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ON THE CLASSES OF (n,m)–POWER D–NORMAL AND (n,m)–POWER D–QUASI–NORMAL OPERATORS

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Abstract. This paper is devoted to the study of some new classes of operators on Hilbert space called (n,m)-power D-normal $\left([(n,m)D\mathbf{N}]\right)$ and (n,m)-power D-quasi-normal $\left([(n,m)D\mathbf{QN}]\right)$, associated with a Drazin invertible operator using its Drazin inverse. Some properties of $[(n,m)D\mathbf{N}]$ and $[(n,m)D\mathbf{QN}]$ are investigated and some examples are also given.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators defined in $\mathscr{H}$. For every $T \in \mathscr{B}(\mathscr{H})$, denote by $\mathscr{R}(T)$, $\mathscr{N}(T)$ and T^* the range, the null space and the adjoint of T, respectively. If $\mathscr{M} \subset \mathscr{H}$ is a closed subspace of $\mathscr{H}$ satisfying $T\mathscr{M} \subset \mathscr{M}$, then $\mathscr{M}$ is called an invariant subspace of T. In addition, if $\mathscr{M}$ also is invariant subspace of T^*, then $\mathscr{M}$ is called a reducing subspace of T. For any arbitrary operator $T \in \mathscr{B}(\mathscr{H})$, we can write

$$T = X + iY \tag{1.1}$$

where

$$X = ReT = \frac{1}{2}\left(T + T^*\right), \quad Y = ImT = \frac{1}{2i}\left(T - T^*\right). \tag{1.2}$$

The operators X and Y are called the real and imaginary parts of T, and the decomposition (1.1) is called the Cartesian decomposition of T and it is unique. We shall write for positive integer m, $C_m^2 = T^{*m}T^m$ and $B_m^2 = T^mT^{*m}$, where B_m and C_m are non-negative definite. For any operator $T \in \mathscr{B}(\mathscr{H})$, $|T| = (T^*T)^{\frac{1}{2}}$ and

$$[T^*,T] = T^*T - TT^* = |T|^2 - |T^*|^2.$$

An operator $T \in \mathscr{B}(\mathscr{H})$ is called normal if it satisfies the following condition $T^*T = TT^*$. The class of quasi-normal operators denoted by $[\mathbf{QN}]$, was first introduced and studied by A. Brown [6] in 1953. The operator T is quasi-normal if T commutes

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with T^*T ($T(T^*T) = (T^*T)T$). The author A. S. Jibril in [15] introduced the class of n-power normal operators as a generalization of normal operators and its denoted by $[n\mathbf{N}]$. The operator T is called n-power normal if $T^nT^* = T^*T^n$. In [21] and [22], the first named author introduced the class of n-power quasi-normal operators denoted by $[n\mathbf{QN}]$, as a generalization of quasi-normal operators. An operator T is called n-power quasi-normal if T^n commutes with T^*T, i.e.; $T^n(T^*T) = (T^*T)T^n$. In [1] and [2], the authors has introduced and studied the classes of (n,m)-normal powers and (n,m)-power quasi-normal operators as follows: an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be (n,m)-powers (or (n,m)-power normal) if $T^n\big(T^m\big)^* = \big(T^m\big)^*T^n$ and it said to be (n,m)-power quasi-normal if $T^nT^{*m}T = T^{*m}TT^n$ where n,m be two nonnegative integers. Let $[(n,m)\mathbf{N}]$ and $[(n,m)\mathbf{QN}]$ the classes constituting of (n,m)-power normal and (n,m)-power-quasi-normal operators respectively. Then $[(n,m)\mathbf{N}] \subset [(n,m)\mathbf{QN}]$.

The Drazin inverse in the setting of bounded linear operators on complex Banach spaces was investigated by Caradus [8] and King [17]. The Drazin inverse has become a useful tool in a number of areas such that differential and difference equations, Markov chains, optimal control and iterative method ([4], [7]).

We recall that the Drazin inverse of the operator $T \in \mathscr{B}(\mathscr{H})$ is the unique operator $T^D \in \mathscr{B}(\mathscr{H})$, provided it exists, satisfying the following conditions

$$T^DT = TT^D, \ \big(T^D\big)^2T = T^D, \qquad T^{\nu+1}T^D = T^{\nu} \ \text{ for some integer } \nu \geqslant 0.$$

The smallest natural number ν satisfying the previous system of equations is known as the index of the operator T and is denoted by $ind(T)$. It is well known ([9]) that the Drazin inverse of the operator $T \in \mathscr{B}(\mathscr{H})$ exists if and only if $0 \notin \overline{\sigma(T)\setminus\{0\}}$ and the point zero, provided $0 \in \sigma(T)$, is a pole of the resolvent $R(T,\mu) := (\mu I - T)^{-1}$. Here $\sigma(T)$ denotes spectrum of the operator T, and for $K \subset \mathbb{C}$ symbol $\overline{K}$ denotes closure of K.

If we define $T^0 = I$, then the previous conditions hold with $\mu = 0$ if and only if T is invertible. We note that if T is nilpotent, then it is Drazin invertible, $T^D = 0$, and $ind(T) = p$, where p is the power of nilpotency of T. When $ind(T) = 1$, T^D is called the group inverse of T and the symbol $T^\sharp$ denote it.

For $T \in \mathscr{B}(\mathscr{H})$, it was observed that the Drazin inverse T^D of T satisfies $(T^*)^D = (T^D)^*$ and $\big(T^k\big)^D = \big(T^D\big)^k$ for positive integer k. The Drazin invertibility of an operator in $\mathscr{B}(\mathscr{H})$ is similarly invariant, i.e. if T is Drazin invertible and $S \in \mathscr{B}(\mathscr{H})$ is an invertible operator, then $S^{-1}TS$ is Drazin invertible and

$$(S^{-1}TS)^D = S^{-1}T^DS.$$

We denote by $\mathscr{B}(\mathscr{H})^D$ the set of all Drazin invertible elements of $\mathscr{B}(\mathscr{H})$.

Very recently, the authors M. Dana and R. Yousfi in [11] has introduced the following classes of operators. Let $T \in \mathscr{B}(\mathscr{H})^D$, T is said to be

(i) D-normal if $T^DT^* = T^*T^D$.

(ii) D-quasi-normal if $T^D(T^*T) = (T^*T)T^D$.

(iii) n-power D-normal if $\left(T^D\right)^n T^* = T^*\left(T^D\right)^n$.

(iv) n-power D-quasi-normal if $\left(T^D\right)^n(T^*T) = (T^*T)\left(T^D\right)^n$.

Let $[D\mathbf{N}], [nD\mathbf{N}], [D\mathbf{QN}]$ and $[nD\mathbf{QN}]$ denote the classes constituting of D-normal, n-power D-normal, D-quasi-normal and n-power D-quasi-normal operators. Then

(i) $[D\mathbf{N}] \subset [D\mathbf{QN}] \subset [nD\mathbf{QN}]$.

(ii) $[D\mathbf{N}] \subset [nD\mathbf{N}] \subset [nD\mathbf{QN}]$.

LEMMA 1.1. *([8], [25]) Let $T,S \in \mathscr{B}(\mathscr{H})^D$. Then the following properties hold.*

(1) TS is Drazin invertible if and only if ST is Drazin invertible. Moreover

$$(TS)^D = T[(ST)^D]^2 S \text{ and } ind(TS) \leqslant ind(ST)+1.$$

(2) If T is idempotent, then $T^D = T^\sharp = T$.

(3) If $TS = ST$, then $(TS)^D = S^D T^D = T^D S^D, T^D S = ST^D$ and $TS^D = S^D T$.

(4) If $TS = ST = 0$, then $\left(T+S\right)^D = T^D + S^D$.

LEMMA 1.2. *([12, Lemma 3.1]) If $T \in \mathscr{B}(\mathscr{H})$ and $S \in \mathscr{B}(\mathscr{H})$ are Drazin invertible with $ind(T) = p$ and $ind(S) = q$. Then $V = \begin{pmatrix} T & R \\ 0 & S \end{pmatrix}$ is also Drazin invertible and $V^D = \begin{pmatrix} T^D & X \\ 0 & S^D \end{pmatrix}$ where*

$$X = \sum_{0 \leqslant j \leqslant q-1} \left(T^D\right)^{j+2} R S^j S^\pi + T^\pi \left(\sum_{0 \leqslant k \leqslant p-1} T^j R \left(S^D\right)^{k+2} \right) - T^D R S^D. \tag{1.3}$$

This paper has been organized in three sections. In section two, we introduce a new class of operators named (n,m)-power D-normal operators associated with a Drazin invertible operator using its Drazin inverse. Our motivation for this study comes from the problem of finding operators that their Drazin inverses are (n,m)-power normal. Some of the basic properties of this class with some examples are studied. Moreover, the product, direct sum, tensor product and the sum of finite numbers of these type are discussed. In section three, the classes of (n,m)-power D-quasi-normal operators which are generalizations of the class of n-power D-quasi-normal operators and (n,m)-power D-normal operators are introduced and also some properties of such classes are given. An investigation of extensions of the Fuglede-Putnam´s theorem for (n,m)-power D-normal operator will be given in section four.

2. (n,m)-power D-normal operators

In this section, the class of (n,m)-power D-normal operators as a generalization of the classes of D-normal and n-power D-normal operators is introduced. In addition, we study several properties for members from this class of operators.

DEFINITION 2.1. Let $T \in \mathscr{B}(\mathscr{H})$ be Drazin invertible operator. We said that T is (n,m)-power D-normal if

$$\left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n \tag{2.1}$$

for some positive integers n and m.This class of operators will denoted by $[(n,m)D\mathbf{N}]$.

REMARK 2.1. (i) If $n = m = 1$, then (n,m)-power D-normal becomes D-normal, i.e.

$$[(1,1)D\mathbf{N}] = [D\mathbf{N}].$$

(ii) If $m = 1$, then $(n,1)$-power D-normal becomes n-power D-normal, i.e.

$$[(n,1)D\mathbf{N}] = [nD\mathbf{N}].$$

(iii) $T \in [(n,m)D\mathbf{N}] \Longleftrightarrow [\left(T^D\right)^n, T^{*m}] = 0$.

REMARK 2.2. Obviously, that the class of (n,m) power D-normal operators includes classes of (n,m)-power-normal and n-power D–normal operators, i.e. the following inclusions holds

$$[(n,m)\mathbf{N}] \subset [(n,m)D\mathbf{N}] \text{ and } [nD\mathbf{N}] \subset [(n,m)D\mathbf{N}].$$

REMARK 2.3. The following inclusions hold.

(i) $[(n,m)D\mathbf{N}] \subset [(2n,m)D\mathbf{N}]$.

(ii) $[(n,m)D\mathbf{N}] \subset [(n,2m)D\mathbf{N}]$.

(iii) $[(n,m)D\mathbf{N}] \subset [(2n,2m)D\mathbf{N}]$.

EXAMPLE 2.1. Consider the matrix $T = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \in \mathscr{B}(\mathbb{C}^3)$. We observe that $T^D = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$. A direct calculation shows that T is (n,m)-D-power normal for all positive integers n and m.

The following examples show that there exists a (n,m)-power D-normal operator which is neither (n,m)-power normal or nor n-power D-normal.

EXAMPLE 2.2. Let $T = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ be an operator on Hilbert space $\mathbb{C}^3$, it is easy to check that $T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. A direct calculation shows that

$$\left(T^D\right)^2 T^* = T^* \left(T^D\right)^2 \text{ and } T^2T^* \neq T^*T^2.$$

This shows that T is $(2,1)$ power-D-normal but it is not $(2,1)$-power normal.

EXAMPLE 2.3. Consider the operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ operator acting on $\mathbb{C}^2$. Then $T^D = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. A Direct calculation shows that T is $(2,2)$-power D-normal but T is not 2-power D-normal.

It is well known that if T is n-power D-normal, then T^n is D-normal. In the following theorem, we extend this result to (n,m)-power D-normal operator as follows.

THEOREM 2.1. *Let $T \in \mathscr{B}(\mathscr{H})^D$. If T is (n,m)-power D-normal, then the following statements hold:*

(i) T^k is D-normal where k is the least common multiple of n and m.

(ii) T^{nm} is D-normal operator.

Proof.

(i) Assume T is (n,m)-power D-normal that is $\left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n$. Let $k = pn$ and $k = qm$. We have

$$\begin{aligned}
\left(T^k\right)^D \left(T^k\right)^* &= \left(T^D\right)^{pn} \left(T^*\right)^{qm} \\
&= \left[\left(T^D\right)^n\right]^p \left[\left(T^*\right)^m\right]^q \\
&= \underbrace{\left(T^D\right)^n . \cdots . \left(T^D\right)^n}_{p-\text{times}} \underbrace{\left(T^*\right)^m \cdots \left(T^*\right)^m}_{q-\text{times}} \\
&= \underbrace{\left(T^*\right)^m \cdots \left(T^*\right)^m}_{q-\text{times}} \underbrace{\left(T^D\right)^n . \cdots . \left(T^D\right)^n}_{p-\text{times}} \\
&= \left(T^*\right)^{qm} \left(T^D\right)^{np} \\
&= \left(T^{qm}\right)^* \left(T^{np}\right)^D \\
&= T^{*k} \left(T^k\right)^D,
\end{aligned}$$

which means that T^k is D-normal.

(ii) By similar way. □

The following proposition collects some of basic properties of (n,m)-power D-normal operators.

PROPOSITION 2.1. *Let $T \in \mathscr{B}(\mathscr{H})^D$. The following properties hold.*

(1) If T is (n,m)-power D-normal, then T^D is (n,m)-power normal.

(2) T is (n,n)-power D-normal if and only if $\left(T^D\right)^n$ is normal.

(3) T is (n,m)-power D-normal if and only if T^* is (n,m)-power D-normal.

(4) If $\mathscr{M}$ is a closed subspace of $\mathscr{H}$ such that $\mathscr{M}$ reduces T, then $\left(T|\mathscr{M}\right)^{nm}$ is D-normal.

(5) If $T,S \in \mathscr{B}(\mathscr{H})^D$ such that S is unitary equivalent to T and T is (n,m)-power D-normal, then S is (n,m)-power D-normal.

Proof.

(1) Assume that T is (n,m)-power D-normal that is $\left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n$. By Lemma 1.1, it follows that

$$\begin{aligned}\left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n &\Rightarrow \left(T^D\right)^n \left(T^{*m}\right)^D = \left(T^{*m}\right)^D \left(T^D\right)^n \\ &\Rightarrow \left(T^D\right)^n \left(\left(T^D\right)^*\right)^m = \Big(\left(\left(T^D\right)^*\right)^m \left(T^D\right)^n.\end{aligned}$$

Therefore T^D is (n,m)-power normal.

(2) Assume that T is (n,n)-power D-normal, then by the statement (1) we have T^D is (n,n)-power normal that is $\left(T^D\right)^n \left(T^D\right)^{*n} = \left(T^D\right)^{*n}\left(T^D\right)^n$ or equivalently

$$\left(T^D\right)^n \left(\left(T^D\right)^n\right)^* = \left(\left(T^D\right)^n\right)^* \left(T^D\right)^n.$$

Thus $\left(T^D\right)^n$ is normal.

Conversely, assume that $\left(T^D\right)^n$ is normal. Since $T^D T = T T^D$, we have

$$\left(T^D\right)^n T = T\left(T^D\right)^n.$$

By Fuglede theorem ([10]), $\left(T^D\right)^{*n} T = T\left(T^D\right)^{*n}$ and it follows that

$$\left(T^D\right)^n \left(T^*\right)^n = \left(T^*\right)^n \left(T^D\right)^n.$$

Therefore T is (n,n)-power D-normal.

(3) We obtain the equivalence by taking conjugate operator.

(4) Since T is (n,m)-power D-normal operator, then by Theorem 2.1, T^{nm} is D-normal. Since $\mathscr{M}$ reduces T , then $T^{nm}/\mathscr{M}$ is D-normal (see [11]). Moreover, $T^{nm}/\mathscr{M} = \left(T/\mathscr{M}\right)^{nm}$, thus $\left(T/\mathscr{M}\right)^{nm}$ is D-normal.

(5) Since S is unitary equivalent to T, then there exists a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that $S = U^*TU$. In view of the property that the Drazin invertibility of an operator is similarly invariant, we have $S^D = \left(U^*TU\right)^D = U^*T^DU$ and furthermore

$\left(U^*T^DU\right)^n = U^*\left(T^D\right)^n U$. On the other hand, it is easily seen that

$$\begin{aligned}
\left(S^D\right)^n (S)^{*m} &= U^*\left(T^D\right)^n U.U^*T^{*m}U = U^*\left(T^D\right)^n T^{*m}U \\
&= U^*T^{*m}\left(T^D\right)^n U \\
&= \left(U^*T^mU\right)^m \left(U^*T^DU\right)^n \\
&= S^{*m}\left(S^D\right)^n.
\end{aligned}$$

Consequently, S is (n,m)-power D-normal. □

The following example shows that (n,m)-power D-normality is not preserved under similarity.

EXAMPLE 2.4. Let $T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then T is $(2,2)$-power D-normal but $S = XTX^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ is not $(2,2)$-power D-normal.

PROPOSITION 2.2. *Let $T \in \mathscr{B}(\mathscr{H})^D$, $X = \left(T^D\right)^n + T^{*m}$ and $Y = \left(T^D\right)^n - T^{*m}$. The following statements hold.*

(1) T is (n,m)-power D-normal if and only if $XY = YX$.

(2) If T is of class $[(n,m)D\mathbf{N}]$, then $Z = \left(T^D\right)^n T^{*m}$ commutes with X and Y.

(3) T is of class $[(n,m)D\mathbf{N}]$ if and only if $\left(T^D\right)^n$ commutes with X.

(4) T is of class $[(n,m)D\mathbf{N}]$ if and only if $\left(T^D\right)^n$ commutes with Y.

Proof.

(1)

$$\begin{aligned}
& XY = YX \\
\Leftrightarrow\ & \left(\left(T^D\right)^n + T^{*m}\right)\left(\left(T^D\right)^n - T^{*m}\right) = \left(\left(T^D\right)^n - T^{*m}\right)\left(\left(T^D\right)^n + T^{*m}\right) \\
\Leftrightarrow\ & \left(T^D\right)^{2n} - \left(T^D\right)^n T^{*m} + T^{*m}\left(T^D\right)^n - T^{*2m} \\
& = \left(T^D\right)^{2n} + \left(T^D\right)^n T^{*m} - T^{*m}\left(T^D\right)^n - T^{*2m} \\
\Leftrightarrow\ & \left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n.
\end{aligned}$$

Hence $XY = YX$ if and only if T is (n,m)-power D-normal.

The proof of statements $(2),(3)$ and (4) are straightforward. $\square$

Recall that a pair of operators $(T,S) \in \mathscr{B}(\mathscr{H})^2$ is said to be a doubly commuting pair if (T,S) satisfies $TS = ST$ and $T^*S = ST^*$.

The following discusses the conditions for product and sum of two (n,m)-power D-normal operators to be (n,m)-power D-normal.

THEOREM 2.2. *Let $T,S \in \mathscr{B}(\mathscr{H})^D$ are (n,m)-power D-normal. If (T,S) is a doubly commuting pair, then the following statements hold.*

(1) TS is (n,m)-power D-normal.

(2) If $TS = ST = 0$, then $T+S$ is (n,m)-power D-normal operator.

Proof.

(1) Since $TS = ST$ and $T^*S = ST^*$, if follows that

$$\begin{aligned}\left((TS)^D\right)^n\left((TS)^*\right)^m &= \left(T^DS^D\right)^n\left(T^*S^*\right)^m = \left(T^D\right)^n\left(S^D\right)^n\left(T^*\right)^m\left(S^*\right)^m\\ &= \left(T^D\right)^nT^{*m}\left(S^D\right)^nS^{*m} = T^{*m}S^{*m}\left(T^D\right)^n\left(S^D\right)^n\\ &= \left((TS)^*\right)^m\left((TS)^D\right)^n.\end{aligned}$$

Thus ST is (n,m)-power D-normal.

(2) Under the assumptions that T and S are (n,m)-power D-normal, it follows by taking into account the statements of Lemma 1.1 that

$$\begin{aligned}\left((T+S)^D\right)^n\left(T+S\right)^{*m} &= \left(\left(T^D\right)^n + \left(S^D\right)^n\right)\left(T^{*m}+S^{*m}\right)\\ &= \left(T^D\right)^nT^{*m} + \left(T^D\right)^nS^{*m} + \left(S^D\right)^nT^{*m} + \left(S^D\right)^nS^{*m}\\ &= T^{*m}\left(T^D\right)^n + S^{*m}\left(T^D\right)^n + T^{*m}\left(S^D\right)^n + S^{*m}\left(S^D\right)^n\\ &= \left(T+S\right)^{*m}\left((T+S)^D\right)^n.\end{aligned}$$

Hence $T+S$ is (n,m)-power D-normal. $\square$

PROPOSITION 2.3. *Let T and S are of class $[(n,m)D\mathbf{N}]$ such that $TS = ST = 0$. Then $(T+S)$ is nm-power D-normal.*

Proof. By the statements (3) and (4) of Lemma 1.1, it is well known that

$$(T+S)^D = T^D + S^D, T^DS = ST^D = 0 \text{ and } TS^D = S^DT = 0.$$

Therefore, we have the following relations

$$\left(T^D\right)^{nm}S = S\left(T^D\right)^{nm} = 0 \text{ and } T\left(S^D\right)^{nm} = \left(S^D\right)^{nm}T = 0.$$

In view of Theorem 2.1, we clearly have T^{nm} is D-normal and S^{nm} is D-normal. Now, since $\left(T^D\right)^{nm}$ and $\left(S^D\right)^{nm}$ are normal by Fuglede theorem we have

$$\left(T^D\right)^{*nm}S = S\left(T^D\right)^{*nm} = 0 \text{ and } T\left(S^D\right)^{*nm} = \left(S^D\right)^{*nm}T^* = 0.$$

We can deduce that

$$\begin{aligned}
\left(\left(T+S\right)^D\right)^{nm}\left(T+S\right)^{*nm} &= \left(\left(T^D\right)^{nm} + \left(S^D\right)^{nm}\right)\left(T^{*nm} + S^{*nm}\right) \\
&= \left(T^D\right)^{nm}T^{*nm} + \left(T^D\right)^{nm}S^{*nm} + \left(S^D\right)^{nm}T^{*nm} + \left(S^D\right)^{nm}S^{*nm} \\
&= T^{*nm}\left(T^D\right)^{nm} + S^{*nm}\left(S^D\right)^{nm} \\
&= \left(T+S\right)^{*nm}\left(\left(T+S\right)^D\right)^{nm}.
\end{aligned}$$

Therefore $(T+S)^{nm}$ is D-normal and consequently $T+S$ is nm-power D-normal as required and the proof is complete. $\square$

The following example shows that the classes $[(n,m)D\mathbf{N}]$ and $[(n+1,m)D\mathbf{N}]$ are not the same.

EXAMPLE 2.5. Let $T = \begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. Then $T^D = \frac{1}{9}\begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}$ and $S^D = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. A Direct calculation shows that T is of class $[(2,2)D\mathbf{N}]$ but $T \notin [(3,2)D\mathbf{N}]$. Moreover S is of class $[(3,2)D\mathbf{N}]$ but $S \notin [(2,2)D\mathbf{N}]$.

In the following proposition, we study the relation between the two classes $[(2,m)D\mathbf{N}]$ and $[(3,m)D\mathbf{N}]$.

PROPOSITION 2.4. *Let $T \in \mathscr{B}(\mathscr{H})$ be Drazin invertible operator such that T is of class $[(2,m)D\mathbf{N}]$ and of class $[(3,m)D\mathbf{N}]$ for some positive integer m, then T is of class $[(n,m)D\mathbf{N}]$ for all positive integer $n \geqslant 4$.*

Proof. We prove the assertion by using the mathematical induction. For $n=4$, it is a consequence of the item (i) of Remark 2.3.

We prove this for $n=5$. Since $T \in [(2,m)D\mathbf{N}]$,

$$(T^D)^2(T^*)^m = (T^*)^m(T^D)^2, \tag{2.2}$$

multiplying (2.2) to the left by $(T^D)^3$ we get

$$(T^D)^5(T^*)^m = (T^D)^3(T^*)^m(T^D)^2.$$

Thus implies

$$(T^D)^5(T^*)^m = (T^*)^m(T^D)^5.$$

Now assume that the result is true for $n \geqslant 5$ that is

$$(T^D)^n(T^*)^m = (T^*)^m(T^D)^n,$$

then

$$\begin{aligned}(T^D)^{n+1}(T^*)^m &= T^D(T^*)^m\left(T^D\right)^n \\ &= T^D(T^*)^m\left(T^D\right)^2\left(T^D\right)^{n-2} \\ &= (T^D)^3(T^*)^m(T^D)^{n-2} \\ &= (T^*)^m(T^D)^{n+1}.\end{aligned}$$

Thus T is of class $[(n+1,m)D\mathbf{N}]$. The proof is complete. □

EXAMPLE 2.6. Consider the operator matrix $T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ acting on $\mathbb{C}^2$. The Drazin inverse of T is $T^D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. By calculations we have that $T \in [(2,2)D\mathbf{N}] \cap [(3,2)D\mathbf{N}]$. Therefore $T \in [(n,2)D\mathbf{N}]$ for $n \geqslant 4$.

PROPOSITION 2.5. *Let $T \in \mathscr{B}(\mathscr{H})^D$. If T is of class $[(n,m)D\mathbf{N}]$ and of class $[(n+1,m)D\mathbf{N}]$, then T is of class $[(n+2,m)D\mathbf{N}]$ for some positive integers n and m. In particular T is of class $[(k,m)D\mathbf{N}]$ for all $k \geqslant n$.*

Proof. Since T is of class $[(n,m)D\mathbf{N}]$ and of class $[(n+1,m)D\mathbf{N}]$, it follows that

$$\left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n \text{ and } \left(T^D\right)^{n+1} T^{*m} = T^{*m}\left(T^D\right)^{n+1}.$$

Note that

$$\begin{aligned}\left(T^D\right)^{n+2}T^{*m} &= \left(T^D\right)\left(T^D\right)^{n+1}T^{*m} = \left(T^D\right)T^{*m}\left(T^D\right)^{n+1} \\ &= \left(T^D\right)\left(T^D\right)^n T^{*m} T^D \\ &= T^{*m}\left(T^D\right)^{n+2}.\end{aligned}$$

Hence T is of class $[(n+2,m)D\mathbf{N}]$. By repeating this process we can prove that T is of class $[(k,m)D\mathbf{N}]$ for all $k \geqslant n$. □

PROPOSITION 2.6. *Let $T \in \mathscr{B}(\mathscr{H})$ be Drazin invertible operator. If T is both of class $[(n,m)D\mathbf{N}]$ and $[(n+1,m)D\mathbf{N}]$ such that T^D is injective, then T is of class $[(1,m)D\mathbf{N}]$.*

Proof. Since T is of class $[(n,m)D\mathbf{N}]$ and of class $[(n+1,m)D\mathbf{N}]$, it follows that

$$(T^D)^n\Big(T^D(T^*)^m-(T^*)^mT^D\Big)=0.$$

If T^D is injective, then so is $(T^D)^n$ and we have $T^D(T^*)^m-(T^*)^mT^D=0$, hence T is of class $[(1,m)D\mathbf{N}]$. □

The following examples show that a (n,m)-power D-normal operator need not be $(n,m+1)$-power D-normal and vice versa.

EXAMPLE 2.7. Let $T=\begin{pmatrix}0&1\\-1&1\end{pmatrix}$ be an operator acting on $\mathbb{C}^2$. The Drazin inverse of T is $T^D=\begin{pmatrix}1&-1\\1&0\end{pmatrix}$. A Direct calculation shows that T is of class $[(2,3)D\mathbf{N}]$ but $T\notin[(2,2)D\mathbf{N}]$.

EXAMPLE 2.8. Let $T=\begin{pmatrix}3&-2\\0&-3\end{pmatrix}$. Then $T^D=\frac{1}{9}\begin{pmatrix}3&-2\\0&-3\end{pmatrix}$. A Direct calculation shows that T is of class $[(2,2)D\mathbf{N}]$ but $T\notin[(2,3)D\mathbf{N}]$.

In the following proposition, we study the relation between the two classes $[(n,2)D\mathbf{N}]$ and $[(n,3)D\mathbf{N}]$.

PROPOSITION 2.7. *Let $T\in\mathscr{B}(\mathscr{H})$ be Drazin invertible operator such that T is of class $[(n,2)D\mathbf{N}]$ and of class $[(n,3)D\mathbf{N}]$ for some positive integer n, then T is of class $[(n,m)D\mathbf{N}]$ for all positive integer $m\geqslant 4$.*

Proof. We omit the proof since the techniques are similar to those of the proof of Proposition 2.4. □

EXAMPLE 2.9. Consider the operator matrix $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ acting on $\mathbb{C}^2$. The Drazin inverse of T is $T^D=\begin{pmatrix}1&-1\\0&1\end{pmatrix}$. By calculations we have that $T\in[(2,2)D\mathbf{N}]\cap[(2,3)D\mathbf{N}]$. Therefore $T\in[(2,m)D\mathbf{N}]$ for $m\geqslant 4$.

The proof of the following proposition is very similar to the proof of proposition 2.5, thus we omitted.

PROPOSITION 2.8. *Let $T\in\mathscr{B}(\mathscr{H})^D$. If T is of class $[(n,m)D\mathbf{N}]$ and of class $[(n,m+1)D\mathbf{N}]$, then T is of class $[(n,m+2)D\mathbf{N}]$ for some positive integers n,m. In particular T is of class $[(n,k)D\mathbf{N}]$ for all $k\geqslant m$.*

In the following proposition, we discuss conditions pertaining to an (n,m)-power D-normal operator to be n-power D-normal.

PROPOSITION 2.9. *Let $T \in \mathscr{B}(\mathscr{H})^D$. If T is both of class $[(n,m)D\mathbf{N}]$ and $[(n,m+1)D\mathbf{N}]$ such that T^* is injective, then T is of class $[(n,1)D\mathbf{N}] = [nD\mathbf{N}]$.*

Proof. Since T is of class $[(n,m)D\mathbf{N}]$ and of class $[(n,m+1)D\mathbf{N}]$, it follows that

$$(T^*)^m\Big(\big(T^D\big)^nT^* - T^*\big(T^D\big)^n\Big) = 0.$$

If T^* is injective, then so is $(T^*)^m$ and we have $\big(T^D\big)^nT^* - T^*\big(T^D\big)^n = 0$, hence T is of class $[(n,1)D\mathbf{N}]$ or equivalently T is of class $[nD\mathbf{N}]$. $\square$

In [15] it was proved that if T is n-power normal which is a partial isometry, then T is $(n+1)$-power normal. In the following theorem we extend this result to (n,m)-power normal operator.

THEOREM 2.3. *Let $T \in \mathscr{B}(\mathscr{H})$ be an (n,m)-power normal for $n \geqslant m$. If T^m is a partial isometry, then T is $(n+m,m)$-power normal.*

Proof. Since T^m is partial isometry, $T^mT^{*m}T^m = T^m$ by [10, p.250].

Hence, we easily get

$$T^mT^{*m}T^n = T^n \quad \text{and} \quad T^nT^{*m}T^m = T^n,$$

which means that $T^nT^{*m}T^m = T^mT^{*m}T^n$. Since T is (n,m)-power normal, we get

$$T^{n+m}T^{*m} = T^{*m}T^{n+m},$$

and the proof is complete. $\square$

THEOREM 2.4. *Let $T \in \mathscr{B}(\mathscr{H})^D$ be of class $[(n,m)D\mathbf{N}]$ for some positive integers n and m for which $n \geqslant m$. If T^m is a partial isometry, then T is of class $[(n+m,m)D\mathbf{N}]$.*

Proof. Firstly observe that since T is a Drazin invertible we have $\big(T^D\big)^2T = T^D$ from which it is easily to obtain that

$$\big(T^D\big)^{2k}T^k = \big(T^D\big)^k \qquad k \geqslant 1.$$

Since T^m is partial isometry, then

$$T^mT^{*m}T^m = T^m. \tag{2.3}$$

Multiplying (2.3) to the left by $\left(T^D\right)^{n+m}$ and to the right by $\left(T^D\right)^{2m}$ we get

$$\left(T^D\right)^n T^{*m}\left(T^D\right)^m = \left(T^D\right)^{n+2m} \tag{2.4}$$

Multiplying (2.3) to the left by $\left(T^D\right)^{2m}$ and to the right by $\left(T^D\right)^{n+m}$ we get

$$\left(T^D\right)^m T^{*m}\left(T^D\right)^n = \left(T^D\right)^{n+2m}. \tag{2.5}$$

In view of (2.4) and (2.5) we have

$$\left(T^D\right)^n T^{*m}\left(T^D\right)^m = \left(T^D\right)^m T^{*m}\left(T^D\right)^n.$$

By taking into account that T is of class $[(n,m)D\mathbf{N}]$, we obtain

$$T^{*m}\left(T^D\right)^{n+m} = \left(T^D\right)^{n+m}T^{*m}.$$

Thus T is of class $[(n+m,m)D\mathbf{N}]$. □

REMARK 2.4. If $m=1$, Theorem 2.4 coincides with [11, Proposition 3.21].

PROPOSITION 2.10. *Let $T \in \mathscr{B}(\mathscr{H})^D$. If T is (n,m) power D-normal operator, then so is T^k for every positive integer k.*

Proof. To prove that T^k is of class $[(n,m)D\mathbf{N}]$, we have to prove that

$$\left(\left(T^k\right)^D\right)^n\left(T^k\right)^{*m} = \left(\left(T^k\right)^{*m}\left(\left(T^k\right)^D\right)^n, \quad k=1,2,\cdots.$$

We prove the statement by using mathematical induction on k. Since T is (n,m)-power D-normal, the result is true for $k=1$. Now we assume that the result is true for k, that is

$$\left(\left(T^k\right)^D\right)^n\left(T^k\right)^{*m} = \left(T^k\right)^{*m}\left(\left(T^k\right)^D\right)^n,$$

and prove it for $k+1$.

$$\begin{aligned}
\left(\left(T^{k+1}\right)^D\right)^n\left(T^{k+1}\right)^{*m} &= \left(\left(T\right)^D\right)^n\left(\left(T^k\right)^D\right)^n\left(T^k\right)^{*m}T^{*m}\\
&= \left(\left(T\right)^D\right)^n\left(T^k\right)^{*m}\left(T^D\right)^n T^{*m}\\
&= \left(T^k\right)^{*m}T^{*m}\left(\left(T\right)^D\right)^n\left(T^D\right)^n\\
&= \left(\left(T^{k+1}\right)^{*m}\left[\left(T^{k+1}\right)^D\right)^n.
\end{aligned}$$

Therefore T^{k+1} is of class $[(n,m)D\mathbf{N}]$. We conclude that the statement of the proposition holds. □

THEOREM 2.5. *The class of all (n,m)-power D-normal on $\mathscr{H}$ is a closed subset of $\mathscr{B}(\mathscr{H})^D$ under scalar multiplication.*

Proof. Let $T \in [(n,m)D\mathbf{N}]$. A simple calculations show that $\alpha T \in [(n,m)D\mathbf{N}]$ for $\alpha \in \mathbb{C}$. On the other hand let $(T_k)_k$ be a sequence in $[(n,m)D\mathbf{N}]$ converges to $T \in \mathscr{B}(\mathscr{H})$ strongly and T_k^D converges to T^D strongly. Then by a simple computation, one can get that

$$\begin{aligned}\|\big(T^D\big)^n T^{*m} - T^{*m}\big(T^D\big)^n\| &= \|\big(T^D\big)^n T^{*m} - \big(T_k^D\big)^n T_k^{*m} + T_k^{*m}\big(T_k^D\big)^n - T^{*m}\big(T^D\big)^n\| \\ &\leqslant \|\big(T^D\big)^n T^{*m} - \big(T_k\big)^n T_k^{*m}\| + \|T_k^{*m}\big(T_k^D\big)^n - T^{*m}\big(T^D\big)^n\|.\end{aligned}$$

By taking $k \longrightarrow \infty$ and tanking into account [24, lemma 3.1 and Theorem 3.6] we get that

$$\big(T^D\big)^n T^{*m} - T^{*m}\big(T^D\big)^n = 0.$$

Therefore T is of class $[(n,m)D\mathbf{N}]$. $\square$

THEOREM 2.6. *Let $(T_k)_{1\leqslant k\leqslant d} \in \big(\mathscr{B}(\mathscr{H})^D\big)^d$ such that each T_k is (n,m)-power D-normal, then*

(1) $T_1 \oplus T_2 \oplus \cdots \oplus T_d$ is a (n,m)-power D-normal.

(2) $T_1 \otimes T_2 \otimes \cdots \otimes T_d$ is a (n,m)-power D-normal.

Proof.

(1) Since each T_k for $k = 1,\cdots,d$ is (n,m)-power D-normal, then $\big(T_k^D\big)^n T_k^{*m} = T_k^{*m}\big(T_k^D\big)^n$ and we have

$$\begin{aligned}&\big((T_1 \oplus T_2 \oplus \cdots \oplus T_d)^D\big)^n \big(T_1 \oplus T_2 \oplus \cdots \oplus T_d\big)^{*m} \\ &= \big(((T_1)^D)^n \oplus ((T_2)^D)^n \oplus \cdots \oplus ((T_d)^D)^n\big)\big(T_1^{*m} \oplus T_2^{*m} \oplus \cdots \oplus T_d^{*m}\big) \\ &= ((T_1)^D)^n T_1^{*m} \oplus ((T_2)^D)^n T_2^{*m} \oplus \cdots \oplus ((T_d)^D)^n T_d^{*m} \\ &= T_1^{*m}((T_1)^D)^n \oplus T_2^{*m}((T_2)^D)^n \oplus \cdots \oplus T_d^{*m}((T_d)^D)^n \\ &= (T_1 \oplus T_2 \oplus \cdots \oplus T_d)^{*m}\big((T_1 \oplus T_2 \oplus \cdots \oplus T_d)^D\big)^n.\end{aligned}$$

Thus $\oplus_{1\leqslant k\leqslant d} T_k$ is a (n,m)-power D-normal.

(2) Since each T_k for $k = 1,\cdots,d$ is (n,m)-power D-normal, then $\big(T_k^D\big)^n T_k^{*m} = T_k^{*m}\big(T_k^D\big)^n$ and we have for $(x_k)_{1\leqslant k\leqslant d} \in \mathscr{H}^d$

$$\begin{aligned}&\big((T_1 \otimes T_2 \otimes \cdots \otimes T_d)^D\big)^n \big(T_1 \otimes T_2 \otimes \cdots \otimes T_d\big)^{*m}(x_1 \otimes x_2 \otimes \cdots \otimes x_d) \\ &= \big(T_1^D\big)^n T_1^{*m} x_1 \otimes \big(T_2^D\big)^n T_2^{*m} x_2 \otimes \cdots \otimes \big(T_d^D\big)^n T^{*m} x_d \\ &= T_1^{*m}\big(T_1^D\big)^n x_1 \otimes T_2^{*m}\big(T_2^D\big)^n x_2 \otimes \cdots \otimes T^{*m}\big(T_d^D\big)^n x_d \\ &= \big(T_1 \otimes T_2 \otimes \cdots \otimes T_d\big)^{*m}\big((T_1 \otimes T_2 \otimes \cdots \otimes T_d)^D\big)^n (x_1 \otimes x_2 \otimes \cdots \otimes x_d).\end{aligned}$$

Hence the result. $\square$

The following theorem characterizes the (n,m)-power D-normal operator matrix.

THEOREM 2.7. *Let* $T,S \in \mathscr{B}(\mathscr{H})^D$ *and let* $V = \begin{pmatrix} T & R \\ 0 & S \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. *Then* V *is of class* $[(n,m)D\mathbf{N}]$ *if and only if* $T \in [(n,m)D\mathbf{N}]$, $S \in [(n,m)D\mathbf{N}]$ *and* $Q_1 = Q_2 = 0$, *where*

$$Q_1 = \sum_{0 \leqslant j \leqslant n-1} \left(T^D\right)^j X \left(S^D\right)^{n-1-j}, \quad Q_2 = \sum_{0 \leqslant j \leqslant m-1} S^{*m-1-j} R^* T^{*m}$$

and X *is given by (1.3).*

Proof. In view of lemma 1.2, V is Drazin invertible and $V^D = \begin{pmatrix} T^D & X \\ 0 & S^D \end{pmatrix}$. It can be easily verified that

$$\left(V^D\right)^n = \begin{pmatrix} \left(T^D\right)^n & \sum_{0 \leqslant j \leqslant n-1} \left(T^D\right)^j X \left(S^D\right)^{n-1-j} \\ 0 & \left(S^D\right)^n \end{pmatrix} = \begin{pmatrix} \left(T^D\right)^n & \mathscr{Q}_1 \\ 0 & \left(S^D\right)^n \end{pmatrix}$$

and

$$V^{*m} = \begin{pmatrix} T^{*m} & 0 \\ \sum_{0 \leqslant j \leqslant m-1} S^{*m-1-j} R^* T^{*m} & S^{*m} \end{pmatrix} = \begin{pmatrix} T^{*m} & 0 \\ Q_2 & S^{*m} \end{pmatrix}.$$

According to the Definition 2.1, we must have

$$\left(V^D\right)^n V^{*m} = V^{*m} \left(V^D\right)^n \Leftrightarrow \begin{cases} \left(T^D\right)^n T^{*m} = T^{*m} \left(T^D\right)^n, \\ \left(S^D\right)^n S^{*m} = S^{*m} \left(S^D\right)^n \\ Q_1 = Q_2 = 0 \end{cases}$$

This finishes the proof of theorem. □

3. (n,m)-power D-quasi-normal operators

In this section, the class of (n,m)-power D-quasi-normal operators as a generalization of the classes of (n,m)-power D-normal operators is introduced. In addition, we make several observations about members from this class.

DEFINITION 3.1. Let $T \in \mathscr{B}(\mathscr{H})^D$. We said that T is (n,m)-power D-quasi normal if

$$\left(T^D\right)^n \left(T^{*m} T\right) = \left(T^{*m} T\right) \left(T^D\right)^n \tag{3.1}$$

for some positive integers n,m. This class of operators will be denoted by $[(n,m)D\mathbf{QN}]$.

REMARK 3.1. We make the following observations.

(1) $[(1,1)D\mathbf{QN}]$ is the class of D-quasi-normal operator, i.e. $[(1,1)D\mathbf{QN}] = [D\mathbf{QN}]$.

(2) $[(n,1)D\mathbf{QN}]$ is the class of n-power D-quasi normal: $[(n,1)D\mathbf{QN}] = [nD\mathbf{QN}]$.

(3) Every n-power D-quasi-normal is an (n,m)-power D-quasi-normal:

$$[nD\mathbf{QN}] \subset [(n,m)D\mathbf{QN}].$$

(4) Every (n,m)-power D-normal is an (n,m)-power D-quasi-normal, i.e.

$$[(n,m)D\mathbf{N}] \subset [(n,m)D\mathbf{QN}].$$

(5) $T \in [(n,m)D\mathbf{QN}] \Longleftrightarrow [\left(T^D\right)^n, T^{*m}T] = 0$.

(6) If T is (n,m)-power D-quasi-normal such that T has a dense range, then T is (n,m)- power D-normal.

(7) If T is (n,m)-power D-quasi-normal, then T is $(2n,m)$-power D-quasi-normal.

REMARK 3.2. Clearly, the class of (n,m)-power D-quasi-normal operators includes class of (n,m)-power quasi-normal, i.e. the following inclusion holds

$$[(n,m)\mathbf{QN}] \subset [(n,m)D\mathbf{QN}].$$

We give the following example to show that there exists a (n,m)-power D-quasi-normal operator which is neither a (n,m)-power D-normal nor n-power D-quasi-normal for some integers n and m.

EXAMPLE 3.1. Let $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ be an operator on Hilbert space $\mathbb{C}^3$, it is easy to check that $T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then by computations we get that

$$\left(T^D\right)^2\left(T^{*2}T\right) = \left(T^{*2}T\right)\left(T^D\right)^2, \quad \left(T^D\right)^2 T^{*2} \neq T^{*2}\left(T^D\right)^2 \text{ and } \left(T^D\right)^2 T^*T \neq T^*T\left(T^D\right)^2$$

This shows that T is $(2,2)$-power D-quasi-normal which is neither $(2,2)$-power D-normal nor 2-power D-quasi-normal.

The following proposition gives a characterization of an (n,m)-power D-quasi-normal operators.

PROPOSITION 3.1. *Let $T \in \mathscr{B}(\mathscr{H})^D$, $A = \left(T^D\right)^n + T^{*m}T$ and $B = \left(T^D\right)^n - T^{*m}T$. Then T is of class $[(n,m)D\mathbf{QN}]$ if and only if A commutes with B.*

Proof. Commutativity of A and B is equivalent to $\left(T^D\right)^n T^{*m}T = T^{*m}T\left(T^D\right)^n$. □

PROPOSITION 3.2. *Let T, A, B be as in Proposition 3.1. If T is of class $[(n,m)D\mathbf{QN}]$, then $\left(T^D\right)^n T^{*m}T$ commutes with A and B.*

Proof. By (3.1) we have that

$$\left(T^D\right)^n T^{*m}T\left(\left(T^D\right)^n \pm T^{*m}T\right) = \left(\left(T^D\right)^n \pm T^{*m}T\right)\left(T^D\right)^n T^{*m}T. \quad \square$$

In general, the two classes $[(n,m)D\mathbf{QN}]$ and $[(n+1,m)D\mathbf{QN}]$ are not the same (see [21]).

PROPOSITION 3.3. *Let $T \in \mathscr{B}(\mathscr{H})^D$. If T is both of class $[(n,m)D\mathbf{QN}]$ and $[(n+1,m)D\mathbf{QN}]$, then it is of class $[(n+2,m)D\mathbf{QN}]$, i.e.*

$$[(n,m)D\mathbf{QN}] \cap [(n+1,m)D\mathbf{QN}] \subset [(n+2,k)D\mathbf{QN}].$$

Proof. Since T is both of class $[(n,m)D\mathbf{QN}]$ and $[(n+1,m)DQN]$, it follows that

$$\left(T^D\right)^{n+1} T^{*m}T = T^{*m}T\left(T^D\right)^{n+1} \text{ and } \left(T^D\right)^n T^{*m}T = T^{*m}T\left(T^D\right)^n.$$

Now

$$\left(T^D\right)^{n+2} T^{*m}T = (T^D)T^{*m}T\left(T^D\right)^{n+1} = \left(T^D\right)^{n+1}(T^{*m}T\left(T^D\right) = T^{*m}T\left(T^D\right)^{n+2}$$

so that $\left(T^D\right)^{n+2} T^{*m}T$ may be transformed into $T^{*m}T\left(T^D\right)^{n+2}$. $\square$

In [21] it was probed that if T is of class $[n\mathbf{QN}]$ such that T is a partial isometry, then T is of class $[(n+1)\mathbf{QN}]$. We extend this result to the class of $[(n,m)\mathbf{QN}]$ as follows.

THEOREM 3.1. *Let $T \in \mathscr{B}(\mathscr{H})$ be of class $[(n,m)\mathbf{QN}]$ for some positive integers n and m such that $n \geqslant m$. If T^m is a partial isometry, then T is of class $[(n+m,m)\mathbf{QN}]$.*

Proof. Under the assumption that T^m is a partial isometry, we have

$$T^m T^{*m} T^m = T^m,$$

or equivalently

$$T^m(T^{*m}T)T^{m-1} = T^m. \tag{3.2}$$

Multiplying (3.2) to the left by T^{n-m} and to the right by T we get

$$T^n(T^{*m}T)T^m = T^{n+1}. \tag{3.3}$$

Multiplying (3.2) to the right by T^{n-m+1} we get

$$T^m(T^{*m}T)T^n = T^{n+1}. \tag{3.4}$$

Combining (3.3) and (3.4) and using the fact that $T \in [(n,m)\mathbf{QN}]$ we get

$$T^{n+m}(T^{*m}T) = (T^{*m}T)T^{n+m}.$$

Therefore T is $(n+m.m)$-power D-quasi-normal. $\square$

THEOREM 3.2. *Let $T \in \mathscr{B}(\mathscr{H})^D$ be an (n,m)-power D-quasi-normal for some integers n and m with $n \geqslant m$. If T^m is a partial isometry, then is of class $[(n+m,m)D\mathbf{QN}]$.*

Proof. Firstly, observe that since T is a Drazin invertible we have $\left(T^D\right)^2 T = T^D$ from which it is easily to obtain that

$$\left(T^D\right)^{2k-1} T^{k-1} = \left(T^D\right)^k \qquad k \geqslant 1.$$

Since T^m is partial isometry by

$$T^m T^{*m} T^m = T^m. \tag{3.5}$$

or equivalently

$$T^m (T^{*m} T) T^{m-1} = T^m. \tag{3.6}$$

Multiplying (3.6) to the left by $\left(T^D\right)^{n+m}$ and to the right by $\left(T^D\right)^{2m-1}$ we get

$$\left(T^D\right)^n (T^{*m} T)\left(T^D\right)^m = \left(T^D\right)^{n+2m-1}. \tag{3.7}$$

Multiplying (3.6) to the left by $\left(T^D\right)^{2m}$ and to the right by $\left(T^D\right)^{n+2m-1}$ we get

$$\left(T^D\right)^m (T^{*m} T)\left(T^D\right)^n = \left(T^D\right)^{n+2m-1}. \tag{3.8}$$

In view of (3.7) and (3.8) we obtain

$$\left(T^D\right)^n (T^{*m} T)\left(T^D\right)^m = \left(T^D\right)^m (T^{*m} T)\left(T^D\right)^n.$$

By taking into account that T is of class $[(n,m)D\mathbf{QN}]$ we obtain

$$(T^{*m} T)\left(T^D\right)^{n+m} = \left(T^D\right)^{n+m} (T^{*m} T).$$

Thus T is of class $[(n+m,m)D\mathbf{QN}]$. $\square$

The class $[(n,m)D\mathbf{QN}]$ has the following properties.

THEOREM 3.3. *The class $[(n,m)D\mathbf{QN}]$ is closed under unitary equivalence.*

Proof. Let $S \in \mathscr{B}(\mathscr{H})$ be unitary equivalent to T. Then there is a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that $T = U^* S U$ which implies that $T^* = U^* S^* U$. Noting that $T^n = U^* S^n U$, $T^{*m} T = U^* S^{m*} S U$ and $\left(U^* T U\right)^D = U^* T^D U$. Inserting $I = U U^*$ suitably, then if T is of class $[(n,m)D\mathbf{QN}]$ we deduce that

$$U^* \left(S^D\right)^n S^{*m} S U = (T^D)^n T^{*m} T = T^{*m} T \left(T^D\right)^n = U^* S^{*m} S \left(S^D\right)^n U.$$

Therefore S is of class $[(n,m)D\mathbf{QN}]$. $\square$

The following examples show that a (n,m)-power D-quasi-normal need not be a $(n+1,m)$-power D-quasi-normal and vice versa.

EXAMPLE 3.2. Let us consider the operator T given in Example 3.1. It easily to check that T is $(2,2)$-power D-quasi-normal but T is not $(3,2)$-power D-quasi-normal.

EXAMPLE 3.3. Let us consider the matrix operator $T=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ acting on $\mathbb{C}^2$. Then $T^D=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. By simple calculations, it follows that T is of class $[(3,2)D\mathbf{QN}]$ but not of class $[(2,2)D\mathbf{QN}]$.

PROPOSITION 3.4. *Let $T\in\mathscr{B}(\mathscr{H})^D$. If T is both of class $[(n,m)D\mathbf{QN}]$ and $[(n+1,m)D\mathbf{QN}]$ such that T^D is injective, then T is of class $[(1,m)D\mathbf{QN}]$.*

Proof. Since T is of class $[(n,m)D\mathbf{QN}]\cap[(n+1,m)D\mathbf{QN}]$, it follows that

$$(T^D)^n\Big(T^DT^{*m}T-T^{*m}TT^D\Big)=0.$$

If T^D is injective, then so is $(T^D)^n$ and we have $T^DT^{*m}T-T^{*m}TT^D=0$, whence T is of class $[(1,m)D\mathbf{QN}]$. □

PROPOSITION 3.5. *Let $T\in\mathscr{B}(\mathscr{H})^D$ such that T is of class $[(2,m)D\mathbf{QN}]\bigcap[(3,m)D\mathbf{QN}]$ for some positive integer m, then T is of class $[(n,m)D\mathbf{QN}]$ for all positive integer $n\geqslant 4$.*

Proof. We prove the assertion by using the mathematical induction. For $n=4$ it is a consequence of the statement (7) of Remark 3.1.

We prove this for $n=5$. Since $T\in[(2,m)D\mathbf{QN}]$,

$$\big(T^D\big)^2T^{*m}T=(T^{*m}T\big(T^D\big)^2, \tag{3.9}$$

multiplying (3.9) to the left by $\big(T^D\big)^3$ we get

$$\big(T^D\big)^5T^{*m}T=\big(T^D\big)^3T^{*m}T\big(T^D\big)^2.$$

Thus implies

$$\big(T^D\big)^5T^{*m}T=T^{*m}T\big(T^D\big)^5.$$

Now assume that the result is true for $n\geqslant 5$ that is

$$\big(T^D\big)^nT^{*m}T=T^{*m}T\big(T^D\big)^n,$$

then

$$\begin{aligned}\left(T^D\right)^{n+1}T^{*m}T &= T^DT^{*m}T\left(T^D\right)^n\\ &= T^DT^{*m}T\left(T^D\right)^2\left(T^D\right)^{n-2}\\ &= \left(T^D\right)^3T^{*m}T\left(T^D\right)^{n-2}\\ &= T^{*m}T\left(T^D\right)^{n+1}.\end{aligned}$$

Thus T is of class $[(n+1,m)D\mathbf{QN}]$. The proof is complete. □

PROPOSITION 3.6. *Let $T\in\mathscr{B}(\mathscr{H})^D$ be of class $[(n,m)D\mathbf{QN}]$ and of class $[(n,m+1)D\mathbf{QN}]$ for some positive integers n and m. If T is injective, then T^* is of class $[nD\mathbf{N}]$.*

Proof. Since $T\in[(n,m)D\mathbf{QN}]\cap[(n,m+1)D\mathbf{QN}]$, we have

$$\begin{aligned}&\left(T^D\right)^nT^{*m+1}T-T^{*m+1}T\left(T^D\right)^n=0\\ \Rightarrow\ &\left(T^D\right)^nT^*T^{*m}T-T^*T^{*m}T\left(T^D\right)^n=0\\ \Rightarrow\ &\left[\left(T^D\right)^nT^*-T^*\left(T^D\right)^n\right]T^{*m}T=0\\ \Rightarrow\ &T^*T^m\left[T\left((T^*)^D\right)^n-\left((T^*)^D\right)^nT\right]=0\\ \Rightarrow\ &T^{*m}T^m\left[T\left((T^*)^D\right)^n-\left((T^*)^D\right)^nT\right]=0\\ \Rightarrow\ &\left[T\left((T^*)^D\right)^n-\left((T^*)^D\right)^nT\right]=0\qquad\left(\text{by }\mathscr{N}(T^{*m}T^m)=\{0\}\right)\\ \Rightarrow\ &T\left((T^*)^D\right)^n=\left((T^*)^D\right)^nT=0.\end{aligned}$$

Therefore T^* is of class $[nD\mathbf{N}]$. □

THEOREM 3.4. *Let $T\in\mathscr{B}(\mathscr{H})^D$ of class $[(n,m)D\mathbf{QN}]$. If T is m-power normal, then T^D is of class $[(n,m)\mathbf{QN}]$.*

Proof. Since T is (n,m)-power D-quasi-normal, it follows in view of lemma 1.1 that

$$\begin{aligned}&\left(T^D\right)^nT^{*m}T-T^{*m}T\left(T^D\right)^n=0\\ \Rightarrow\ &\left(T^D\right)^n\left(T^{*m}T\right)^D-\left(T^{*m}T\right)^D\left(T^D\right)^n=0\\ \Rightarrow\ &\left(T^D\right)^n((T)^D)^{*m}T^D-((T)^D)^{*m}T^D\left(T^D\right)^n=0\quad(\text{since }\ T^{*m}T=TT^{*m}).\end{aligned}$$

Hence T^D is (n,m)-power quasi-normal operator as required. □

PROPOSITION 3.7. *The set of all (n,m)-power D-quasi-normal operators on $\mathscr{H}$ is closed subset of $\mathscr{B}(\mathscr{H})^D$ with is closed under scalar multiplication.*

THEOREM 3.5. *Let $(T_k)_{1\leqslant k\leqslant d}\in\left(\mathscr{B}(\mathscr{H})^D\right)^d$ such that each T_k is (n,m)-power D-quasi-normal, then*

(1) $T_1 \oplus T_2 \oplus \cdots \oplus T_d$ *is a* (n,m)*-power* D*-quasi-normal.*

(2) $T_1 \otimes T_2 \otimes \cdots \otimes T_d$ *is a* (n,m)*-power* D*-quasi-normal.*

Proof. The proof of this theorem is formally the same as the proof of Theorem 2.4 with suitable changes and thus we omit the details. □

In the following theorem, we collect some further of basic properties of the class $[(n,m)D\mathbf{QN}]$.

THEOREM 3.6. *Let* T *is of class* $[(n,m)D\mathbf{QN}]$*. Then the following properties hold.*

(1) If $\mathscr{M}$ *is a closed subspace of* $\mathscr{H}$ *such that* T *is a reducing subspace for* T*, then* $T_{/\mathscr{M}}$ *is of class* $[(n,m)D\mathbf{QN}]$.

(2) If $S \in \mathscr{B}(\mathscr{H})$ *is of class* $[(n,m)D\mathbf{QN}]$ *such that* $[T,S] = [T,S^*] = 0$*, then* TS *is of class* $[(n,m)D\mathbf{QN}]$.

(3) If S *is of class* $[\mathbf{N}]$ *such that* $[T,S] = 0$*, then* TS *is of class* $[(n,m)D\mathbf{QN}]$.

(4) If S *is of class* $[(n,m)D\mathbf{QN}]$ *such that* $[T,S] = [T,S^*] = 0$*, then* $T+S$ *is of class* $[(n,m)D\mathbf{QN}]$.

(5) If S *is of class* $[\mathbf{N}]$ *such that* $[T,S] = 0$*, then* $T+S$ *is of class* $[(n,m)D\mathbf{QN}]$.

Proof. By analogous arguments as in the proof of [11, Theorem 4.7], we show the statements $(1)-(5)$. □

Now we discuss (n,m)-power D-quasi-normality of an operator under some commuting conditions on the real and imaginary part of its Drazin inverse.

THEOREM 3.7. *Let* $T \in \mathscr{B}(\mathscr{H})^D$ *such that* $\mathscr{R}(T^{m-1})$ *is dense. Then* T *is of class* $[(n,m)D\mathbf{QN}]$ *if and only if* C_m *commutes with* $Re\big(T^D\big)^n$ *and* $Im\big(T^D\big)^n$.

Proof. Let T be (n,m)-power D-quasi-normal, i.e.

$$\big(T^D\big)^n\big(T^{*m}T\big) = \big(T^{*m}T\big)\big(T^D\big)^n,$$

it follows that

$$\big(T^D\big)^n\big(T^{*m}T^m\big) = \big(T^{*m}T^m\big)\big(T^D\big)^n,$$

i.e. $C_m^2\big(ReT^D\big)^n = \big(ReT^D\big)^n C_m^2$. Since C_m is non-negative definite, it follows that

$$C_m Re\big(T^D\big)^n = \big(ReT^D\big)^n C_m.$$

In a Similar way we can prove that $C_m\big(ImT^D\big)^n = Im\big(T^D\big)^n C_m$.

Conversely, assume that $C_m Re\big(T^D\big)^n = Re\big(T^D\big)^n C_m$ and $C_m Im\big(T^D\big)^n = Im\big(T^D\big)^n C_m$. Then

$$C_m^2 Re\big(T^D\big)^n = \big(ReT^D\big)^n C_m^2 \quad\text{and}\quad C_m^2 Im\big(T^D\big)^n = Im\big(T^D\big)^n C_m^2.$$

Hence

$$C_m^2\big(Re\big(T^D\big)^n + iIm\big(T^D\big)^n\big) = \big(Re\big(T^D\big)^n + iIm\big(T^D\big)^n\big)C_m^2$$

and we have $C_m^2\big(T^D\big)^n = \big(T^D\big)^n C_m^2$.

On the other hand, we have

$$\begin{aligned} C_m^2\big(T^D\big)^n = \big(T^D\big)^n C_m^2 &\Leftrightarrow T^{*m}T^m\big(T^D\big)^n - \big(T^D\big)^n T^{*m}T^m = 0 \\ &\Leftrightarrow \Big(T^{*m}T\big(T^D\big)^n - \big(T^D\big)^n T^{*m}T\Big)T^{m-1} = 0 \\ &\Leftrightarrow T^{*m}T\big(T^D\big)^n - \big(T^D\big)^n T^{*m}T = 0 \quad \big(\overline{\mathscr{R}(T^{m-1})} = \mathscr{H}\big). \end{aligned}$$

Therefore T is of class $[(n,m)D\mathbf{QN}]$. $\square$

THEOREM 3.8. *Let T is of class $[(n,m)D\mathbf{QN}]$ for some positive integers n and m for which $n \geqslant m$. Assume that $\mathscr{R}(T^{m-1})$ is dense and $C_m^2\big(T^D\big)^n = \big(T^D\big)^n B_m^2$, then B_m commutes with $Re\big(T^D\big)^n$ and $Im\big(T^D\big)^n$.*

Proof. Since T is of class $[(n,m)D\mathbf{QN}]$, then

$$\big(T^D\big)^n T^{*m}T^n = T^{*m}T^n\big(T^D\big)^n \text{ and } \big(T^D\big)^n T^{*m}T^n = T^{*m}T^n\big(T^D\big)^n.$$

$$\begin{aligned} B_m^2 Re\big(T^D\big)^n &= \frac{1}{2}T^mT^{*m}\Big(\big(T^D\big)^n + \big(T^D\big)^{*n}\Big) \\ &= \frac{1}{2}\Big(T^mT^{*m}\big(T^D\big)^n + T^mT^{*m}\big(T^D\big)^{*n}\Big) \\ &= \frac{1}{2}\Big(T^mT^{*m}T^n\big(T^D\big)^{2n} + T^m\big(T^D\big)^{*n}T^{*m}\Big) \\ &= \frac{1}{2}\Big(T^m\big(T^D\big)^{2n}T^{*m}T^n + T^mT^{*n}\big(T^D\big)^{*2n}T^{*m}\Big) \\ &= \frac{1}{2}\Big(\big(T^D\big)^{2n-m}(T^{*m}T^m)T^{n-m} + T^mT^{*m}\big(T^D\big)^{*n}T^{*n-m}\big(T^D\big)^{*n}T^{*m}\Big) \\ &= \frac{1}{2}\Big(T^{*m}T^m\big(T^D\big)^{2n-m}T^{n-m} + \big(T^D\big)^{*n}T^{*m}T^m\big(T^D\big)^{*m}T^{*m}\Big) \\ &= \frac{1}{2}\Big(T^{*m}T^m\big(T^D\big)^n + T^{*m}T^m\big(T^D\big)^{*n}\Big) \\ &= \frac{1}{2}\Big(\big(T^D\big)^n T^mT^{*m} + \big(T^D\big)^{*n}T^mT^{*m}\Big) \\ &= Re\big(T^D\big)^n B_m^2. \end{aligned}$$

Consequently, $B_m Re\left(T^D\right)^n = Re\left(T^D\right)^n B_m$.

In a similar way, we can prove that $B_m Im\left(T^D\right)^n = Im\left(T^D\right)^n B_m$. □

PROPOSITION 3.8. *Let $T \in \mathscr{B}(\mathscr{H})^D$. If T is of class $[(n,m)D\mathbf{QN}]$ such that $\mathscr{N}(T^*) \subset \mathscr{N}(T^D)$, then T is of class $[(n,m)D\mathbf{N}]$.*

Proof. We consider the following two cases:

Case I: If $\mathscr{R}(T)$ is dense, then T is of class $[(n,m)D\mathbf{N}]$.

Case II: If $\mathscr{R}(T)$ is not dense. By the fact that T is of class $[(n,m)D\mathbf{QN}]$ it is easily to see that

$$\left(\left(T^D\right)^n T^{*m} - T^{*m}\left(T^D\right)^n\right) T = 0,$$

which implies that

$$\left(T^D\right)^n T^{*m} - T^{*m}\left(T^D\right)^n = 0 \text{ on } \overline{\mathscr{R}(T)}.$$

We now apply the condition $\mathscr{N}(T^*) \subset \mathscr{N}(T^D)$ to conclude that

$$\left(T^D\right)^n T^{*m} - T^{*m}\left(T^D\right)^n = 0 \text{ on } \mathscr{N}(T^*).$$

Combining these we have that T belong to $[(n,m)D\mathbf{N}]$. □

4. Fuglede-Putnam theorem for (n,m)-power D-normal operatoes

The Fuglede-Putnam theorem plays a major role in the theory of bounded operators. Many authors have worked on it since the papers of B. Fuglede [13] and then by C. R. Putnam [20]. A. Bachir and M. W. Altanji [3] generalized that theorem for (p,k)-quasiposinormal operators, M. Hichem Mortad [19] generalized this theorem to isometry and co-isometry operators. There were various generalizations of Fuglede-Putnam's theorem to nonnormal operators, we only cite [5, 14, 23].

The famous Fuglede-Putnam's theorem is as follows:

THEOREM 4.1. *(Fuglede-Putnam). Let $M,N \in \mathscr{B}(\mathscr{H})$ be normal and $T \in \mathscr{B}(\mathscr{H})$. If $TN = MT$, then $TN^* = M^*T$.*

REMARK 4.1. If $N = M$, this is Fuglede's theorem.

REMARK 4.2. Here, we give out an example that if $A \in \mathscr{B}(\mathscr{H})$ and T is (n,m)-power D-normal satisfying $TA = AT$, we can not get $T^*A = AT^*$.

To see this, just consider the following example.

EXAMPLE 4.1. Let T the operator represented by the matrix $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A direct computation shows the following

- T is D-normal and $AT = TA$ but $AT^* \neq T^*A$.
- T^D is normal and $AT^D = T^DA$ but $AT^* \neq T^*A$.

PROPOSITION 4.1. *Let T be (n,m)-power D-normal operators, for some positive integers n and m. If there exists another operator A, such that, $TA = AT$, then we have $\left(T^D\right)^{*k}A = A\left(T^D\right)^{*k}$ where k is the least common multiple of n and m. In particular $\left(T^D\right)^{*nm}A = A\left(T^D\right)^{*nm}$.*

Proof. From the hypothesis $TA = AT$, we have $T^kA = AT^k$. In view of lemma 1.1, we have $\left(T^D\right)^kA = A\left(T^D\right)^k$. By applying Theorem 2.1 we have $\left(T^D\right)^k$ is normal and so that $\left(T^D\right)^{*k}A = A\left(T^D\right)^{*k}$ (by Fuglede theorem). □

PROPOSITION 4.2. *Let T,S be (n,m)-power D-normal operators, for some positive integers n and m. If there exists another operator A, such that, $T^DA = AS^D$, then*

(i) $\left(T^D\right)^{*k}A = A\left(S^D\right)^{*k}$ *where k is the least common multiple of n and m.*

(ii) $\left(T^D\right)^{*nm}A = A\left(S^D\right)^{*nm}$.

Proof.

(i) By the assumption that $T^DA = AS^D$, it is easy to see that

$$\left(T^D\right)^kA = A\left(S^D\right)^k.$$

Now, from the statement (2) of Proposition 2.1, it follows that, T^D and S^D are (n,m)-power normal. In particular, $\left(T^D\right)^k$ and $\left(S^D\right)^k$, are normal operators. Thus, the pair, $\left(\left(S^D\right)^k, \left(T^D\right)^k\right)$, do satisfy the Fuglede-Putnam theorem.

The statement (ii) follows by similar arguments. □

THEOREM 4.2. *Let $T \in [(n,m)D\mathbf{N}]$, for some positive integers n and m, and $S \in [D\mathbf{N}]$. If there exists another operator A, such that, $\left(T^D\right)^kA = AS^D$ where k is the least common multiple of n and m. Then we have*

$$\left(T^D\right)^*A = A\left(S^D\right)^*.$$

Proof. Under the assumptions we have

$$T \in [(n,m)D\mathbf{N}] \Rightarrow T^D \in [(n,m)\mathbf{N}] \Rightarrow \left(T^D\right)^k \in [\mathbf{N}] \Leftrightarrow T^D \in [(k,1)\mathbf{N}].$$

and

$$S \in [D\mathbf{N}] \Leftrightarrow S^D \in [\mathbf{N}].$$

Since $\left(T^D\right)^k A = AS^D$, it follows from [16] that $\left(T^D\right)^* A = A\left(S^D\right)^*$. $\square$

COROLLARY 4.1. *Let $T \in [(n,m)D\mathbf{N}]$, for some positive integers n and m, and $S \in [D\mathbf{N}]$. If there exists another operator A, such that, $\left(T^D\right)^{nm} A = AS^D$, then*

$$\left(T^D\right)^* A = A\left(S^D\right)^*.$$

THEOREM 4.3. *Let T, S be injective (n,m)-power D-normal operators, for some positive integers n and m. If there exists another operator A, such that,*

$$\left(T^D\right)^k A = A\left(S^D\right)^k \text{ where } k \text{ is the least common multiple of } n \text{ and } m,$$

then

$$\left(T^D\right)^* A = A\left(S^D\right)^*.$$

Proof. Firstly, we not that, the assumptions that T and S are injective implies that T^D and S^D are injective. In view of Proposition 2.1 we know that $T^D \in [(n,m)\mathbf{N}]$ and $S^D \in [(n,m)\mathbf{N}]$. Hence $\left(T^D\right)^k$ and $\left(S^D\right)^k$ are normal operator (Theorem 2.1), i.e; T^D and S^D are k-power normal. As $\left(T^D\right)^k A = A\left(S^D\right)^k$, it follows from [16] that $\left(T^D\right)^* A = A\left(S^D\right)^*$. $\square$

We omit the proof of the following corollary since the techniques are similar to those of the proof of Theorem 4.3.

COROLLARY 4.2. *Let T, S be injective (n,m)-power D-normal operators, for some positive integers n and m. If there exists another operator A, such that, $\left(T^D\right)^{nm} A = A\left(S^D\right)^{nm}$, then $\left(T^D\right)^* A = A\left(S^D\right)^*$.*

THEOREM 4.4. *Let $T \in [D\mathbf{N}]$ and $A, B \in \mathscr{B}(\mathscr{H})$ satisfying*

$$AT = TB \tag{4.1}$$

$$BT = TA, \tag{4.2}$$

$$(A-B)T^D = -T^D(A-B), \tag{4.3}$$

Then

$$\left(T^D\right)^* A = B\left(T^D\right)^* \text{ and } \left(T^D\right)^* B = A\left(T^D\right)^*.$$

Proof. By adding (4.1) and (4.2) we get

$$(A+B)T = T(A+B) \tag{4.4}$$

and we deduce from Lemma 1.1 that $(A+B)T^D = T^D(A+B)$. Since T^D is normal, it follows by Fuglede theorem that

$$(A+B)\left(T^D\right)^* = \left(T^D\right)^*(A+B)$$

and so that

$$A\left(T^D\right)^* - \left(T^D\right)^*B = \left(T^D\right)^*A - B\left(T^D\right)^*. \tag{4.5}$$

Also by (4.3) we have by same reason

$$(A-B)\left(T^D\right)^* = -\left(T^D\right)^*(A-B), \tag{4.6}$$

By adding (4.5) with (4.6) we get

$$2A\left(T^D\right)^* - 2\left(T^D\right)^*B.$$

Therefore

$$A\left(T^D\right)^* = \left(T^D\right)^*B.$$

On the other hand, By subtracting (4.5) from (4.6) we get

$$2\left(T^D\right)^*A - 2B\left(T^D\right)^* = 0$$

and so that

$$\left(T^D\right)^*A = B\left(T^D\right)^*.$$

The proof is completed. □

COROLLARY 4.3. *Let $T \in [(n,m)D\mathbf{N}]$ and $A, B \in \mathscr{B}(\mathscr{H})$ satisfying*

$$AT^k = T^kB, \tag{4.7}$$

$$BT^k = T^kA, \tag{4.8}$$

$$(A-B)\left(T^D\right)^k = -\left(T^D\right)^k(A-B), \tag{4.9}$$

where k is the least common multiple of n and m. Then

$$\left(T^D\right)^{*k}A = B\left(T^D\right)^{*k} \text{ and } \left(T^D\right)^{*k}B = A\left(T^D\right)^{*k},$$

Proof. The proof goes along the same lines as the proof of Theorem 4.4.

Under the assumption that T is of class $[(n,m)D\mathbf{N}]$, it follows that T^k is of class $[D\mathbf{N}]$ (Proposition 2.1). Moreover T^k satisfies the conditions of Theorem 4.4, the conclusion of Corollary 4.3 holds. □

COROLLARY 4.4. *Let* $T \in [(n,m)D\mathbf{N}]$ *and* $A,B \in \mathscr{B}(\mathscr{H})$ *satisfying*

$$AT^{nm} = T^{nm}B, \tag{4.10}$$

$$BT^{nm} = T^{nm}A, \tag{4.11}$$

$$(A-B)\left(T^D\right)^{nm} = -\left(T^D\right)^{nm}(A-B), \tag{4.12}$$

Then

$$\left(T^D\right)^{*nm}A = B\left(T^D\right)^{*nm} \text{ and } \left(T^D\right)^{*nm}B = A\left(T^D\right)^{*nm}.$$

Proof. Since $T \in [(n,m)DN]$, it follows that T^{nm} is D-normal by Theorem 2.1. However T^{nm} satisfy the conditions of Theorem 4.4 and consequently the statements of the corollary are proved. □

Recall that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be decomposable if for every open cover $\{U,V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathscr{N}$ and $\mathscr{M}$ such that $H = \mathscr{N} + \mathscr{M}$, $\sigma(T|\mathscr{N}) \subset \overline{U}$ and $\sigma(T|\mathscr{M}) \subset \overline{V}$. Remark that T is decomposable if and only if T and T^* have the property (β) ([18, Theorem 2.5.19]).

THEOREM 4.5. *Let* $T \in \mathscr{B}(\mathscr{H})^D$ *be* (n,m)*-power* D*-normal, then* T^D *is decomposable.*

Proof. Since T is (n,m)-power D-normal by Proposition 2.1 and Theorem 2.1 T^{nm} and T^{*nm} are D-normal or equivalently $\left(T^D\right)^{nm}$ and $\left(T^D\right)^{*nm}$ are normal. Hence $\left(T^D\right)^{nm}$ is decomposable. In view of [18, Theorem 3.3.9], we deduce that T^D is decomposable. □

Conclusion

Here by concluded that this paper shows that there is a lot of work that can be done in the area of Drazin invertible operators on a Hilbert space. In further studies this work will be extended and serves as a tool for other works.

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