10]. They outline the argument given above in Theorem 3.2 for Sturm-Liouville problems (see [10, Section 2, equations (2.23), (2.24), and (2.25)]) as well as for general Shin-Zettl quasi-differential operators (see [10, Section 4; in particular equations (4.57)–(4.61)]). The authors are certain that the most general situation (when T_0 has finite and equal deficiency indices), which we prove in Theorem 3.2, was known to Everitt and Markus but we cannot find an exact reference in their joint work.

4. Maximal and minimal operators in $H \oplus W$

We remind the reader that $T_0 : \mathscr{D}(T_0) \subseteq H \to H$ is a closed, symmetric operator with equal, finite deficiency indices $\text{def}(T_0)$ and adjoint operator T_1 satisfying $T_1^* = T_0 \subseteq T_0^* = T_1$. In this section, we identify a *family* of minimal operators $\widehat{T}_0 : \mathscr{D}(\widehat{T}_0) \subseteq H \oplus W \to H \oplus W$ and an associated family of *maximal* operators $\widehat{T}_1 : \mathscr{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$ in the extended space $H \oplus W$ generated by, respectively, the minimal operator T_0 and the maximal operator T_1 in the base space H. We show that each $\widehat{T}_0$ is a closed, symmetric operator in $H \oplus W$ with equal deficiency indices and $\text{def}(\widehat{T}_0) = \text{def}(T_0)$. Moreover, the operators $\widehat{T}_0$ and $\widehat{T}_1$ are adjoints of each other just as in the classical case with T_0 and T_1.

A fundamental assumption in our development of the maximal and minimal operators in $H \oplus W$ is the following dimensionality requirement for the extension space:

CONDITION 4.1. $\dim(W) \leqslant \text{def}(T_0)$.

Fix a partial GKN set

$$\{t_j \mid j = 1, \ldots, \dim(W)\} \subseteq \mathscr{D}(T_1); \tag{4.1}$$

recall, from Remark 3.1 and Condition 4.1, that this set exists and satisfies the two conditions

$$\sum_{j=1}^{\dim(W)} \alpha_j t_j \in \mathscr{D}(T_0) \Longrightarrow \alpha_j = 0 \qquad (j = 1, \ldots, \dim(W)) \tag{4.2}$$

and

$$[t_i, t_j]_H = 0 \qquad (i, j = 1, \ldots, \dim(W)). \tag{4.3}$$

It is clear that the maximal operator T_1 in the base space is symmetric on

$$\Delta_0 := \mathscr{D}(T_0) + \operatorname{span}\{t_j \mid j = 1, \ldots, \dim(W)\} \subseteq \mathscr{D}(T_1). \tag{4.4}$$

Now let

$$\{\xi_j \mid j = 1, \ldots, \dim(W)\} \subseteq W$$

be an orthonormal basis of W and define $\Psi : \Delta_0 \to W$ by

$$\begin{aligned}
\Psi(t_j) &= \xi_j \quad (j = 1, \ldots, \dim(W)) \\
\Psi(s) &= 0 \quad (s \in \mathscr{D}(T_0)).
\end{aligned} \tag{4.5}$$

and extend Ψ linearly to Δ_0; that is to say

$$\Psi\left(s + \sum_{j=1}^{\dim(W)} \alpha_j t_j\right) = \sum_{j=1}^{\dim(W)} \alpha_j \xi_j.$$

Note the key fact that Ψ maps the partial GKN set $\{t_j \mid j = 1, \ldots, \dim(W)\}$ onto W.

Lastly, fix an arbitrary self-adjoint operator $B : W \to W$ in the extension space W. With these definitions and conditions in place, we are now in position to define a *minimal operator* $\widehat{T}_0$ in $H \oplus W$ generated by T_0.

DEFINITION 4.1. The minimal operator $\widehat{T}_0 : \mathscr{D}\left(\widehat{T}_0\right) \subseteq H \oplus W \to H \oplus W$ is defined to be

$$\widehat{T}_0(x, a) = (T_1 x, Ba) \tag{4.6}$$

$$(x, a) \in \mathscr{D}(\widehat{T}_0) := \{(x, \Psi x) \mid x \in \Delta_0\}. \tag{4.7}$$

At this point, it is unclear why we call $\widehat{T}_0$ the *minimal* operator generated by T_0; we will justify this terminology in Remark 4.1. In Theorem 4.1 below we show that the minimal operator $\widehat{T}_0$ is, in fact, a densely defined operator which is both closed and symmetric. Moreover, in Theorem 4.2, where it is shown that $(\widehat{T}_0)^* = \widehat{T}_1$, we introduce the important linear transformation $\Omega : \mathscr{D}(T_1) \to W$ defined by

$$\Omega x := \sum_{j=1}^{\dim(W)} [x, t_j]_H \xi_j \quad (x \in \mathscr{D}(T_1)). \tag{4.8}$$

Observe, by definition of the partial GKN set $\{t_j \mid j = 1, \ldots, \dim W\}$ and Lemma 3.3, that

$$\Omega x = 0 \quad (x \in \Delta_0). \tag{4.9}$$

With this transformation Ω, we are now ready to introduce the maximal operator $\widehat{T}_1$.

DEFINITION 4.2. *The maximal operator* $\widehat{T}_1 : \mathscr{D}\left(\widehat{T}_1\right) \subseteq H \oplus W \to H \oplus W$ *is defined by*

$$\widehat{T}_1(x,a) = (T_1 x, Ba - \Omega x) \tag{4.10}$$

$$(x,a) \in \mathscr{D}(\widehat{T}_1) := \{(x,a) \mid x \in \mathscr{D}(T_1), a \in W\}. \tag{4.11}$$

Note that if $(x, \Psi x) \in \mathscr{D}(\widehat{T}_0)$, then $(x, \Psi x) \in \mathscr{D}(\widehat{T}_1)$. Moreover, in this case, $\Omega x = 0$ by (4.9) so

$$\widehat{T}_1(x, \Psi x) = (T_1 x, B\Psi x - \Omega x) = (T_1 x, B\Psi x) = \widehat{T}_0(x, \Psi x);$$

that is

$$\widehat{T}_0 \subseteq \widehat{T}_1. \tag{4.12}$$

REMARK 4.1. The term 'maximal' is appropriate; indeed, observe that $\mathscr{D}(\widehat{T}_1)$ is the largest linear manifold in $H \oplus W$ on which an operator representation of T_1 makes sense. Moreover, once we establish the fact that $(\widehat{T}_0)^* = \widehat{T}_1$, we see that the term 'minimal' is appropriate for the operator $\widehat{T}_0$.

PROPOSITION 4.1. *The extension* $J : \mathscr{D}(J) \subseteq H \to H$ *of the minimal operator* T_0, *defined by*

$$Jx := T_1 x$$
$$x \in \mathscr{D}(J) := \Delta_0,$$

is a closed symmetric operator.

Proof. Since T_0 is densely defined and $\mathscr{D}(T_0) \subseteq \Delta_0$, it is clear that $\mathscr{D}(J)$ is dense in H. Now, from Lemma 3.3 and (3.7), we see that

$$[x,y]_H = 0 \quad (x,y \in \Delta_0 \subseteq \mathscr{D}(T_1)).$$

Hence, from Definition 3.3,

$$0 = [x,y]_H = \langle T_1 x, y \rangle_H - \langle x, T_1 y \rangle_H = \langle Jx, y \rangle_H - \langle x, Jy \rangle_H \quad (x,y \in \mathscr{D}(J)),$$

establishing that J is symmetric in H. To show that J is closed, suppose first that $\dim(\Delta_0) \pmod{\mathscr{D}(T_0)} = 1$; that is

$$\mathscr{D}(J) = \mathscr{D}(T_0) + \operatorname{span}\{t_1\}, \tag{4.13}$$

where $t_1 \in \mathscr{D}(T_1) \backslash \mathscr{D}(T_0)$. Consider a sequence $\{x_j\} \subseteq \Delta_0$ and vectors $x, y \in H$ such that $x_j \to x$ and $T_1 x_j = Jx_j \to y$ where the convergence of both sequences is in H. Of course, we need to show

$$x \in \mathscr{D}(J) \tag{4.14}$$

and

$$Jx = y. \tag{4.15}$$

Since $\mathscr{D}(J) \subseteq \mathscr{D}(T_1)$ and T_1 is closed, we know that $x \in \mathscr{D}(T_1)$ and $T_1 x = y$. Hence (4.15) will be established once we show (4.14). Now, by (4.13) and Theorem 2.1, we can write

$$x_j = v_j + \alpha_j t_1 \tag{4.16}$$
$$= (\alpha_j t_1^0 + v_j) + \alpha_j t_1^+ + \alpha_j t_1^- \quad (j \in \mathbb{N})$$

where $t_1 = t_1^0 + t_1^+ + t_1^-$ and v_j, $t_1^0 \in \mathscr{D}(T_0)$, $t_1^\pm \in X_\pm$ and $x = x^0 + x^+ + x^-$, where $x^0 \in \mathscr{D}(T_0)$ and $x^\pm \in X_\pm$. Since $x_j \to x$ and $T_1 x_j \to T_1 x$, we see that

$$\left\| x_j - x \right\|_H^* \to 0 \text{ as } j \to \infty,$$

where $\|\cdot\|_H^*$ is the graph norm given in (2.3). Since

$$\left(\left\| x_j - x \right\|_H^* \right)^2 = \left(\left\| \alpha_j t_1^0 + v_j - x^0 \right\|_H^* \right)^2 + \left(\left\| \alpha_j t_1^+ - x^+ \right\|_H^* \right)^2 + \left(\left\| \alpha_j t_1^- - x^- \right\|_H^* \right)^2,$$

we see, from (2.3) and (2.4), that

$$\alpha_j t_1^0 + v_j \to x^0 \text{ in } H \tag{4.17}$$
$$\alpha_j t_1^+ \to x^+ \text{ in } H \tag{4.18}$$
$$\alpha_j t_1^- \to x^- \text{ in } H. \tag{4.19}$$

Since t_1^+ and t_1^- both cannot be zero (otherwise, $t_1 = t_1^0 \in \mathscr{D}(T_0)$, contradicting our choice of t_1), we see from either (4.18) or (4.19) that there exists $\alpha \in \mathbb{C}$ with $\alpha_j \to \alpha$. It follows that $\alpha_j t_1 \to \alpha t_1$ in H. Then, from (4.16) and (4.17), we see that

$$x_j = v_j + \alpha_j t_1 = \alpha_j t_1^0 + v_j + \alpha_j t_1 - \alpha_j t_1^0 \to (x^0 - \alpha t_1^0) + \alpha t_1 \in \mathscr{D}(T_0) + \operatorname{span}\{t_1\}.$$

Hence we see that $x = (x^0 - \alpha t_1^0) + \alpha t_1 \in \mathscr{D}(J)$, as required. The general proof of this proposition follows by induction on $\dim(W)$. $\quad\square$

REMARK 4.2. Proposition 4.1 shows that, on Δ_0, the maximal operator T_1 is a closed, symmetric operator. Of course, T_1 is not, in general, symmetric on $\mathscr{D}(T_1)$.

THEOREM 4.1. *The operator $\widehat{T}_0$ is a closed, densely defined symmetric operator in $H \oplus W$.*

Proof. (i) $\widehat{T}_0$ is Hermitian:
Let $(x, \Psi x)$, $(y, \Psi y) \in \mathscr{D}(\widehat{T}_0)$. Then, by Proposition 4.1 and the fact that B is symmetric in W, we see that

$$\left\langle \widehat{T}_0(x, \Psi x), (y, \Psi y) \right\rangle_{H \oplus W} = \langle (T_1 x, B\Psi x), (y, \Psi y) \rangle_{H \oplus W} = \langle T_1 x, y \rangle_H + \langle B\Psi x, \Psi y \rangle_W$$

$$= \langle x, T_1 y \rangle_H + \langle \Psi x, B\Psi y \rangle_W = \left\langle (x, \Psi x), \widehat{T}_0(y, \Psi y) \right\rangle_{H \oplus W}.$$

Hence $\widehat{T}_0$ is Hermitian.

(ii) $\underline{\mathscr{D}(\widehat{T}_0)}$ is dense in $H \oplus W$:

Since $\mathscr{D}(T_0)$ is dense in H and Ψ is surjective, it is clear that $\mathscr{D}(\widehat{T}_0)$ is dense in $H \oplus W$.

(iii) $\underline{\widehat{T}_0}$ is symmetric in $H \oplus W$:

This follows immediately from (i) and (ii).

(iv) $\underline{\widehat{T}_0}$ is closed in $H \oplus W$:

Suppose that $\{(x_n, \Psi x_n)\} \subseteq \mathscr{D}(\widehat{T}_0)$ is such that

$$(x_n, \Psi x_n) \to (x, a) \text{ in } H \oplus W \tag{4.20}$$

and

$$\widehat{T}_0(x_n, \Psi x_n) \to (y, b) \text{ in } H \oplus W. \tag{4.21}$$

These conditions in (4.20) and (4.21) are equivalent to

$$x_n \to x \text{ and } T_1 x_n \to y \text{ in } H \tag{4.22}$$

and

$$\Psi x_n \to a \text{ and } B\Psi x_n \to b \text{ in } W. \tag{4.23}$$

We need to show that $(x, a) \in \mathscr{D}(\widehat{T}_0)$ and $\widehat{T}_0(x, a) = (y, b)$; that is to say, we need to prove:

$$x \in \Delta_0 \tag{4.24}$$

$$T_1 x = y \tag{4.25}$$

$$\Psi x = a \tag{4.26}$$

and

$$Ba = b. \tag{4.27}$$

Since $\{x_n\} \subseteq \Delta_0$, we see that $T_1 x_n = J x_n$ so, by Proposition 4.1,

$$x \in \Delta_0 \text{ and } J x = T_1 x = y,$$

establishing (4.24) and (4.25). For the remainder of this proof, write

$$x = x_0 + t \tag{4.28}$$

and

$$x_n = x_{n,0} + t_n, \tag{4.29}$$

where $x_0,\ x_{n,0} \in \mathscr{D}(T_0)$,

$$t = \sum_{j=1}^{\dim W} \alpha_j t_j,$$

and

$$t_n = \sum_{j=1}^{\dim W} \alpha_{n,j} t_j.$$

From (4.29) and the definition of Ψ, we see that

$$\Psi x_n = \Psi(x_{n,0} + t_n) = \Psi t_n = \sum_{j=1}^{\dim W} \alpha_{n,j} \xi_j$$

so that, from (4.23),

$$\alpha_{n,j} = (\Psi x_n, \xi_j)_W \to \langle a, \xi_j \rangle_W.$$

It follows that

$$t_n \to \widehat{t} := \sum_{j=1}^{\dim W} \langle a, \xi_j \rangle_W t_j \qquad (4.30)$$

so that $T_1 t_n \to T_1 \widehat{t}$. Notice that

$$\Psi t_n \to \Psi \widehat{t} = \sum_{j=1}^{\dim W} \langle a, \xi_j \rangle_W \xi_j = a. \qquad (4.31)$$

From (4.22), (4.29) and (4.30), we deduce that

$$x_{n,0} = x_n - t_n \to x - \widehat{t} = x_0 + t - \widehat{t} \text{ in } H$$
$$T_0 x_{n,0} = T_1 x_n - T_1 t_n \to y - T_1 \widehat{t} \text{ in } H.$$

Since T_0 is closed, we see that $x_0 + t - \widehat{t} \in \mathscr{D}(T_0)$ and, in particular, that $t - \widehat{t} \in \mathscr{D}(T_0)$. By definition of the partial GKN set $\{t_j \mid 1 \leqslant j \leqslant \dim W\}$, we must have

$$t = \widehat{t}. \qquad (4.32)$$

Combining (4.31) and (4.32), we obtain

$$\Psi x_n = \Psi t_n \to \Psi \widehat{t} = \Psi t = \Psi(x - x_0) = \Psi x = a;$$

establishing (4.26). Finally,

$$B\Psi x_n \to B\Psi x = Ba$$

so, by (4.23), $Ba = b$ which proves (4.27). This completes the proof that $\widehat{T_0}$ is closed. $\square$

This brings us to the proof of the fundamental relation between the maximal and minimal operators $\widehat{T_1}$ and $\widehat{T_0}$.

THEOREM 4.2. $(\widehat{T_0})^* = \widehat{T_1}$.

Proof. For $(x,a), (y,b) \in \mathscr{D}(\widehat{T_1})$, a calculation shows that

$$\left\langle \widehat{T_1}(x,a),(y,b) \right\rangle_{H \oplus W} - \left\langle (x,a), \widehat{T_1}(y,b) \right\rangle_{H \oplus W} = [x,y]_H - \langle \Omega x, b \rangle_W + \langle a, \Omega y \rangle_W. \quad (4.33)$$

(see Definition 4.50). Notice that when $y = t_k$ and $b = \xi_k$, we obtain

$$\langle \Omega x, \xi_j \rangle_W = \sum_{k=1}^{\dim W} \langle [x, t_k]_H \xi_k, \xi_j \rangle_W = [x, t_j]_H \tag{4.34}$$

since $\{\xi_j \mid j = 1, \ldots, \dim W\}$ is an orthonormal basis of W. Suppose now that $(y, b) \in \mathcal{D}(\widehat{T}_0)$ so $y \in \Delta_0$ and $b = \Psi y$. Then

$$y = y_0 + \widetilde{t},$$

where $y_0 \in \mathcal{D}(T_0)$,

$$\widetilde{t} := \sum_{j=1}^{\dim W} \alpha_j t_j$$

and

$$b = \Psi y = \Psi \widetilde{t} = \sum_{j=1}^{\dim W} \alpha_j \xi_j.$$

By (4.12), $\widehat{T}_0(y, \Psi y) = \widehat{T}_1(y, \Psi y)$ so, from (4.33), we obtain

$$\left\langle \widehat{T}_1(x, a), (y, \Psi y) \right\rangle_{H \oplus W} - \left\langle (x, a), \widehat{T}_0(y, \Psi y) \right\rangle_{H \oplus W}$$
$$= [x, y]_H - \langle \Omega x, b \rangle_W + \langle a, \Omega y \rangle_W$$
$$= [x, y_0]_H + [x, \widetilde{t}]_H - \langle \Omega x, \Psi y_0 \rangle_W - \langle \Omega x, \Psi \widetilde{t} \rangle_W + \langle a, \Omega y \rangle_W. \tag{4.35}$$

We now deal with each of the five terms in (4.35). First, from Lemma 3.3,

$$[x, y_0]_H = 0. \tag{4.36}$$

From (4.34), we see that

$$[x, \widetilde{t}]_H - \langle \Omega x, \Psi \widetilde{t} \rangle_W = \sum_{j=1}^{\dim W} \alpha_j \{ [x, t_j]_H - \langle \Omega x, \xi_j \rangle_W \} = 0. \tag{4.37}$$

From (4.5), $\Psi y_0 = 0$ so

$$\langle \Omega x, \Psi y_0 \rangle_W = 0. \tag{4.38}$$

Likewise, from (4.9), we see that $\Omega y = 0$ so

$$\langle a, \Omega y \rangle_W = 0. \tag{4.39}$$

Together, (4.36), (4.37), (4.38) and (4.39) show that

$$\left\langle \widehat{T}_1(x, a), (y, \Psi y) \right\rangle_{H \oplus W} = \left\langle (x, a), \widehat{T}_0(y, \Psi y) \right\rangle_{H \oplus W} \tag{4.40}$$

for $(x, a) \in \mathcal{D}(\widehat{T}_1)$ and $(y, \Psi y) \in \mathcal{D}(\widehat{T}_0)$. Hence, we obtain

$$\widehat{T}_1 \subseteq (\widehat{T}_0)^*. \tag{4.41}$$

To show $(\widehat{T}_0)^* \subseteq \widehat{T}_1$, let $(x,a) \in \mathscr{D}((\widehat{T}_0)^*)$ and set $(x^*,a^*) = (\widehat{T}_0)^*(x,a)$. Then for $(y, \Psi y) \in \mathscr{D}(\widehat{T}_0)$,

$$\langle (x^*,a^*),(y,\Psi y) \rangle_{H \oplus W} = \left\langle (\widehat{T}_0)^*(x,a),(y,\Psi y) \right\rangle_{H \oplus W}$$
$$= \left\langle (x,a),\widehat{T}_0(y,\Psi y) \right\rangle_{H \oplus W} \qquad (4.42)$$

since $\widehat{T}_0$ is closed. Written out, the identity in (4.42) gives

$$\langle x^*,y \rangle_H + \langle a^*, \Psi y \rangle_W = \langle x, T_0 y \rangle_H + \langle a, B\Psi y \rangle_W. \qquad (4.43)$$

In particular, if $y \in \mathscr{D}(T_0)$, then (4.43) reduces to

$$\langle x^*,y \rangle_H = \langle x, T_0 y \rangle_H.$$

Thus $x \in \mathscr{D}(T_0^*) = \mathscr{D}(T_1)$ and
$$T_1 x = x^*. \qquad (4.44)$$

Substituting (4.44) into (4.43) and recalling that B is symmetric in W yields

$$\langle a^*, \Psi y \rangle_W = \langle x, T_0 y \rangle_H + \langle a, B\Psi y \rangle_W - \langle T_1 x, y \rangle_H$$
$$= -[x,y]_H + \langle Ba, \Psi y \rangle_W. \qquad (4.45)$$

In particular, let $y = t_k$ so $\Psi y = \xi_k$. From (4.34), we see that $[x,t_k]_H = \langle \Omega x, \xi_k \rangle_W$. Hence, we find that (4.45) becomes

$$\langle a^*, \xi_k \rangle_W = -\langle \Omega x, \xi_k \rangle_W + \langle Ba, \xi_k \rangle_W \quad (k = 1,2,\ldots,\dim W). \qquad (4.46)$$

Since $\{\xi_k \mid k = 1,\ldots,\dim W\}$ is a basis for W, we can conclude from (4.46) that

$$a^* = Ba - \Omega x. \qquad (4.47)$$

Consequently, from (4.44) and (4.47), we see that

$$(x^*,a^*) = (T_1 x, Ba - \Omega x) = \widehat{T}_1(x,a)$$

so
$$(\widehat{T}_0)^* \subseteq \widehat{T}_1. \qquad (4.48)$$

Combining (4.41) and (4.48), we obtain $(\widehat{T}_0)^* = \widehat{T}_1$.   □

Together Theorem 4.1 and Theorem 4.2 establish the following fundamental operator relationship between $\widehat{T}_0$ and $\widehat{T}_1$.

THEOREM 4.3. $\widehat{T}_0 = \overline{\widehat{T}_0} \subseteq (\widehat{T}_0)^* = \widehat{T}_1.$

Consequently we may apply the Stone-von Neumann theory to the minimal operator $\widehat{T}_0$. Accordingly we define the *positive* and *negative deficiency spaces* associated with $\widehat{T}_0$ in $H \oplus W$.

DEFINITION 4.3. (Deficiency spaces in the extended space $H \oplus W$)

$$Y_{\pm} := \{(x,a) \in \mathscr{D}(\widehat{T_1}) \mid \widehat{T_1}(x,a) = \pm i(x,a)\}.$$

Remarkably, as we shall see in the next result, the deficiency spaces $Y_{\pm} \subseteq H \oplus W$ and $X_{\pm} \subseteq H$ are *isomorphic*. We note that, since $B : W \to W$ is self-adjoint, then $B \pm iI$ is invertible.

LEMMA 4.1. $(x,a) \in Y_{\pm}$ *if and only if* $x \in X_{\pm}$ *and* $a = (B \mp iI)^{-1}\Omega x$. *Moreover, the deficiency indices of* $\widehat{T_0}$ *are equal and finite and satisfy* $\mathrm{def}(\widehat{T_0}) = \mathrm{def}(T_0)$.

Proof. Let $(x,a) \in Y_{\pm}$. Then $T_1 x = \pm ix$ and $Ba - \Omega x = \pm ia$. Therefore $x \in X_{\pm}$ and $a = (B \mp iI)^{-1}\Omega x$. Conversely if $x \in X_{\pm}$ and $a = (B \mp iI)^{-1}\Omega x$ then $Ba - \Omega x = \pm ia$ so $\widehat{T_1}(x,a) = \pm i(x,a)$. We see that the mappings $X_{\pm} \to Y_{\pm}$ given by $x \to (x, (B \mp iI)^{-1}\Omega x)$ are vector space isomorphisms. In particular, $\dim(X_{\pm}) = \dim(Y_{\pm})$. This shows that the deficiency indices of the minimal operator $\widehat{T_0}$ are finite and equal with

$$\dim(Y_+) = \dim(Y_-) = \mathrm{def}(\widehat{T_0}) < \infty. \quad \square \qquad (4.49)$$

In particular, equation (4.49) guarantees the GKN-EM theorem applies to $\widehat{T_0}$. We now define the (degenerate) symplectic form in $H \oplus W$ associated with the operators $\widehat{T_0}$ and $\widehat{T_1}$. We remark that, in equation (4.33), we actually already computed this symplectic form.

DEFINITION 4.4. (General symplectic form)

$$[(x,a),(y,b)]_{H \oplus W} := [x,y]_H - \langle \Omega x, b \rangle_W + \langle a, \Omega y \rangle_W \qquad ((x,a),(y,b) \in \mathscr{D}(\widehat{T_1})), \tag{4.50}$$

where $[\cdot,\cdot]_H$ is the symplectic form defined in (3.3) and where the mapping Ω is defined in (4.8).

We are now in position to apply the GKN-EM theorem (Theorem 3.2) to the minimal operator $\widehat{T_0}$ in $H \oplus W$ and, as a result, characterize all self-adjoint extensions (respectively, restrictions) of $\widehat{T_0}$ (respectively, the maximal operator $\widehat{T_1}$).

THEOREM 4.4. (GKN-EM theorem in $H \oplus W$) *We have the following assumptions/definitions*:

(i) T_0 *and* T_1 *are, respectively, the minimal and maximal operators in the Hilbert space* $(H, \langle \cdot, \cdot \rangle_H)$, *called the base (complex) Hilbert space, with domains* $\mathscr{D}(T_0)$ *and* $\mathscr{D}(T_1)$; T_0 *is a closed, symmetric operator satisfying* $T_0 \subseteq T_1$ *with* $T_0^* = T_1$ *and* $T_1^* = T_0$;

(ii) *The deficiency indices of* T_0 *are assumed to be equal and finite and denoted by* $\mathrm{def}(T_0)$;

(iii) $[\cdot,\cdot]_H$ *is the symplectic form given by*

$$[x,y]_H = \langle T_1 x, y \rangle_H - \langle x, T_1 y \rangle_H \quad (x,y \in \mathscr{D}(T_1)),$$

(*see Definition* 3.3);

(iv) $(W, \langle \cdot, \cdot \rangle_W)$, *the extension space, is a finite dimensional complex Hilbert space with* $\dim W \leqslant \operatorname{def}(T_0)$ (*Condition* 4.1) *and orthonormal basis* $\{\xi_j \mid j = 1, \ldots, \dim W\}$;

(v) $B : W \to W$ *is a self-adjoint operator;*

(vi) $H \oplus W$, *the extended space, is the Hilbert space defined in* (1.4) *with inner product* (1.5);

(vii) $P = \{t_j \mid j = 1, \ldots, \dim W\}$ *is a partial GKN set* (*see* (4.2) *and* (4.3));

(viii) $\Delta_0 = \mathscr{D}(T_0) + \operatorname{span}\{t_j \mid j = 1, \ldots, \dim W\}$ (*see* (4.4));

(ix) $\Psi : \Delta_0 \to W$ *is defined to be*

$$\Psi\left(x_0 + \sum_{j=1}^{\dim W} \alpha_j t_j\right) = \sum_{j=1}^{\dim W} \alpha_j \xi_j \quad (x_0 \in \mathscr{D}(T_0))$$

(*see* (4.5));

(x) $\Omega : \mathscr{D}(T_1) \to W$ *is defined to be*

$$\Omega x = \sum_{j=1}^{\dim W} [x, t_j]_H \xi_j$$

(*see* (4.8));

(xi) $\widehat{T}_0 : \mathscr{D}(\widehat{T}_0) \subseteq H \oplus W \to H \oplus W$ *is the minimal operator in* $H \oplus W$ *defined by*

$$\widehat{T}_0(x, a) = (T_1 x, Ba)$$
$$(x, a) \in \mathscr{D}(\widehat{T}_0) = \{(x, \Psi x) \mid x \in \Delta_0\}$$

(*see Definition* 4.1);

(xii) $\widehat{T}_1 : \mathscr{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$ *is the maximal operator in* $H \oplus W$ *defined by*

$$\widehat{T}_1(x, a) = (T_1 x, Ba - \Omega x)$$
$$\mathscr{D}(\widehat{T}_1) = \{(x, a) \mid x \in \mathscr{D}(T_1); a \in W\}$$

(*see* 4.2);

(xiii) $[\cdot,\cdot]_{H\oplus W}$ *is the symplectic form given by*

$$[(x,a),(y,b)]_{H\oplus W} := [x,y]_H - \langle\Omega x,b\rangle_W + \langle a,\Omega y\rangle_W \qquad ((x,a),(y,b)) \in \mathscr{D}(\widehat{T}_1),$$

(see 4.50*).*

Under these definitions and assumptions, we obtain the following results:

(a) $\widehat{T}_0$ *is a closed, symmetric operator satisfying* $\widehat{T}_0 \subseteq \widehat{T}_1$ *with* $(\widehat{T}_0)^* = \widehat{T}_1$ *and* $(\widehat{T}_1)^* = \widehat{T}_0$ *(see Theorems* 4.1 *and* 4.2*);*

(b) *The deficiency indices of* $\widehat{T}_0$ *are equal and finite and* $\mathrm{def}(\widehat{T}_0) = \mathrm{def}(T_0)$ *(see Lemma* 4.1*);*

(c) *Suppose* $\widehat{T}$ *is a self-adjoint extension of* $\widehat{T}_0$ *(equivalently,* $\widehat{T}$ *is a self-adjoint restriction of* $\widehat{T}_1$*) satisfying* $\widehat{T}_0 \subseteq \widehat{T} \subseteq \widehat{T}_1$. *Then there exists a GKN set* $\{(x_j,a_j) \mid j = 1,\ldots,\mathrm{def}(T_0)\} \subseteq \mathscr{D}(\widehat{T}_1)$ *(see Remark* 3.1*) satisfying the two conditions*

(α) $\{(x_j,a_j) \mid j = 1,\ldots,\mathrm{def}(T_0)\}$ *is linearly independent modulo* $\mathscr{D}(\widehat{T}_0)$,

(β) $[(x_j,a_j),(x_k,a_k)]_{H\oplus W} = 0$ *for* $j,k = 1,\ldots,\mathrm{def}(T_0)$

such that

$$\widehat{T}(x,a) = (T_1 x, Ba - \Omega x) \tag{4.51}$$
$$\mathscr{D}(\widehat{T}) = \{(x,a) \in \mathscr{D}(\widehat{T}_1) \mid [(x,a),(x_j,a_j)]_{H\oplus W} = 0 \ (j = 1,\ldots,\mathrm{def}(T_0))\}; \tag{4.52}$$

(d) *If* $\widehat{T}$ *is defined by* (4.51) *and* (4.52) *where* $\{(x_j,a_j) \mid j = 1,\ldots,\mathrm{def}(T_0)\} \subseteq \mathscr{D}(\widehat{T}_1)$ *is a GKN set satisfying conditions* (α) *and* (β), *then* $\widehat{T}$ *is a self-adjoint extension of* $\widehat{T}_0$ *(equivalently,* $\widehat{T}$ *is a self-adjoint restriction of* $\widehat{T}_1$*) in* $H \oplus W$.

5. Examples

5.1. Example 1: The Legendre type self-adjoint operator

Throughout this example, we let $H = L^2(-1,1)$ (with its usual inner product) and $W = \mathbb{C}^2$, endowed with the weighted Euclidean inner product

$$\langle(a_1,b_1),(a_2,b_2)\rangle_W = \frac{a_1\overline{a}_2 + b_1\overline{b}_2}{A} \qquad ((a_1,b_1),(a_2,b_2) \in W); \tag{5.1}$$

here A is a fixed, positive constant. Let $\{\xi_1,\xi_2\}$ be the orthonormal basis in W given by

$$\xi_1 = (\sqrt{A},0) \text{ and } \xi_2 = (0,\sqrt{A}) \tag{5.2}$$

and, for this example, suppose $B : W \to W$ is the zero self-adjoint operator.

In [3, 4, 12], the authors discuss the spectral analysis of the Legendre type differential expression defined earlier in (1.3); that is,

$$\ell_{LT}[y](u) := \left((1-u^2)^2 y''(u)\right)'' - \left((8+4A(1-u^2))y'(u)\right)' \tag{5.3}$$

where A is the same constant appearing in (5.1). This differential expression was first discovered by H. L. Krall [13, 14]. When

$$\lambda_n = n(n+1)(n^2+n+4A-2) \quad (n \in \mathbb{N}_0), \tag{5.4}$$

the equation $\ell_{LT}[y] = \lambda_n y$ has a polynomial solution $y = P_{n,A}(u)$ of degree n; that is,

$$\ell_{LT}[P_{n,A}] = \lambda_n P_{n,A} \quad (n \in \mathbb{N}_0). \tag{5.5}$$

The sequence $\{P_{n,A}\}_{n=0}^{\infty}$ is called the *Legendre type polynomials*; they form a complete orthogonal sequence in the Hilbert space $L_\mu^2[-1,1]$, where

$$L_\mu^2[-1,1] = \{f : [-1,1] \to \mathbb{C} \mid f \text{ is Lebesgue measurable}$$

$$\text{with } \int_{[-1,1]} |f|^2 \, d\mu < \infty\},$$

with inner product

$$\langle f, g \rangle_\mu = \frac{f(-1)\overline{g}(-1)}{A} + \int_{-1}^{1} f(u)\overline{g}(u)du + \frac{f(1)\overline{g}(1)}{A},$$

and where $d\mu$ is the Lebesgue-Stieltjes measure given by

$$d\mu = dx + \frac{1}{A}\delta(x+1) + \frac{1}{A}\delta(x-1).$$

When $y = P_{n,A}$, we see from (5.3) and (5.5) that

$$\mp 8A P_{n,A}'(\pm 1) = \lambda_n P_{n,A}(\pm 1). \tag{5.6}$$

Various properties of the Legendre type polynomials can be found in [12].

Because the measure μ has jumps at $u = \pm 1$, the classic GKN theory is not immediately applicable in finding a self-adjoint operator representation T of $\ell_{LT}[\cdot]$ in $L_\mu^2[-1,1]$. In order to construct T, Everitt and Littlejohn [3, 4] first studied properties of functions in the maximal domain

$$\mathscr{D}(T_1) = \{x : (-1,1) \to \mathbb{C} \mid x, x', x'', x''' \in AC_{\text{loc}}(-1,1);$$

$$x, \ell_{LT}[x] \in L^2(-1,1)\},$$

where T_1 is the maximal operator, generated by $\ell_{LT}[\cdot]$, in the Hilbert space $L^2(-1,1)$. They establish the remarkable smoothness property

$$x \in \mathscr{D}(T_1) \implies x'' \in L^2(-1,1) \tag{5.7}$$

and hence, upon making the natural identifications

$$x(\pm 1) = \lim_{u \to \pm 1} x(u) \qquad x'(\pm 1) = \lim_{u \to \pm 1} x'(u), \tag{5.8}$$

we can say that

$$x \in \mathscr{D}(T_1) \Longrightarrow x, x' \in AC[-1, 1].$$

Moreover, they prove that the associated sesquilinear form has the simple formulation

$$[x, y]_H = 8(x(1)\overline{y}'(1) - x'(1)\overline{y}(1) + x'(-1)\overline{y}(-1) - x(-1)\overline{y}'(-1)) \tag{5.9}$$

for $x, y \in \mathscr{D}(T_1)$. Considering this last formula and Lemma 3.3, it is apparent that the minimal domain associated with $\ell_{LT}[\cdot]$ is explicitly given by

$$\mathscr{D}(T_0) = \left\{ x \in \mathscr{D}(T_1) \mid x(\pm 1) = x'(\pm 1) = 0 \right\}. \tag{5.10}$$

The deficiency index of the minimal operator T_0, generated by $\ell_{LT}[\cdot]$, in $L^2(-1, 1)$ is $\mathrm{def}(T_0) = 2$. This follows since each endpoint $u = \pm 1$ is in the limit-3 case which can be shown by a Frobenius analysis. We emphasize that we are not seeking to find self-adjoint extensions of T_0 in $L^2(-1, 1)$ but instead we want to find a self-adjoint representation of $\ell_{LT}[\cdot]$ in $L_\mu^2[-1, 1]$ which produces the Legendre type polynomials $\{P_{n,A}\}_{n=0}^\infty$ as eigenfunctions. By analyzing functions in $\mathscr{D}(T_1)$, Everitt and Littlejohn show that the operator $T : \mathscr{D}(T) \subseteq L_\mu^2[-1, 1] \to L_\mu^2[-1, 1]$ defined by

$$Tx(u) = \begin{cases} -8Ax'(-1) & u = -1 \\ \ell_{LT}[x](u) & -1 < u < 1 \\ 8Ax'(1) & u = 1 \end{cases} \tag{5.11}$$
$$x \in \mathscr{D}(T) := \mathscr{D}(T_1)$$

is self-adjoint, has the Legendre type polynomials $\{P_{n,A}\}_{n=0}^\infty$ as eigenfunctions, and has discrete spectrum

$$\sigma(T) = \sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N}_0\},$$

where each λ_n is given in (5.4). It is surprising that the maximal domain $\mathscr{D}(T_1)$ is the domain of a self-adjoint operator in $L_\mu^2[-1, 1]$. By the GKN theorem, $\mathscr{D}(T_1)$ cannot be the domain of a self-adjoint extension of T_0 in $L^2(-1, 1)$.

We now show, using the results developed in this paper, how to construct the self-adjoint operator T given in (5.11) in the direct sum space $H \oplus W$. Indeed, below, we construct a self-adjoint operator $\widehat{T}$ that is, essentially, the operator T defined in (5.11). With this alternative approach, we will see how *continuity* is a GKN-EM boundary condition that produces the Legendre type self-adjoint operator $\widehat{T}$.

The first step in our analysis is to observe that the space $L_\mu^2[-1, 1]$ is isometrically isomorphic to the direct sum

$$H \oplus W = \{(x, (a, b)) \mid x \in H; (a, b) \in W\}.$$

Next define $t_j \in \mathscr{D}(T_1)$ $(j = 1,2)$ by

$$t_1(u) = \begin{cases} \sqrt{A} & u \text{ near} -1 \\ 0 & u \text{ near } 1 \end{cases} \quad t_2(u) = \begin{cases} 0 & u \text{ near} -1 \\ \sqrt{A} & u \text{ near } 1; \end{cases}$$

we remark that such functions in $\mathscr{D}(T_1)$ exist by Naimark's Patching Lemma [15, Lemma 2, Section 17.3]. It is straightforward to see, using (5.9) and (5.10), that $\{t_1, t_2\}$ is a GKN set for T_0. Consequently, we see that

$$\Delta_0 = \{x_0 + c_1 t_1 + c_2 t_2 \mid x_0 \in \mathscr{D}(T_0); c_1, c_2 \in \mathbb{C}\}, \tag{5.12}$$

where Δ_0 is defined in (4.4). Moreover,

$$\Psi(x_0 + c_1 t_1 + c_2 t_2) = c_1 \xi_1 + c_2 \xi_2 = \langle c_1 \sqrt{A}, c_2 \sqrt{A} \rangle, \tag{5.13}$$

where $\{\xi_1, \xi_2\}$ is defined in (5.2) and where $\Psi : \Delta_0 \to W$ is the map defined in (4.5). Using (5.9), calculations show that

$$[x, t_1]_H = 8\sqrt{A}x'(-1), \ [x, t_2]_H = -8\sqrt{A}x'(1) \quad (x \in \mathscr{D}(T_1)).$$

It follows that

$$\Omega x = 8\sqrt{A}x'(-1)\xi_1 - 8\sqrt{A}x'(1)\xi_2 = (8Ax'(-1), -8Ax'(1)) \quad (x \in \mathscr{D}(T_1)), \tag{5.14}$$

where $\Omega : \mathscr{D}(T_1) \to W$ is the mapping defined in (4.8).

The minimal operator $\widehat{T}_0 : \mathscr{D}(\widehat{T}_0) \subseteq H \oplus W \to H \oplus W$, in this example, is given by

$$\widehat{T}_0(x, \Psi x) = (T_1 x, B\Psi x) = (\ell_{LT}[x], (0,0)) \tag{5.15}$$

$$\mathscr{D}(\widehat{T}_0) = \{(x, \Psi x) \mid x \in \Delta_0\}. \tag{5.16}$$

From the theory we established in Section 4, $\widehat{T}_0$ is a closed, symmetric operator in $H \oplus W$ with $\operatorname{def}(\widehat{T}_0) = 2$.

Using (5.14), we see that the associated maximal operator $\widehat{T}_1 : \mathscr{D}(\widehat{T}_1) \subseteq H \oplus W \to H \oplus W$ is given explicitly by

$$\widehat{T}_1(x, (a,b)) = (\ell_{LT}[x], (-8Ax'(-1), 8Ax'(1))) \tag{5.17}$$

$$\mathscr{D}(\widehat{T}_1) = \{(x, (a,b)) \mid x \in \mathscr{D}(T_1); \ a, b \in \mathbb{C}\}. \tag{5.18}$$

From (5.1), (5.9) and (5.14), a calculation shows that the symplectic form $[\cdot, \cdot]_{H \oplus W}$, defined in (4.50), is given by

$$
\begin{aligned}
&[(x, (a_1, b_1)), (y, (a_2, b_2))]_{H \oplus W} \\
&= [x, y]_H - \langle \Omega x, (a_2, b_2) \rangle_W + \langle (a_1, b_1), \Omega y \rangle_W \\
&= 8(x(1)\overline{y}'(1) - x'(1)\overline{y}(1) + x'(-1)\overline{y}(-1) - x(-1)\overline{y}'(-1)) \\
&\quad - 8x'(-1)\overline{a}_2 + 8x'(1)\overline{b}_2 + 8A\overline{y}'(-1)a_1 - 8A\overline{y}'(1)b_1
\end{aligned}
\tag{5.19}
$$

for $(x,(a_1,b_1))$, $(y,(a_2,b_2)) \in \mathscr{D}(\widehat{T}_1)$. Define $x_j \in \mathscr{D}(T_1)$, for $j = 1,2$, by

$$x_1(u) = \begin{cases} 0 & u \text{ near} -1 \\ \sqrt{A}(u-1) & u \text{ near } 1 \end{cases}, \quad x_2(u) = \begin{cases} \sqrt{A}(u+1) & u \text{ near} -1 \\ 0 & u \text{ near } 1. \end{cases}$$

From (5.9) and (5.10), we see that $\{x_1, x_2\}$ is a GKN set for T_0. Calculations also show that

$$[x,x_1]_H = 8\sqrt{A}x(1) \text{ and } [x,x_2]_H = -8\sqrt{A}x(-1) \quad (x \in \mathscr{D}(T_1)). \tag{5.20}$$

In addition, we see that

$$\Omega x_1 = (0, -8A\sqrt{A}) \text{ and } \Omega x_2 = (8A\sqrt{A}, 0). \tag{5.21}$$

We now claim that

PROPOSITION 5.1. $\{(x_1, (0,0)), (x_2, (0,0))\}$ *is a GKN set for* $\widehat{T}_0$.

Proof. Suppose that

$$c_1(x_1, (0,0)) + c_2(x_2, (0,0)) = (c_1 x_1 + c_2 x_2, (0,0)) \in \mathscr{D}(\widehat{T}_0).$$

By definition of $\mathscr{D}(\widehat{T}_0)$, *we see that* $\Psi(c_1 x_1 + c_2 x_2) = (0,0)$. *This implies that* $c_1 x_1 + c_2 x_2 \in \mathscr{D}(T_0)$. *However since* $\{x_1, x_2\}$ *is a GKN set for* T_0, *we must have* $c_1 = c_2 = 0$. *Hence* $\{(x_1, (0,0)), \langle x_2, (0,0)\rangle\}$ *is linearly independent modulo* $\mathscr{D}(\widehat{T}_0)$. *Next, using* (5.19), *we see that*

$$[(x_1, (0,0)), (x_2, (0,0))]_{H \oplus W} = [x_1, x_2]_H = 0.$$

Calculations also show that

$$[(x_j, (0,0)), (x_j, (0,0))]_{H \oplus W} = 0 \quad (j = 1,2).$$

This completes the proof of the proposition. □

We now find the appropriate self-adjoint operator $\widehat{T}$ in $H \oplus W$ having the Legendre type polynomial vectors $\{(P_{n,A}, (P_{n,A}(-1), P_{n,A}(1)))\}_{n=0}^{\infty}$ as eigenfunctions. Indeed, using Theorem 4.4 part (d), the operator $\widehat{T} : \mathscr{D}(\widehat{T}) \subseteq H \oplus W \to H \oplus W$, defined by

$$\widehat{T}(x,(a,b)) = (\ell_{LT}[x], (-8Ax'(-1), 8Ax'(1))) \tag{5.22}$$
$$\mathscr{D}(\widehat{T}) = \{(x,(a,b)) \in \mathscr{D}(\widehat{T}_1) \mid [(x,(a,b)), (x_j, (0,0))]_{H \oplus W} = 0 \ (j = 1,2)\} \tag{5.23}$$

is self-adjoint.

We now investigate each of the two boundary conditions in (5.23). From (5.19), (5.20) and (5.21), a calculation shows that

$$0 = [(x,(a,b)), (x_1, (0,0))]_{H \oplus W} = [x,x_1]_H - \langle \Omega x, (0,0)\rangle_W + \langle (a,b), \Omega x_1 \rangle_W$$

$$= [x,x_1]_H + \langle (a,b), (0, -8A\sqrt{A})\rangle_W = 8\sqrt{A}x(1) - \frac{8A\sqrt{A}b}{A} = 8\sqrt{A}x(1) - 8\sqrt{A}b,$$

implying

$$b = x(1). \tag{5.24}$$

A similar calculation, using (5.20) and (5.21), yields

$$0 = [(x,(a,b)),(x_2,(0,0))]_{H \oplus W} = -8\sqrt{A}x(-1) + 8\sqrt{A}a$$

which establishes

$$a = x(-1). \tag{5.25}$$

Hence the domain of $\widehat{T}$, given in (5.23), simplifies to

$$\mathscr{D}(\widehat{T}) = \{(x,(x(-1),x(1))) \mid x \in \mathscr{D}(T_1)\}. \tag{5.26}$$

Notice that this domain (5.26) extends the continuity of each $x \in \mathscr{D}(T_1)$ from $(-1,1)$ to the closure $[-1,1]$. It is remarkable that, in this sense, *continuity* is a GKN-EM boundary condition. Furthermore, from (5.5), (5.6) and (5.22), notice that

$$\begin{aligned}
\widehat{T}(P_{n,A},(P_{n,A}(-1),P_{n,A}(1))) &= (\ell_{LT}[P_{n,A}],(-8AP'_{n,A}(-1),8AP'_{n,A}(1))) \\
&= (\lambda_n P_{n,A},(\lambda_n P_{n,A}(-1),\lambda_n P_{n,A}(1))) \\
&= \lambda_n(P_{n,A},(P_{n,A}(-1),P_{n,A}(1))).
\end{aligned}$$

Moreover, $\widehat{T}$ is the same operator as T defined in (5.11).

REMARK 5.1. If $B : W \to W$ is an arbitrary self-adjoint operator in W, the operator $\widehat{S} : \mathscr{D}(\widehat{S}) \subseteq H \oplus W \to H \oplus W$, defined by

$$\widehat{S}(x,(a,b)) = (\ell_{LT}[x],B(a,b) + (-8Ax'(-1),8Ax'(1)))$$

$$\mathscr{D}(\widehat{S}) = \{(x,(x(-1),x(1))) \mid x \in \mathscr{D}(T_1)\},$$

is self-adjoint in $H \oplus W$. However, it is the case that the Legendre type polynomial vectors $\{(P_{n,A},(P_{n,A}(-1),P_{n,A}(1)))\}_{n=0}^{\infty}$ are eigenfunctions of $\widehat{S}$ if and only if $B = 0$.

5.2. Example 2: A simple first-order differential operator

Let $H = L^2[0,1]$ be endowed with the standard L^2 inner product

$$\langle x,y \rangle_H = \int_0^1 x(u)\overline{y}(u)du$$

and let $W = \mathbb{C}$ have the usual Euclidean inner product

$$\langle a,b \rangle_W := a\overline{b} \quad (a,b \in W).$$

In this example, we show how to construct a self-adjoint operator in $H \oplus W$ generated by the first-order Lagrangian symmetric differential expression

$$\ell[x](u) = ix'(u).$$

Our construction can be modified to find numerous other self-adjoint operators in $H \oplus W$ generated by $\ell[\cdot]$.

The maximal and minimal domains in H associated with $\ell[\cdot]$ are respectively given by

$$T_1 x = ix'$$
$$\mathscr{D}(T_1) = \{x : [0,1] \to \mathbb{C} \mid x \in AC[0,1]; \; x' \in H\},$$

and

$$T_0 x = ix'$$
$$\mathscr{D}(T_0) = \{x \in \mathscr{D}(T_1) \mid x(0) = x(1) = 0\}; \tag{5.27}$$

see [17, Chapter 13, Example 13.4]. The symplectic form $[\cdot,\cdot]_H$ associated with T_1 is given by

$$[x,y]_H = ix(1)\overline{y}(1) - ix(0)\overline{y}(0).$$

An elementary calculation shows that the deficiency indices of T_0 are equal with $\mathrm{def}(T_0) = 1$.

Choose $\{\xi_1 = 1\}$ as the orthonormal basis for W. All self-adjoint operators $B : W \to W$ have the form $Ba = \alpha a$ for some real number α; we fix one such an operator. Define $t_1 \in \mathscr{D}(T_1)$ by

$$t_1(u) \equiv 1 \quad (0 \leqslant u \leqslant 1)$$

and note that

$$[x,t_1]_H = i(x(1) - x(0)). \tag{5.28}$$

It is clear, from (5.27), that t_1 is not in the minimal domain; moreover, from (5.28), we see that

$$[t_1,t_1]_H = 0;$$

this shows that $\{t_1\}$ is a GKN set for T_0 in H. Moreover, with this GKN set, the reader can readily verify, using Theorem 2.3, that the operator $T : \mathscr{D}(T) \subset H \to H$ defined by

$$Tx = ix'$$
$$\mathscr{D}(T) = \{x \in \mathscr{D}(T_1) \mid x(0) = x(1)\}$$

is self-adjoint in H; see [17, Chapter 13, Example 13.4] for an interesting direct proof of the self-adjointness of T.

The operators $\Omega : \mathscr{D}(T_1) \to W$ and $\Psi : \Delta_0 \to W$ from Section 4 are given by

$$\Omega x = [x,t_1]_H \xi_1 = i(x(1) - x(0))$$

and

$$\Psi(t_0 + at_1) = a \quad (t_0 \in \mathscr{D}(T_0)).$$

The maximal operator $\widehat{T}_1 : \mathscr{D}(\widehat{T}_1) \subset H \oplus W \to H \oplus W$ and the minimal operator $\widehat{T}_0 : \mathscr{D}(\widehat{T}_0) \subset H \oplus W \to H \oplus W$ can now both be defined. Indeed,

$$\widehat{T}_1(x,a) = (ix', Ba - \Omega x) = (ix', \alpha a - i(x(1) - x(0)))$$
$$\mathscr{D}(\widehat{T}_1) = \{(x,a) \mid x \in \mathscr{D}(T_1), a \in W\}$$

and

$$\widehat{T}_0(x, \Psi x) = (ix', B\Psi x) = (ix', \alpha \Psi x)$$
$$\mathscr{D}(\widehat{T}_0) = \{(x, \Psi x) \mid x \in \Delta_0\}.$$

The symplectic form $[\cdot, \cdot]_{H \oplus W}$ associated with $\widehat{T}_1$ is given by

$$[(x, a), (y, b)]_{H \oplus W} = [x, y]_H - \langle \Omega x, b \rangle_W + \langle a, \Omega y \rangle_W \tag{5.29}$$
$$= i(x(1)\overline{y}(1) - x(0)\overline{y}(0)) - i(x(1) - x(0))\overline{b} - ai(\overline{y}(1) - \overline{y}(0))$$

for (x, a), $(y, b) \in \mathscr{D}(\widehat{T}_1)$.

Define $x_1 \in \mathscr{D}(T_1)$ by

$$x_1(u) = \begin{cases} 1 & u \text{ near } 1 \\ 0 & u \text{ near } 0; \end{cases}$$

from (5.29), we see that

$$[(x, a), (x_1, 1/2)]_{H \oplus W} = ix(1) - i(x(1) - x(0))/2 - ai. \tag{5.30}$$

We now show that $\{(x_1, 1/2)\}$ is a GKN set for $\widehat{T}_0$. From (5.30), note that

$$[(x_1, 1/2), (x_1, 1/2)]_{H \oplus W} = 0; \tag{5.31}$$

We claim that $(x_1, 1/2) \notin \mathscr{D}(\widehat{T}_0)$; indeed, otherwise, we have

$$x_1(u) = t_0(u) + \frac{1}{2}t_1(u) \quad (u \in [0, 1]) \tag{5.32}$$

for some $t_0 \in \mathscr{D}(T_0)$. However, choosing $u = 0$ or $u = 1$ shows that (5.32) is not possible. It now follows that $\{(x_1, 1/2)\}$ is a GKN set for $\widehat{T}_0$.

From (5.30), we see that

$$[(x, a), (x_1, 1/2)]_{H \oplus W} = 0 \text{ if and only if } a = \frac{x(0) + x(1)}{2}. \tag{5.33}$$

It now follows, from Theorem 3.2 and (5.33), that the operator $\widehat{T} : \mathscr{D}(\widehat{T}) \subset H \oplus W \to H \oplus W$ defined by

$$\widehat{T}(x, (x(0) + x(1))/2) = (ix', \alpha(x(0) + x(1))/2 - i(x(1) - x(0))) \tag{5.34}$$
$$\mathscr{D}(\widehat{T}) = \{(x, (x(0) + x(1))/2) \mid x \in \mathscr{D}(T_1)\}$$

is self-adjoint.

5.3. Example 3: Variations on the Fourier self-adjoint operator

For this example, we consider the well known Fourier differential expression

$$\ell_F[x](u) = -x''(u) \quad (u \in [a, b]) \tag{5.35}$$

where $[a,b]$ is a compact interval. Here, the Hilbert space is $H = L^2[a,b]$ and, in the sub-examples below, we will consider W to be either $\mathbb{C}$ or $\mathbb{C}^2$ with a weighted Euclidean inner product.

The maximal operator $T_1 : \mathscr{D}(T_1) \subset H \to H$ is defined by

$$T_1 x = \ell_F[x]$$
$$\mathscr{D}(T_1) = \{x : [a,b] \to \mathbb{C} \,|\, x, x' \in AC[a,b]; x'' \in L^2[a,b]\}$$

while the minimal operator $T_0 : \mathscr{D}(T_0) \subset H \to H$ is given by

$$T_0 x = \ell_F[x]$$
$$\mathscr{D}(T_0) = \{x \in \mathscr{D}(T_1) | x(a) = x'(a) = x(b) = x'(b) = 0\}.$$

The symplectic form $[\cdot,\cdot]_H$ associated with T_1 is given by

$$[x,y]_H = x(b)\overline{y}'(b) - x'(b)\overline{y}(b) + x'(a)\overline{y}(a) - x(a)\overline{y}'(a) \quad (x,y \in \mathscr{D}(T_1)). \qquad (5.36)$$

Because $\ell_F[\cdot]$ is regular, the deficiency index of T_0 is $\mathrm{def}(T_0) = 2$. Consequently, by the GKN theorem, every self-adjoint extension of T_0 in H will be a certain restriction of the maximal operator defined by two appropriate boundary conditions. One such self-adjoint operator is the classical *Fourier trigonometric self-adjoint operator* in H, generated by $\ell_F[f]$, with domain

$$\{x \in \mathscr{D}(T_1) \,|\, x'(a) = x'(b), x(a) = x(b)\}.$$

We list several examples of self-adjoint operators, generated by $\ell_F[\cdot]$ in $H \oplus W$.

5.3.1. One dimensional extension spaces

Consider the one dimensional extension space W, with basis $\{\xi_1 = 1\}$, given by

$$W = \mathbb{C}$$
$$\langle z_1, z_2 \rangle_W = z_1 \overline{z_2} \quad (z_1, z_2 \in W).$$

Every self-adjoint operator $B : W \to W$ has the form $Bz = \alpha z$ for some $\alpha \in \mathbb{R}$; for this example, we fix such a B. Observe the Hilbert space $H \oplus W$ is suggested in a natural way by the inner product

$$\int_a^b x(u)\overline{y}(u)\,du + x(b)\overline{y}(b).$$

With this particular inner product in mind, $H \oplus W$ is isomorphic to the Lebesgue-Stieltjes integration space generated by the discontinuous Lebesgue-Stieltjes measure

$$d\mu = du + \delta(u - b).$$

EXAMPLE 3.1. With the partial GKN set $\{t_1\}$ for T_0 in H given by

$$t_1(u) = \begin{cases} 1 & u \text{ near } b \\ 0 & u \text{ near } a, \end{cases}$$

a calculation shows that $\{(x_1,0),(x_2,0)\}$ is a GKN set for $\widehat{T}_0$ in $H \oplus W$ where

$$x_1(u) = \begin{cases} u - b \ u \text{ near } b \\ 0 \quad\ u \text{ near } a \end{cases} \text{ and } x_2(u) = \begin{cases} 0 \quad\ u \text{ near } b \\ u - a \ u \text{ near } a. \end{cases}$$

We leave it to the reader to check that

$$\text{(i) } [(x,z),(x_1,0)]_{H \oplus W} = x(b) - z$$
$$\text{(ii) } [(x,z),(x_2,0)]_{H \oplus W} = -x(a).$$

From these equations, we see that the operator

$$\widehat{T}(x,x(b)) = (-x'', \alpha x(b) + x'(b))$$
$$\mathscr{D}(\widehat{T}) = \{(x,x(b)) \mid x \in \mathscr{D}(T_1); x(a) = 0\}$$

is self-adjoint in $H \oplus W$.

EXAMPLE 3.2. By picking the partial GKN set $\{t_1\}$ for T_0 in H, where

$$t_1(u) = \begin{cases} 0 \quad\ u \text{ near } b \\ u - a \ u \text{ near } a, \end{cases}$$

and the GKN set $\{(x_1,0),(x_2,0)\}$ for $\widehat{T}_0$ in $H \oplus W$, where

$$x_1(u) = \begin{cases} 0 \ u \text{ near } b \\ 1 \ u \text{ near } a \end{cases} \text{ and } x_2(u) = \begin{cases} u - b \ u \text{ near } b \\ 0 \quad\ u \text{ near } a, \end{cases}$$

the reader can check that the operator

$$\widehat{T}(x,x'(a)) = (-x'', \alpha x'(a) + x(a))$$
$$\mathscr{D}(\widehat{T}) = \{(x,x'(a)) \mid x \in \mathscr{D}(T_1); x(b) = 0\}$$

is self-adjoint in $H \oplus W$.

5.3.2. Two dimensional extension spaces

For the last three examples, let $W = \mathbb{C}^2$ have the weighted inner product

$$\langle (z_1,z_2),(z_1',z_2') \rangle_W = \frac{z_1 \overline{z_1'}}{M} + \frac{z_2 \overline{z_2'}}{N},$$

where $M,N > 0$. Let $\{\xi_1 = (\sqrt{M},0), \xi_2 = (0,\sqrt{N})\}$ be a basis for W. The reader can check that the most general form of a self-adjoint operator $B : W \to W$, using this inner product, has the matrix representation

$$B = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} N/M & \gamma \end{pmatrix}, \tag{5.37}$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

EXAMPLE 3.3. Define $t_i, x_i \in \mathscr{D}(T_1)$ $(i = 1, 2)$ by

$$t_1(u) = \begin{cases} \sqrt{M} \ u \text{ near } a \\ 0 \quad u \text{ near } b \end{cases}, \qquad t_2(u) = \begin{cases} 0 \quad u \text{ near } a \\ \sqrt{N} \ u \text{ near } b \end{cases},$$

$$x_1(u) = \begin{cases} \sqrt{M}(u - a) \ u \text{ near } a \\ 0 \qquad\qquad u \text{ near } b \end{cases}, \qquad x_2(u) = \begin{cases} 0 \qquad\qquad u \text{ near } a \\ \sqrt{N}(u - b) \ u \text{ near } b. \end{cases}$$

It is the case that $\{t_1, t_2\}$ is a GKN set for T_0 in H and $\{(x_1, 0), (x_2, 0)\}$ is a GKN set for $\widehat{T}_0$ in $H \oplus W$. Moreover, using (5.36), we see that

$$[x, t_1]_H = x'(a)\sqrt{M}; \ [x, t_2]_H = -x'(b)\sqrt{N}$$
$$[x, x_1]_H = -x(a)\sqrt{M}; \ [x, x_2]_H = x(b)\sqrt{N}$$

and

$$\Omega x = [x, t_1]\xi_1 + [x, t_2]\xi_2 = (x'(a)M, -x'(b)N) \quad (x \in \mathscr{D}(T_1)).$$

In particular,

$$\Omega x_1 = (M^{3/2}, 0) \text{ and } \Omega x_2 = (0, -N^{3/2}).$$

With $z = (z_1, z_2) \in W$, calculations show that

$$0 = [(x, z), (x_1, 0)]_{H \oplus W} = -x(a)\sqrt{M} + z_1\sqrt{M} \tag{5.38}$$

and

$$0 = [(x, z), (x_2, 0)]_{H \oplus W} = x(b)\sqrt{N} - z_2\sqrt{N} \tag{5.39}$$

yielding

$$z_1 = x(a) \text{ and } z_2 = x(b).$$

That is, the boundary conditions expressed in (5.38) and (5.39) yield continuity of functions in the domain of the self-adjoint operator $\widehat{T} : \mathscr{D}(\widehat{T}) \subset H \oplus W \to H \oplus W$ defined by

$$\widehat{T}(x, (x(a), x(b))) = (-x'', B(x(a), x(b)) - (x'(a)M, -x'(b)N))$$

$$\mathscr{D}(\widehat{T}) = \{(x, (x(a), x(b))) \mid x \in \mathscr{D}(T_1)\},$$

where B is given in (5.37) and

$$B(x(a), x(b)) := \begin{pmatrix} \alpha & \beta \\ \overline{\beta}N/M & \gamma \end{pmatrix} \begin{pmatrix} x(a) \\ x(b) \end{pmatrix}.$$

In this case, the setting $H \oplus W$ can be identified with the Hilbert function space $L_\sigma^2[a, b]$, given by

$$L_\sigma^2[a, b] = \{f : [a, b] \to \mathbb{C} \mid f \text{ is Lebesgue measurable on } [a, b]$$
$$\text{and } \|f\|_{H \oplus W} < \infty\},$$

where $\|\cdot\|_{H \oplus W}$ is the norm generated by the inner product

$$\langle x, y \rangle_{H \oplus W} = \int_a^b x(u)\overline{y}(u)du + \frac{x(a)\overline{y}(a)}{M} + \frac{x(b)\overline{y}(b)}{N}$$

and $d\sigma$ is the Lebesgue-Stieltjes measure generated by the distribution function

$$\sigma(u) = \begin{cases} -\dfrac{1}{M} + a & u \leqslant a \\ u & a < u < b \\ \dfrac{1}{N} + b & u \geqslant b. \end{cases}$$

EXAMPLE 3.4. For this example, we switch the roles of $\{t_1, t_2\}$ and $\{x_1, x_2\}$ which are defined in example 5.3.2. Note that, in this case, $\{(t_1, 0), (t_2, 0)\}$ is a GKN set for $\widehat{T_0}$ and $\{x_1, x_2\}$ is a GKN set for T_0. The calculations given in the previous example hold with the exception

$$\Omega x = (-x(a)M, x(b)M) \quad (x \in \mathscr{D}(T_1)).$$

Again, with $z = (z_1, z_2)$, we find that

$$0 = [(x, z), (t_1, 0)]_{H \oplus W} = x'(a)\sqrt{M} - z_1 \sqrt{M}$$

and

$$0 = [(x, z), (t_2, 0)]_{H \oplus W} = -x'(b)\sqrt{N} + z_2 \sqrt{N}$$

so that

$$z_1 = x'(a) \text{ and } z_2 = x'(b).$$

In this case, the operator $\widehat{T} : \mathscr{D}(\widehat{T}) \subset H \oplus W \to H \oplus W$, defined by

$$\widehat{T}(x, (x'(a), x'(b))) = (-x'', B(x'(a), x'(b)) - (-x(a)M, x(b)M)),$$

with domain

$$\mathscr{D}(\widehat{T}) = \{(x, (x'(a), x'(b))) \mid x \in \mathscr{D}(T_1)\}$$

is self-adjoint. In this case, for $x, y \in \mathscr{D}(\widehat{T})$, the inner product on $H \oplus W$ simplifies to the discrete Sobolev inner product

$$\langle x, y \rangle_{H \oplus W} = \int_a^b x(u)\overline{y}(u)du + \frac{x'(a)\overline{y}'(a)}{M} + \frac{x'(b)\overline{y}'(b)}{N}. \tag{5.40}$$

Notice that, because of the derivatives in the discrete part of (5.40), the closure of $\mathscr{D}(\widehat{T})$ is not a function space and there is no positive Borel measure generating this inner product.

EXAMPLE 3.5. For our last example, we consider a variation of the last two examples. Indeed, define $t_i, x_i \in \mathscr{D}(T_1)$ $(i = 1, 2)$ by

$$t_1(u) = \begin{cases} \sqrt{M} & u \text{ near } a \\ 0 & u \text{ near } b \end{cases}, \qquad t_2(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N}(u - b) & u \text{ near } b \end{cases},$$

$$x_1(u) = \begin{cases} 0 & u \text{ near } a \\ \sqrt{N} & u \text{ near } b \end{cases}, \qquad x_2(u) = \begin{cases} \sqrt{M}(u-a) & u \text{ near } a \\ 0 & u \text{ near } b \end{cases}$$

In this case $\{t_1, t_2\}$ is a GKN set for T_1 and $\{(x_1, 0), (x_2, 0)\}$ is a GKN set for $\widehat{T}_0$. Moreover, a calculation shows

$$\Omega x = (x'(a)M, x(b)N).$$

With $z = (z_1, z_2)$, the two boundary conditions

$$0 = [(x, z), (x_1, 0)]_{H \oplus W} = -x'(b)\sqrt{N} + z_2\sqrt{N}$$

$$0 = [(x, z), (x_2, 0)]_{H \oplus W} = -x(a)\sqrt{M} + z_1\sqrt{M}$$

yield

$$z_1 = x(a) \text{ and } z_2 = x'(b).$$

These calculation show that the operator $\widehat{T}$, given by

$$\widehat{T}(x, (x(a), x'(b))) = (-x'', B(x(a), x'(b)) - (x'(a)M, x(b)N))$$

with domain

$$\mathscr{D}(\widehat{T}) = \left\{ (x, (x(a), x'(b))) \mid x \in \mathscr{D}(T_1) \right\},$$

is self-adjoint in $H \oplus W$. For each $x, y \in \mathscr{D}(\widehat{T})$, the 'mixed' inner product in $H \oplus W$ reduces to

$$\langle x, y \rangle_{H \oplus W} = \int_a^b x(u)\overline{y}(u)du + \frac{x(a)\overline{y}(a)}{M} + \frac{x'(b)\overline{y}'(b)}{N}.$$

As in the last example, no positive Borel measure generates this inner product and the closure of $\mathscr{D}(\widehat{T})$ in the topology from $\langle \cdot, \cdot \rangle_{H \oplus W}$ is not a function space.

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