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# 2–LOCAL $*$–LIE AUTOMORPHISMS OF SEMI–FINITE FACTORS

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*Abstract.* Let $\mathscr{M}$ be a semi-finite von Neumann algebra factor on a complex Hilbert space $H$ with dimension greater than 3. Then every surjective 2-local $*$-Lie automorphism $\Phi$ of $\mathscr{M}$ is of the form $\Phi = \Psi + \tau$, where $\Psi$ is a $*$-automorphism or the negative of a $*$-anti-automorphism of $\mathscr{M}$, and $\tau$ is a mapping from $\mathscr{M}$ into $\mathbb{C}I$ vanishing on every sum of commutators.

## 1. Introduction and preliminaries

Let $\mathscr{A}$ be an algebra over the complex field $\mathbb{C}$. Recall that a linear map $\theta$ on $\mathscr{A}$ is called a local isomorphism (respectively, local derivation) if for each $A \in \mathscr{A}$, there exists an isomorphism (respectively, a derivation) $\theta_A$, depending on $A$, such that $\theta(A) = \theta_A(A)$. The local map problem was initiated by Kadison [1] and Larson and Sourour [2] in 1990. In the past decades, the study of local maps has attracted much attention of scholars. There exists a vast literature on local isomorphisms and local derivations. Some results on them are contained in [3, 4, 5, 6, 7, 8].

In 1997, Šemrl [9] introduced the notion of 2-local maps. Recall that a (non-necessarily linear) map $\theta$ on an algebra $\mathscr{A}$ is called a 2-local isomorphism (respectively, 2-local derivation) if for any $A, B \in \mathscr{A}$, there exists an isomorphism (respectively, a derivation) $\theta_{A,B}$, depending on $A$ and $B$, such that $\theta(A) = \theta_{A,B}(A)$ and $\theta(B) = \theta_{A,B}(B)$. Recently, 2-local maps have been studied on different operator algebras by many authors. In [9], Šemrl studied 2-local isomorphisms and 2-local derivations on the algebra of all bounded linear operators on an infinite dimensional separable Hilbert space. Ayupov and Kudaybergenov [10] studied 2-local derivations and automorphisms on $B(H)$ and in [11] they described 2-local derivations on von Neumann algebras. We can refer to [12, 13, 14, 15, 16] for more about 2-local maps.

Let $\mathscr{A}$ and $\mathscr{B}$ be Banach $*$-algebras. Recall that a (non-necessarily linear) bijection $\phi : \mathscr{A} \to \mathscr{B}$ is called a Lie $*$-isomorphism if $\phi([A,B]) = [\phi(A), \phi(B)], \phi(A^*) = \phi(A)^*$ and called a $*$-Lie isomorphism if $\phi([A,B^*]) = [\phi(A), \phi(B)^*]$ for $A, B \in \mathscr{A}$, where $[A,B] = AB - BA$ is the usual Lie product of $A$ and $B$. Contrary to Lie $*$-isomorphisms, a $*$-Lie isomorphism need not preserve Lie products nor involution. A $*$-Lie isomorphism is more general and complicated than a Lie $*$-isomorphism. Therefore, in the process of dealing with details of a $*$-Lie isomorphism, we need more skills

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and tools. Recently, Bai and Du [17, 18] studied the nonlinear $*$-Lie isomorphism. Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebra factors on a complex Hilbert space $H$ with dimension greater than 3. Bai and Du [17] proved that if $\phi : \mathscr{M} \to \mathscr{N}$ is a non-linear $*$-Lie isomorphism, then $\phi$ is of the form $\sigma + \tau$, where $\sigma$ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism of $\mathscr{M}$ onto $\mathscr{N}$ and $\tau$ is a mapping from $\mathscr{M}$ into $\mathbb{C}I$ vanishing on every commutator. Li and Lu [19] studied 2-local $*$-Lie isomorphisms. Recall that a map $\phi$ on an operator algebra $\mathscr{A}$ is called a 2-local $*$-Lie isomorphism if for each $A, B \in \mathscr{A}$, there exists a linear $*$-Lie isomorphism $\phi_{A,B}$ on $\mathscr{A}$ such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$. Let $H$ be a complex Hilbert space of dimension greater than 3. Li and Lu [19] proved that every surjective 2-local $*$-Lie isomorphism $\Phi$ of $B(H)$ has the form $\Phi = \Psi + \tau$, where $\Psi$ is a $*$-isomorphism or the negative of a $*$-anti-isomorphism of $B(H)$ and $\tau$ is a homogeneous map from $B(H)$ into $\mathbb{C}I$ vanishing on every sum of commutators.

In this paper, we generalize the result of Li and Lu to semi-finite von Neumann algebra factors on a complex Hilbert space $H$ with dimension greater than 3. The main result in the paper reads as follows. Let $\mathscr{M}$ be a semi-finite von Neumann algebra factor on a complex Hilbert space $H$ with dimension greater than 3. Then every surjective 2-local $*$-Lie automorphism $\Phi$ of $\mathscr{M}$ is of the form $\Phi = \Psi + \tau$, where $\Psi$ is a $*$-automorphism or the negative of a $*$-anti-automorphism of $\mathscr{M}$, and $\tau$ is a homogeneous mapping from $\mathscr{M}$ into $\mathbb{C}I$ vanishing on every sum of commutators.

Throughout the paper, let $\mathscr{M}_\tau$ be the set of all elements $A \in \mathscr{M}$ such that $\tau(|A|) < \infty$. Then $\mathscr{M}_\tau$ is a $*$-algebra, and moreover, $\mathscr{M}_\tau$ is a two-sided ideal of $\mathscr{M}$. Suppose that $P$ is an arbitrary projection in $\mathscr{M}$, then we set $P^\perp = I - P$.

## 2. Main results

In this section, we characterize surjective 2-local $*$-Lie automorphisms on semi-finite von Neumann algebra factors on a complex Hilbert space with dimension greater than 3. For this, we need some lemmata as follows.

Let $\mathscr{M}$ be a von Neumann algebra and let $\Phi : \mathscr{M} \to \mathscr{M}$ be a 2-local $*$-Lie automorphism. For $A, B \in \mathscr{M}$, the symbol $\Phi_{A,B}$ stands for a $*$-Lie automorphism of $\mathscr{M}$ satisfying $\Phi(A) = \Phi_{A,B}(A)$ and $\Phi(B)=\Phi_{A,B}(B)$.

LEMMA 2.1. *Let $\mathscr{M}$ be a von Neumann algebra factor and let $\Phi : \mathscr{M} \to \mathscr{M}$ be a surjective 2-local $*$-Lie automorphism. Then*

*(1) $\Phi$ is homogeneous and injective;*

*(2) $\Phi^{-1}$ is also a 2-local $*$-Lie automorphism;*

*(3) $\Phi(0) = 0$ and $\Phi(\mathbb{C}I) = \mathbb{C}I$.*

*Proof.*

(1) Let $\lambda \in \mathbb{C}$ and $A \in \mathscr{M}$. Then we have

$$\Phi(\lambda A) = \Phi_{A,\lambda A}(\lambda A) = \lambda \Phi_{A,\lambda A}(A) = \lambda \Phi(A).$$

Hence $\Phi$ is homogeneous.

If $\Phi(A) = \Phi(B)$, then $\Phi_{A,B}(A) = \Phi_{A,B}(B)$ and so $A = B$. Hence $\Phi$ is injective.

(2) For any $C, D \in \mathscr{M}$, there exist $A, B \in \mathscr{M}$ such that $\Phi(A) = C$ and $\Phi(B) = D$. Then there is a $*$-Lie automorphism $\Phi_{A,B}(A) : \mathscr{M} \to \mathscr{M}$ such that $C = \Phi(A) = \Phi_{A,B}(A)$ and $D = \Phi(B) = \Phi_{A,B}(B)$. By (1), $A = \Phi^{-1}(C) = \Phi_{A,B}^{-1}(C)$ and $B = \Phi^{-1}(D) = \Phi_{A,B}^{-1}(D)$. Notice that $\Phi_{A,B}^{-1}$ is also a $*$-Lie automorphism of $\mathscr{M}$. Hence $\Phi^{-1}$ is a 2-local $*$-Lie automorphism.

(3) By the homogeneity of $\Phi$, it is clear that $\Phi(0) = 0$. Let $\lambda \in \mathbb{C}$ and any $A \in \mathscr{M}$, we have

$$[\Phi(A), \Phi(\lambda I)^*] = [\Phi_{\lambda I,A}(A), \Phi_{\lambda I,A}(\lambda I)^*] = \Phi_{\lambda I,A}([A, \overline{\lambda} I]) = 0.$$

Since $\Phi$ is surjective, it follows that $\Phi(\lambda I)^* C = C\Phi(\lambda I)^*$ for any $C \in \mathscr{M}$. Hence $\Phi(\lambda I) \in \mathbb{C}I$, which implies $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$. Notice that $\Phi^{-1}$ is also a surjective 2-local $*$-Lie automorphism. Thus, $\Phi^{-1}(\mathbb{C}I) \subseteq \mathbb{C}I$. Hence $\Phi(\mathbb{C}I) = \mathbb{C}I$. $\square$

LEMMA 2.2. [20, Theorem 6] *Let $\mathscr{M}$ be a von Neumann algebra and $T \in \mathscr{M}$. Then $T = P + \lambda I$ for some idempotent $P \in \mathscr{M}$ and $\lambda \in \mathbb{C}$, if and only if*

$$[[[X, T], T], T] = [X, T]$$

*for every $X \in \mathscr{M}$.*

LEMMA 2.3. [17, Main theorem] *Let $\mathscr{M}, \mathscr{N}$ be von Neumann algebra factors on a complex Hilbert space $H$ with dimension greater than 3. If $\Phi : \mathscr{M} \to \mathscr{N}$ is a linear $*$-Lie isomorphism, then $\Phi$ is of the form $\Phi = \sigma + \tau$, where $\sigma$ is a linear $*$-isomorphism or the negative of a linear $*$-anti-isomorphism, and $\tau$ is a linear map from $\mathscr{M}$ into $\mathbb{C}I$ which maps commutators to zero.*

LEMMA 2.4. [21, Proposition 8.5.3 and Theorem 8.5.7] *Suppose that $\mathscr{M}$ is a von Neumann algebra factor which is not type III, then there is a faithful normal semi-finite tracial weight $\rho$ on $\mathscr{M}$ and every such weight is a positive scalar multiple of $\rho$. In particular, if $\mathscr{M}$ is a finite von Neumann algebra factor, then there is a unique state $\rho_0$ on $\mathscr{M}$ and $\rho_0$ is faithful and normal.*

LEMMA 2.5. *Let $\mathscr{M}$ be a semi-finite von Neumann algebra factor with a faithful normal semi-finite trace $\tau$ and let $\Phi$ be a 2-local $*$-Lie automorphism of $\mathscr{M}$. Then for any $A \in \mathscr{M}$ and $B = PXP^{\perp}$, where $X \in \mathscr{M}_\tau$ and $P$ is an arbitrary projection in $\mathscr{M}$, there exists $\lambda > 0$ such that $\tau(\Phi(A)\Phi(B)) = \lambda \tau(AB)$. In particular, if $\mathscr{M}$ is a finite von Neumann algebra factor, then $\tau(\Phi(A)\Phi(B)) = \tau(AB)$. Similarly, $\Phi^{-1}$ satisfies the same conclusion.*

*Proof.* For $A, B$ above, there exists a $*$-Lie automorphism $\Phi_{A,B}$ of $\mathscr{M}$ such that $\Phi(A) = \Phi_{A,B}(A)$ and $\Phi(B) = \Phi_{A,B}(B)$. Noticing that $B \in \mathscr{M}_\tau$ is a commutator, by Lemma 2.3, there exist a linear $*$-automorphism or the negative of a linear $*$-anti-automorphism $\pi_{A,B}$, and $\lambda \in \mathbb{C}$ such that $\Phi_{A,B}(A) = \pi_{A,B}(A) + \lambda I$ and $\Phi_{A,B}(B) = \pi_{A,B}(B)$.

Here, we claim that $\Phi(B), \Phi_{A,B}(B), \pi_{A,B}(B) \in \mathscr{M}_\tau$.

Indeed, either $\phi$ a $*$-automorphism or a $*$-anti automorphism, $\tau(\phi(\cdot))$ is a faithful normal semi-finite trace of $\mathscr{M}$. By Lemma 2.4, there exists $\lambda_0 > 0$ such that $\tau(\phi(\cdot)) = \lambda_0 \tau(\cdot)$. Notice that $B \in \mathscr{M}_\tau$. Hence $\tau(\phi(B)) = \lambda_0 \tau(B) < \infty$ and then $\phi(B) \in \mathscr{M}_\tau$. Thus, if $\pi_{A,B}$ is a linear $*$-automorphism, then $\tau(\Phi(B)) = \tau(\pi_{A,B}(B)) < \infty$; if $\pi_{A,B}$ is the negative of a linear $*$-anti-automorphism, then $\tau(\Phi(B)) = \tau(\Phi_{A,B}(B)) = \tau(\pi_{A,B}(B)) = -\tau(-\pi_{A,B}(B)) < \infty$, which implies that $\Phi(B), \Phi_{A,B}(B), \pi_{A,B}(B) \in \mathscr{M}_\tau$.

Since $\mathscr{M}_\tau$ is a two-sided ideal of $\mathscr{M}$, there exists $\lambda \in \mathbb{C}$ such that

$$\begin{aligned}\tau(\Phi(A)\Phi(B)) &= \tau(\Phi_{A,B}(A)\Phi_{A,B}(B)) = \tau((\pi_{A,B}(A) + \lambda I)\pi_{A,B}(B)) \\ &= \tau(\pi_{A,B}(A + \lambda I)\pi_{A,B}(B)),\end{aligned}$$

where $\pi_{A,B}$ is either a linear $*$-automorphism or the negative of a linear $*$-anti--automorphism. So we shall discuss the equality by two cases.

**Case 1.** If $\pi_{A,B}$ is a linear $*$-automorphism, then $\tau(\Phi(A)\Phi(B)) = \tau(\pi_{A,B}((A + \lambda I)B))$. Notice that $\tau(\pi_{A,B}(\cdot))$ is a faithful normal semi-finite trace of $\mathscr{M}$.

**Case 2.** If $\pi_{A,B}$ is the negative of a linear $*$-anti-automorphism, then $\tau(\Phi(A)\Phi(B)) = \tau(-\pi_{A,B}(B(A + \lambda I)))$. Notice that $\tau(-\pi_{A,B}(\cdot))$ is a faithful normal semi-finite trace of $\mathscr{M}$.

Hence, by Lemma 2.4, either Case 1 or Case 2, there exists $\lambda > 0$ such that $\tau(\Phi(A)\Phi(B)) = \lambda\tau(((A + \lambda I)B)) = \lambda\tau(AB)$. In particular, if $\mathscr{M}$ is a finite factor, then $\lambda = 1$. It follows that $\tau(\Phi(A)\Phi(B)) = \tau(AB)$. $\square$

LEMMA 2.6. *Let $\mathscr{M}$ be a von Neumann algebra factor and $x \in \mathscr{M}$. Suppose that $Q$ is a fixed projection in $\mathscr{M}$. If $Q_1 QxQ = QxQQ_1$ for any $Q_1 \in \mathscr{P}(\mathscr{M})$ with $Q_1 < Q$, then $QxQ \in \mathbb{C}Q$.*

*Proof.* We observe that $Q\mathscr{M}Q$ is also a von Neumann algebra with unit $Q$ and $QxQ \in Q\mathscr{M}Q$. By $Q_1 QxQ = QxQQ_1$, for any projection $Q_1 < Q$, we can get that $QxQ$ commutes with all projections in $Q\mathscr{M}Q$. Since the linear span of all projections of $Q\mathscr{M}Q$ is norm dense in $Q\mathscr{M}Q$, it follows that $QxQ$ is in the center of $Q\mathscr{M}Q$. Noticing that $\mathscr{M}$ is a factor, it follows that the center of $Q\mathscr{M}Q$ is $\mathbb{C}Q$. Hence $QxQ \in \mathbb{C}Q$. $\square$

Our main result reads as follows.

THEOREM 2.7. *Let $\mathscr{M}$ be a semi-finite von Neumann algebra factor on a complex Hilbert space $H$ with dimension greater than 3. Then every surjective 2-local $*$-Lie automorphism $\Phi$ of $\mathscr{M}$ is of the form $\Phi = \Psi + \tau$, where $\Psi$ is a $*$-automorphism or the negative of a $*$-anti-automorphism of $\mathscr{M}$, and $\tau$ is a homogeneous map from $\mathscr{M}$ into $\mathbb{C}I$ vanishing on every sum of commutators.*

We will prove the theorem by checking several claims as follows.

**Claim 1.** *Let $\mathscr{M}$ be a semi-finite von Neumann algebra factor. Suppose that $\Phi : \mathscr{M} \to \mathscr{M}$ is a surjective 2-local $*$-Lie automorphism. Then $\Phi(A+C)-\Phi(A)-\Phi(C) \in \mathbb{C}I$ for any $A,C \in \mathscr{M}$. Similarly, $\Phi^{-1}$ satisfies the same conclusion.*

Let $\tau$ be a faithful normal semi-finite trace on $\mathscr{M}$. Suppose that $A,C \in \mathscr{M}$ and $B = PXP^{\perp}$, where $X \in \mathscr{M}_\tau$ and $P$ is an arbitrary projection in $\mathscr{M}$. Now we claim that

$$\tau(\Phi(A+C)\Phi(B)) = \tau((\Phi(A)+\Phi(C))\Phi(B)).$$

It is obvious for $B=0$. So we need to prove the case $B \neq 0$. Notice that $B$ is a commutator. For $A+C, B \in \mathscr{M}$, by Lemma 2.3, there exist a $*$- automorphism or the negative of a $*$-anti-automorphism $\pi_{A+C,B}$ and $\lambda \in \mathbb{C}$ such that $\Phi(A+C) = \pi_{A+C,B}(A+C)+\lambda I$ and $\Phi(B) = \pi_{A+C,B}(B)$. We only prove the case in which $\pi_{A+C,B}(\cdot)$ is a linear $*$-automorphism. The case in which $\pi_{A+C,B}(\cdot)$ is the negative of a linear $*$-anti-automorphism is similar (we refer to the proof of Lemma 2.5).

We know from the proof of Lemma 2.5 that $\tau(\pi_{A+C,B}(\cdot))$ is also a faithful normal semi-finite trace on $\mathscr{M}$ and $\Phi(B), \Phi_{A+C,B}(B), \pi_{A+C,B}(B) \in \mathscr{M}_\tau$. Notice that $\mathscr{M}_\tau$ is a two-sided ideal of $\mathscr{M}$. By Lemma 2.4 and Lemma 2.5, there exist $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} > 0$ such that $\tau(\pi_{A+C,B}(\cdot)) = \lambda^{(1)}\tau(\cdot)$, $\tau(\Phi(A)\Phi(B)) = \lambda^{(2)}\tau(AB)$ and $\tau(\Phi(C)\Phi(B)) = \lambda^{(3)}\tau(CB)$. It follows that

$$\begin{aligned}
\tau(\Phi(A+C)\Phi(B)) =& \tau(\Phi_{A+C,B}(A+C)\Phi_{A+C,B}(B)) = \tau((\pi_{A+C,B}(A+C)+\lambda I)\pi_{A+C,B}(B)) \\
=& \tau(\pi_{A+C,B}((A+C+\lambda I)B)) = \lambda^{(1)}\tau((A+C+\lambda I)B) \\
=& \lambda^{(1)}\tau((A+C)B) = \lambda^{(1)}\tau(AB) + \lambda^{(1)}\tau(CB) \\
=& \frac{\lambda^{(1)}}{\lambda^{(2)}}\tau(\Phi(A)\Phi(B)) + \frac{\lambda^{(1)}}{\lambda^{(3)}}\tau(\Phi(C)\Phi(B)).
\end{aligned}$$

For $B = PXP^{\perp} \neq 0$, where $X \in \mathscr{M}_\tau$, we claim that $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)}$.

Indeed, for any $X_1, X_2 \in \mathscr{M}$ and $B = PXP^{\perp} \neq 0$, where $X \in \mathscr{M}_\tau$, observing that $B$ is a commutator, then $\Phi(B) = \Phi_{B,X_1}(B) = \pi_{B,X_1}(B)$ and $\Phi(B) = \Phi_{B,X_2}(B) = \pi_{B,X_2}(B)$ by Lemma 2.3, where $\pi_{B,X_1}$ is either a linear $*$-automorphism or the negative of a linear $*$-anti-automorphism and so is $\pi_{B,X_2}$. Thus $\pi_{B,X_1}(B) = \pi_{B,X_2}(B)$, and $\pi_{B,X_1}(B^*) = \pi_{B,X_2}(B^*)$. In the following, we will deal with the problem in four cases.

**Case 1.** If $\pi_{B,X_1}, \pi_{B,X_2}$ are both linear $*$-automorphisms, then we have $\pi_{B,X_1}(B B^*) = \pi_{B,X_2}(BB^*)$. Notice that $\tau(\pi_{B,X_1}(\cdot)), \tau(\pi_{B,X_2}(\cdot))$ are both faithful normal semi-finite traces of $\mathscr{M}$.

**Case 2.** If $\pi_{B,X_1}, \pi_{B,X_2}$ are both the negative of linear $*$-automorphisms, then we have $-\pi_{B,X_1}(BB^*) = -\pi_{B,X_2}(BB^*)$. Notice that $\tau(-\pi_{B,X_1}(\cdot)), \tau(-\pi_{B,X_2}(\cdot))$ are both faithful normal semi-finite traces of $\mathscr{M}$.

**Case 3.** If $\pi_{B,X_1}$ is a linear $*$-automorphism and $\pi_{B,X_2}$ is the negative of a linear $*$-automorphism, then we have $\pi_{B,X_1}(B^*B) = -\pi_{B,X_2}(BB^*)$. Notice that $\tau(\pi_{B,X_1}(\cdot))$, $\tau(-\pi_{B,X_2}(\cdot))$ are both faithful normal semi-finite traces of $\mathscr{M}$.

**Case 4.** If $\pi_{B,X_1}$ is the negative of a linear $*$-automorphism and $\pi_{B,X_2}$ is a linear $*$-automorphism, then we have $-\pi_{B,X_1}(B^*B) = \pi_{B,X_2}(BB^*)$. Notice that $\tau(-\pi_{B,X_1}(\cdot))$, $\tau(\pi_{B,X_2}(\cdot))$ are both faithful normal semi-finite traces of $\mathscr{M}$.

We only discuss Case 1, and the rest cases are similar.

Notice that $\pi_{B,X_1}(BB^*)$, $\pi_{B,X_2}(BB^*) \in \mathscr{M}_\tau$ from the proof of Lemma 2.5. It follows that $\tau(\pi_{B,X_1}(BB^*)) = \tau(\pi_{B,X_2}(BB^*))$. Moreover, by Lemma 2.4, there exist $\lambda', \lambda''$ such that $\tau(\pi_{B,X_1}(BB^*)) = \lambda'\tau(BB^*)$, and $\tau(\pi_{B,X_2}(BB^*)) = \lambda''\tau(BB^*)$. Hence $\lambda'\tau(BB^*) = \lambda''\tau(BB^*)$. Noticing $B \neq 0$, we get $\lambda' = \lambda''$. Then it follows that $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)}$. Thus,

$$\tau(\Phi(A+C)\Phi(B)) = \tau(\Phi(A)\Phi(B)) + \tau(\Phi(C)\Phi(B)) = \tau((\Phi(A)+\Phi(C))\Phi(B)),$$

and then

$$\tau((\Phi(A+C)-\Phi(A)-\Phi(C))\Phi(B)) = 0$$

for any $A, C \in \mathscr{M}$ and $B = PXP^\perp$, where $X \in \mathscr{M}_\tau$ and $P$ is an arbitrary projection in $\mathscr{M}$. By Lemma 2.5,

$$0 = \tau((\Phi(A+C)-\Phi(A)-\Phi(C))\Phi(B)) = \lambda\tau(\Phi^{-1}(\Phi(A+C)-\Phi(A)-\Phi(C))B),$$

for some $\lambda > 0$. It follows that

$$\tau(\Phi^{-1}(\Phi(A+C)-\Phi(A)-\Phi(C))B) = 0.$$

Let $Y = \Phi^{-1}(\Phi(A+C)-\Phi(A)-\Phi(C))$, then $\tau(YPXP^\perp) = 0$. It follows that

$$\tau(P^\perp YPXP^\perp) = 0.$$

Now take a monotone net $\{P_\alpha\}$ of projections in $\mathscr{M}$ with $\tau(P_\alpha) < \infty$ which converges strongly to $I$. Since $\mathscr{M}_\tau$ is an ideal and $P_\alpha \subseteq \mathscr{M}_\tau$, the elements $P_\alpha PY^*$ also belong to $\mathscr{M}_\tau$. Hence, we obtain

$$\tau(P^\perp YPP_\alpha(P^\perp YP)^*) = 0.$$

As $\tau$ is normal, $\tau(P^\perp YPP_\alpha(P^\perp YP)^*) \to \tau(P^\perp YP(P^\perp YP)^*)$, which implies

$$\tau(P^\perp YP(P^\perp YP)^*) = 0.$$

Since $\tau$ is faithful, $P^\perp YP = 0$. By the arbitrariness of $P$, we also have $PYP^\perp = 0$. The above two equations lead to $PY = YP$. As the linear span of all projections of $\mathscr{M}$ is norm dense in $\mathscr{M}$, it follows that $AY = YA$ for all $A \in \mathscr{M}$, that is, $Y$ is in the center of $\mathscr{M}$. Noticing that $\mathscr{M}$ is a factor, we have $Y \in \mathbb{C}I$. Hence $\Phi^{-1}(\Phi(A+C)-\Phi(A)-\Phi(C)) \in \mathbb{C}I$. By Lemma 2.1, we get $\Phi(A+C)-\Phi(A)-\Phi(C) \in \mathbb{C}I$.

**Claim 2.** *Let $\mathscr{M}$ be a von Neumann algebra factor. Suppose that $\Phi : \mathscr{M} \to \mathscr{M}$ is a surjective 2-local $*$-Lie automorphism. If $E \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$, then $\Phi(E) = F + \lambda_E I$, where $\lambda_E \in \mathbb{C}$ and $F \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$. Moreover, the projection $F$ and the scalar $\lambda_E$ are unique. Similarly, $\Phi^{-1}$ satisfies the same conclusion.*

Let $E \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$. Since $[[[T,E],E],E] = [T,E]$ for all $T \in \mathscr{M}$, then

$$[[[\Phi_{T,E}(T), \Phi_{T,E}(E)^*], \Phi_{T,E}(E)^*], \Phi_{T,E}(E)^*] = [\Phi_{T,E}(T), \Phi_{T,E}(E)^*],$$

and thus

$$[[[\Phi(T), \Phi(E)^*], \Phi(E)^*], \Phi(E)^*] = [\Phi(T), \Phi(E)^*].$$

Noticing that $\Phi$ is surjective, it follows that

$$[[[S, \Phi(E)^*], \Phi(E)^*], \Phi(E)^*] = [S, \Phi(E)^*]$$

for all $S \in \mathscr{M}$. By Lemma 2.2, we have

$$\Phi(E)^* = F + \mu_E I,$$

where $F$ is an idempotent in $\mathscr{M}$ and $\mu_E \in \mathbb{C}$. Morever,

$$0 = \Phi_{E,E}([E,E]) = [\Phi_{E,E}(E), \Phi_{E,E}(E)^*] = [\Phi(E), \Phi(E)^*] = [F, F^*], \tag{2.1}$$

which implies that $F$ is normal. Thus $F = F^*$ and then $F \in \mathscr{P}(\mathscr{M})$. We claim that $F \neq 0$ or $I$. Otherwise, $\Phi(E) = \lambda_E I$ or $(\lambda_E + 1)I$ for some $\lambda_E \in \mathbb{C}$. Then by Lemma 2.1, we have $E = 0$ or $I$, which is a contradiction. Next we prove that $F$ and $\lambda_E$ are unique. Suppose that $\Phi(E) = F + \lambda_E I = F' + \lambda_E' I$, where $F'$ is also a projection and $\lambda_E' \in \mathbb{C}$. Then $F = F' + (\lambda_E' - \lambda_E)I$. Thus $\sigma(F) = \{0,1\} = \sigma(F' + (\lambda_E' - \lambda_E)I) = \{\lambda_E' - \lambda_E, \lambda_E' - \lambda_E + 1\}$. Hence $\lambda_E' = \lambda_E$ and $F = F'$.

By Claim 2, we can define a map $\hat{\Phi} : \mathscr{P}(\mathscr{M}) \setminus \{0, I\} \to \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$ by $\hat{\Phi}(E) = \Phi(E) - \lambda_E I$.

**Claim 3.** *$\hat{\Phi}$ is bijective and $\widehat{\Phi^{-1}} = \hat{\Phi}^{-1}$.*

First, we prove that $\hat{\Phi}$ is bijective.

Let $E, F \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$, then $\hat{\Phi}(E) = \Phi(E) - \lambda_E I$ and $\hat{\Phi}(F) = \Phi(F) - \lambda_F I$. If $\hat{\Phi}(E) = \hat{\Phi}(F)$, then $\Phi(E) - \Phi(F) \in \mathbb{C}I$. By Claim 1, $\Phi(E - F) \in \mathbb{C}I$, which implies $E - F \in \mathbb{C}I$ by Lemma 2.1. Then $E = F + \mu I$, and $\sigma(E) = \{0,1\} = \sigma(F + \mu I) = \{\mu, 1 + \mu\}$ for some $\mu \in \mathbb{C}$. Hence $\mu = 0$ and then $E = F$, which implies that $\hat{\Phi}$ is injective.

For any $F \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$, by Claim 2, $\Phi^{-1}(F) = E + \lambda_F I$, where $E \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$ and $\lambda_F \in \mathbb{C}$, then $F = \Phi(E + \lambda_F I)$. By Lemma 2.1 and Claim 1, there exists $\mu \in \mathbb{C}$ such that $F = \Phi(E + \lambda_F I) = \Phi(E) + \mu I$, then $\Phi(E) = F - \mu I$. Thus by the uniqueness in Claim 2, we get $F = \hat{\Phi}(E)$, which implies $\hat{\Phi}$ is surjective.

Finally, we prove that $\widehat{\Phi^{-1}} = \hat{\Phi}^{-1}$.

Indeed, for any $F \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$, by Lemma 2.5, we observe that $\Phi^{-1}(F) = E + \lambda_F I$, where $E \in \mathscr{P}(\mathscr{M}) \setminus \{0, I\}$ and $\lambda_F \in \mathbb{C}$. Then $\widehat{\Phi^{-1}}(F) = E$. On the other hand, since $\Phi^{-1}(F) = E + \lambda_F I$, then $F = \Phi(E + \lambda_F I) = \Phi(E) + \lambda I$ for some $\lambda \in \mathbb{C}$ by 2.1(3) and Claim 1. By the uniqueness in Lemma 2.5, we have that $F = \hat{\Phi}(E)$ and $\hat{\Phi}^{-1}(F) = E$. Hence $\widehat{\Phi^{-1}} = \hat{\Phi}^{-1}$.

**Claim 4.** *Suppose that $\mathscr{M}$ is a semi-finite von Neumann algebra factor. If $P_1,P_2 \in \mathscr{P}(\mathscr{M})\setminus\{0,I\}$ and $P_1+P_2=I$, then $\hat{\Phi}(P_1)+\hat{\Phi}(P_2)=I$.*

By Lemma 2.1 and Claim 1, we obtain that $\Phi(P_1+P_2)=\Phi(I)\in\mathbb{C}I$ and $\Phi(P_1+P_2)-\Phi(P_1)-\Phi(P_2)\in\mathbb{C}I$. Thus $\Phi(P_1)+\Phi(P_2)\in\mathbb{C}I$ and then $\hat{\Phi}(P_1)+\hat{\Phi}(P_2)\in\mathbb{C}I$. Then there exists $\lambda\in\mathbb{C}$ such that $\hat{\Phi}(P_1)+\hat{\Phi}(P_2)=\lambda I$. Observing that $\hat{\Phi}(P_1),\hat{\Phi}(P_2)\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$, we have that $\sigma(\hat{\Phi}(P_1))=\{0,1\}=\sigma(\lambda I-\hat{\Phi}(P_2))=\{\lambda,\lambda-1\}$. Hence $\lambda=1$, which implies that $\hat{\Phi}(P_1)+\hat{\Phi}(P_2)=I$.

Let $P_1,P_2\in\mathscr{P}(\mathscr{M})$. We say $P_1\leqslant P_2$ if $P_1P_2=P_1=P_2P_1$, and $P_1<P_2$ if $P_1\leqslant P_2$ and $P_1\neq P_2$.

**Claim 5.** *Let $\mathscr{M}$ be a von Neumann algebra factor and $P_1,P_2\in\mathscr{P}(\mathscr{M})$ with $0<P_1<P_2<I$. Set $Q_i=\hat{\Phi}(P_i), i=1,2$. Then either $0<Q_1<Q_2<I$ or $0<Q_2<Q_1<I$. Moreover, let $\mathscr{M}$ be a semi-finite factor and $P_1,P_2,P_3\in\mathscr{P}(\mathscr{M})$ with $0<P_1<P_2<P_3<I$. Set $Q_i=\hat{\Phi}(P_i), i=1,2,3$. Then*

*(1) If $Q_1<Q_2$, then $Q_1<Q_2<Q_3$.*

*(2) If $Q_2<Q_1$, then $Q_1>Q_2>Q_3$.*

First, we give the proof of the first part. By the definition of $\hat{\Phi}$ and Claim 2, $P_1,P_2,P_2-P_1\notin\mathbb{C}I$ implies $Q_1,Q_2,Q_2-Q_1\notin\mathbb{C}I$, in particular $Q_2\neq Q_1$. Since $[P_1,P_2]=0$, we have

$$\begin{aligned}0=&\Phi_{P_1,P_2}[P_1,P_2]=[\Phi_{P_1,P_2}(P_1),\Phi_{P_1,P_2}(P_2)^*]=[\Phi(P_1),\Phi(P_2)^*]=[\hat{\Phi}(P_1),\hat{\Phi}(P_2)]\\ =&[Q_1,Q_2],\end{aligned}$$

which implies $Q_1Q_2=Q_2Q_1$. Then $(Q_2-Q_1)^3=Q_2-Q_1$. Thus, if $Q_2$ and $Q_1$ are not comparable, then $\sigma(Q_2-Q_1)=\{-1,1\}$ or $\{-1,0,1\}$. On the other hand, noticing that $P_2-P_1\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$, then by Claim 2, we have $\Phi(P_2-P_1)\in\mathscr{P}(\mathscr{M})+\mathbb{C}I$. Hence $Q_2-Q_1=\hat{\Phi}(P_2)-\hat{\Phi}(P_1)=\Phi(P_2)-\lambda_{P_2}I-\Phi(P_1)+\lambda_{P_1}I=\Phi(P_2-P_1)+\lambda I\in\mathscr{P}(\mathscr{M})+\mathbb{C}I$ for some $\lambda$ by Claim 1. Thus there exists a scalar $\mu$ such that $\sigma(Q_2-Q_1)=\{\mu,\mu+1\}$, which is a contradiction. Hence $0<Q_1<Q_2<I$ or $0<Q_2<Q_1<I$.

Next we give the proof of the second part. We only prove the case $Q_1<Q_2$. The case $Q_1>Q_2$ is similar. Assume that $Q_1<Q_2$. The projections $Q_1,Q_2,Q_3$ are distinct and mutually comparable by the first part. By simple calculation, we have that $P_1+P_3-P_2\in\mathscr{P}(\mathscr{M})$ and so $Q_1+Q_3-Q_2\in\mathscr{P}(\mathscr{M})+\mathbb{C}I$ by Claim 1. Hence $\sigma(Q_1+Q_3-Q_2)=\{\lambda,\lambda+1\}$ for some $\lambda\in\mathbb{C}$. If $Q_1<Q_3<Q_2$ or $Q_3<Q_1<Q_2$, then $(Q_1+Q_3-Q_2)^3=Q_1+Q_3-Q_2$. It follows that $\sigma(Q_1+Q_3-Q_2)=\{-1,0,1\}$, which is a contradiction.

REMARK 1. Similarly, $\widehat{\Phi^{-1}}(=\hat{\Phi}^{-1})$ also satisfies the same conclusion of Claim 5.

In the following, from Claim 6 to Claim 12, let $\mathscr{M}$ be a semi-finite von Neumann algebra factor on a complex Hilbert space $H$ with dimension greater than 3 and let $P_1$ be a fix projection. Set $P_2=I-P_1$ and $Q_i=\hat{\Phi}(P_i), i=1,2$. Notice that $Q_1,Q_2\in$

$\mathscr{P}(\mathscr{M})\setminus\{0,I\}$ and $Q_1+Q_2=I$ by Claim 4. Let $\mathscr{M}_{ij}=P_i\mathscr{M}P_j$ and $\mathscr{N}_{ij}=Q_i\mathscr{M}Q_j$, $i,j=1,2$. Then $\mathscr{M}=\sum_{i,j=1}^{2}\mathscr{M}_{ij}=\sum_{i,j=1}^{2}\mathscr{N}_{ij}$.

**Claim 6.**

(1) *If there exists $E_1\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$ such that $E_1<P_1$ and $\hat{\Phi}(E_1)<\hat{\Phi}(P_1)=Q_1$ or $E_1>P_1$ and $\hat{\Phi}(E_1)>\hat{\Phi}(P_1)=Q_1$, then for any $E\in\mathscr{P}(\mathscr{M})\setminus\{0,I\},E<P_1$ implies $\hat{\Phi}(E)<Q_1$ and $E>P_1$ implies $\hat{\Phi}(E)>Q_1$. Moreover, $E<P_2$ implies $\hat{\Phi}(E)<Q_2$ and $E>P_2$ implies $\hat{\Phi}(E)>Q_2$.*

(2) *If there exists $E_1\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$ such that $E_1<P_1$ and $\hat{\Phi}(E_1)>\hat{\Phi}(P_1)=Q_1$ or $E_1>P_1$ and $\hat{\Phi}(E_1)<\hat{\Phi}(P_1)=Q_1$, then for any $E\in\mathscr{P}(\mathscr{M})\setminus\{0,I\},E<P_1$ implies $\hat{\Phi}(E)>Q_1$ and $E>P_1$ implies $\hat{\Phi}(E)<Q_1$. Moreover, $E<P_2$ implies $\hat{\Phi}(E)>Q_2$ and $E>P_2$ implies $\hat{\Phi}(E)<Q_2$.*

We only prove (1). The proof of (2) is similar. Assume that there exists $E_1\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$ such that $E_1<P_1$ and $\hat{\Phi}(E_1)<\hat{\Phi}(P_1)=Q_1$. For any $E\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$, if $E>P_1$, then $E_1<P_1<E$. By Claim 5, we can get $\hat{\Phi}(E_1)<Q_1<\hat{\Phi}(E)$. If $E<P_1$, by Claim 5, either $\hat{\Phi}(E)>Q_1$ or $\hat{\Phi}(E)<Q_1$. If $\hat{\Phi}(E)>Q_1$, then we have $\hat{\Phi}(E)>Q_1>\hat{\Phi}(E_1)$. By Remark 1, applying $\hat{\Phi}^{-1}$ to Claim 5, we have $E_1<P_1<E$, which is a contradiction. Hence $\hat{\Phi}(E)<Q_1$. The case that there exists $E_1\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$ such that $E_1>P_1$ and $\hat{\Phi}(E_1)>\hat{\Phi}(P_1)=Q_1$ is dealt with in the same way.

If $E<P_2$, then $I-E>P_1$. By Claim 4 and the above proof, we have $\hat{\Phi}(I-E)=I-\hat{\Phi}(E)>Q_1$. Hence $\hat{\Phi}(E)<1-Q_1=Q_2$. The proof of the case $E>P_2$ is similar.

REMARK 2. If $\Phi$ satisfies the assumption of Claim 6(1), then for any $F\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$, $F<Q_1$ implies $\widehat{\Phi^{-1}}(F)<P_1$ and $F>Q_1$ implies $\widehat{\Phi^{-1}}(F)>P_1$. Similarly, if $\Phi$ satisfies the assumption of Claim 6(2), then for any $F\in\mathscr{P}(\mathscr{M})\setminus\{0,I\}$, $F<Q_1$ implies $\widehat{\Phi^{-1}}(F)>P_1$ and $F>Q_1$ implies $\widehat{\Phi^{-1}}(F)<P_1$.

By Claim 6, we may extend the definition of $\hat{\Phi}$ to all of $\mathscr{P}(\mathscr{M})$ by $\hat{\Phi}(0)=0,\hat{\Phi}(I)=I$ if $\hat{\Phi}$ satisfies Claim 6(1), and $\hat{\Phi}(0)=I,\hat{\Phi}(I)=0$ if $\hat{\Phi}$ satisfies Claim 6(2).

Up to now, we have proved that, if $\Phi$ satisfies the assumption of Theorem 2.7, then either Claim 6(1) or Claim 6(2) occurs. So we will prove Theorem 2.7 in two cases.

**Case 1.** Assume that the case of Claim 6(1) occurs.

**Claim 7.** *Let $\Phi$ be a surjective 2-local $*$-Lie automorphism satisfying Claim 6(1). Then $\Phi(\mathscr{M}_{ij})=\mathscr{N}_{ij}$, $1\leqslant i\neq j\leqslant 2$.*

We only prove the case of $i=1,j=2$. The proof of the case of $i=2,j=1$ is similar. For any $A_{12}\in\mathscr{M}_{12}$, noticing that $A_{12}=[A_{12},P_2]$, we have that

$$\begin{aligned}\Phi(A_{12})=&\Phi_{A_{12},P_2}(A_{12})=\Phi_{A_{12},P_2}([A_{12},P_2])=[\Phi_{A_{12},P_2}(A_{12}),\Phi_{A_{12},P_2}(P_2)^*]\\=&[\Phi(A_{12}),\Phi(P_2)^*]=[\Phi(A_{12}),Q_2]=Q_1\Phi(A_{12})Q_2-Q_2\Phi(A_{12})Q_1.\end{aligned}$$

Multiplied by $Q_2$ on the left and $Q_1$ on the right hand side of the above equality, we obtain $Q_2\Phi(A_{12})Q_1=0$, which implies $\Phi(\mathscr{M}_{12})\subseteq\mathscr{N}_{12}$.

On the other hand, noticing that $\widehat{\Phi^{-1}} = \hat{\Phi}^{-1}$ by Claim 3 and applying the same argument to $\Phi^{-1}$, we can prove that $\mathscr{N}_{12} \subseteq \Phi(\mathscr{M}_{12})$. Hence $\mathscr{N}_{12} = \Phi(\mathscr{M}_{12})$.

**Claim 8.** *Let $\mathscr{M}$ be a semi-finite von Neumann algebra factor on a complex Hilbert space with dimension of $H$ greater than 3 and let $\Phi$ be a surjective 2-local $*$-Lie automorphism of $\mathscr{M}$ satisfying Claim 6(1). Then there exists a homogeneous map $f_i : \mathscr{M}_{ii} \to \mathbb{C}$ such that $\Phi(A_{ii}) - f_i(A_{ii}) \in \mathscr{N}_{ii}$, for all $A_{ii}$ in $\mathscr{M}_{ii}, i = 1,2$, and $f_i$ is unique. Moreover, for each $B_{ii} \in \mathscr{N}$, there exists $A_{ii} \in \mathscr{M}_{ii}$ such that $\Phi(A_{ii}) = B_{ii} + f_i(A_{ii})I$.*

We only prove the case $i = 1$. The proof of the case $i = 2$ is similar.

Let $\Phi(A_{11}) = B_{11} + B_{12} + B_{21} + B_{22}$, for any $A_{11} \in \mathscr{M}_{11}$. Then

$$0 = \Phi_{A_{11},P_1}([A_{11},P_1]) = [\Phi_{A_{11},P_1}(A_{11}), \Phi_{A_{11},P_1}(P_1)^*] = [\Phi(A_{11}), \Phi(P_1)^*] = [\Phi(A_{11}), Q_1].$$

For the above equality, multiplied by $Q_1$ on the left and $Q_2$ on the right, it follows that $Q_1\Phi(A_{11})Q_2 = 0$, and similarly, $Q_2\Phi(A_{11})Q_1 = 0$, which implies that $B_{12} = B_{21} = 0$. Thus $\Phi(A_{11}) = B_{11} + B_{22}$. On the other hand, for each $E \in \mathscr{P}(\mathscr{M})$ with $E < P_2$, by Claim 6, we have $\hat{\Phi}(E) < Q_2$. Notice that $[A_{11}, E] = 0$, then

$$\begin{aligned} 0 =& \Phi_{A_{11},E}([A_{11},E]) = [\Phi_{A_{11},E}(A_{11}), \Phi_{A_{11},E}(E)^*] = [\Phi(A_{11}), \Phi(E)^*] = [B_{11} + B_{22}, \hat{\Phi}(E)] \\ =& [B_{22}, \hat{\Phi}(E)]. \end{aligned}$$

That is, $B_{22}\hat{\Phi}(E) = \hat{\Phi}(E)B_{22}$ for all projections $E \in \mathscr{M}$ with $E < P_2$. Combining $\hat{\Phi}(E) < Q_2$ with Lemma 2.6, we have $B_{22} \in \mathbb{C}Q_2$. Thus there exists $f_1(A_{11}) \in \mathbb{C}$ such that $B_{22} = f_1(A_{11})Q_2$. Hence,

$$\Phi(A_{11}) = B_{11} + f_1(A_{11})Q_2 = B_{11} - f_1(A_{11})Q_1 + f_1(A_{11})I,$$

then $\Phi(A_{11}) - f_1(A_{11})I \in \mathscr{N}_{11}$.

We claim that $f_1$ is unique. Otherwise, for any $A_{11} \in \mathscr{M}_{11}$, suppose that

$$\Phi(A_{11}) = f_1(A_{11})I + B_{11}, \Phi(A_{11}) = f_1^{'}(A_{11})I + B_{11}^{'}.$$

Multiplied by $Q_2$ on one side of the two above equalities, we can get $f_1(A_{11})Q_2 = f_1^{'}(A_{11})Q_2$ and then $f_1(A_{11}) = f_1^{'}(A_{11})$. It follows that $f_1 = f_1^{'}$.

It is obvious that $f_1$ is homogeneous. Indeed, let $A_{11} \in \mathscr{M}_{11}$ and $\lambda \in \mathbb{C}$. Then

$$\Phi(A_{11}) - f_1(A_{11})I \in \mathscr{N}_{11}, \Phi(\lambda A_{11}) - f_1(\lambda A_{11})I \in \mathscr{N}_{11}.$$

It follows that $(f_1(\lambda A_{11}) - \lambda f_1(A_{11}))I \in \mathscr{N}_{11}$ by the homogeneity of $\Phi$. This forces that $f_1(\lambda A_{11}) = \lambda f_1(A_{11})$.

Apply the preceding proof to $\Phi^{-1}$ for any $B_{11} \in \mathscr{N}_{11}$. Thus for $B_{11} \in \mathscr{N}_{11}$, there exist $A_{11}$ and $\lambda \in \mathbb{C}$ such that $\Phi^{-1}(B_{11}) = A_{11} + \lambda I$ and then $\Phi(A_{11} + \lambda I) = B_{11}$. By lemma 2.1 and Claim 1, we can get $\Phi(A_{11}) = B_{11} + \mu I$ for some $\mu \in \mathbb{C}$. Hence $\Phi(A_{11}) - \mu I \in \mathscr{N}_{11}$. By the uniqueness of $f_1$, we have that $\mu = f_1(A_{11})$.

By the uniqueness of $f_1$ and $f_2$ in Claim 8, we can define a map $\Psi : \mathscr{M} \to \mathscr{M}$ by $\Psi(A) = \sum_{i,j=1}^{2} \Phi(A_{ij}) - f_1(A_{11})I - f_2(A_{22})I$ for any $A = \sum_{i,j=1}^{2} A_{ij} \in \mathscr{M}$.

**Claim 9.** *For $\Psi$ above, let $A_{ij} \in \mathscr{M}_{ij}, 1 \leqslant i,j \leqslant 2$. Then*

*(1)* $\Psi(A_{ij}) = \Phi(A_{ij}), 1 \leqslant i \neq j \leqslant 2$*;*

*(2)* $\Psi(\mathscr{M}_{ij}) = \mathscr{N}_{ij},\ i,j = 1,2$*;*

*(3)* $\Psi(P_i) = Q_i, i = 1,2$*;*

*(4)* $\Psi(\sum_{i,j=1}^{2} A_{ij}) = \sum_{i,j=1}^{2} \Psi(A_{ij})$*;*

*(5)* $\Psi$ *is homogeneous and bijective;*

*(6) For any projection* $P \in \mathscr{M}_{11} \bigcup \mathscr{M}_{22}$*,* $\Psi(P) = \hat{\Phi}(P)$*.*

(1) and (4) can be easily obtained. For (2), if $1 \leqslant i \neq j \leqslant 2$, the equality is clear by (1) and Claim 7. Otherwise, we only prove the case $i = j = 1$. The case $i = j = 2$ is similar. Let $A_{11} \in \mathscr{M}_{11}$, by Claim 8, $\Psi(A_{11}) = \Phi(A_{11}) - f_1(A_{11})I \in \mathscr{N}_{11}$. On the contrary, also by Claim 8, for any $B_{11} \in \mathscr{N}_{11}$, there exist $A_{11} \in \mathscr{M}$ and $\mu \in \mathbb{C}$ such that $\Phi(A_{11}) = B_{11} + \mu I$. Hence, by the uniqueness in Claim 8, $\Psi(A_{11}) = B_{11}$.

For (3), by Claim 8, $\Phi(P_i) = \Psi(P_i) + f_i(P_i)I$. On the other hand, by the definition of $\hat{\Phi}$, we have $\Phi(P_i) = \hat{\Phi}(P_i) + \lambda_{P_i} I = Q_i + \lambda_{P_i} I$. Noticing that $Q_i \in \mathscr{M}_{ii}$, by the uniqueness in Claim 8 , we get $\Psi(P_i) = Q_i, i = 1,2$.

For (5), the homogeneity of $\Psi$ can be obtained directly by the homogeneity of $\Phi, f_1$ and $f_2$. We only need to prove that $\Psi$ is injective. Let $A, B \in \mathscr{M}$ and $\Psi(A) = \Psi(B)$. By the definition of $\Psi$, we have

$$\Psi(A) = (\Phi(A_{11}) - f_1(A_{11})I) + \Phi(A_{12}) + \Phi(A_{21}) + (\Phi(A_{22}) - f_2(A_{22})I),$$

and

$$\Psi(B) = (\Phi(B_{11}) - f_1(B_{11})I) + \Phi(B_{12}) + \Phi(B_{21}) + (\Phi(B_{22}) - f_2(B_{22})I).$$

Multiplied by $Q_1$ on the left and $Q_2$ on the right of the above equalities, we get

$$Q_1 \Psi(A) Q_2 = Q_1 \Phi(A_{12}) Q_2, Q_1 \Psi(B) Q_2 = Q_1 \Phi(B_{12}) Q_2.$$

Since $\Psi(A) = \Psi(B)$, $Q_1(\Phi(A_{12}) - \Phi(B_{12}))Q_2 = 0$. By Claim 7, $\Phi(A_{12}) - \Phi(B_{12}) \in \mathscr{N}_{12}$, and then $\Phi(A_{12}) - \Phi(B_{12}) = 0$. Since $\Phi$ is bijective, $A_{12} = B_{12}$. Similarly, $A_{21} = B_{21}$. Multiplied by $Q_1$ on the two sides of the above equalities, we get

$$\Phi(A_{11}) - f_1(A_{11})I = \Phi(B_{11}) - f_1(B_{11})I.$$

Then $\Phi(A_{11}) - \Phi(B_{11}) = \lambda I$ for some $\lambda \in \mathbb{C}$. By Claim 1, $\Phi(A_{11} - B_{11}) = \mu I$ for some $\mu \in \mathbb{C}$. It follows that $A_{11} - B_{11} \in \mathbb{C}I \cap \mathscr{M}_{11}$ by Lemma 2.1. This forces that $A_{11} = B_{11}$. Similarly, $A_{22} = B_{22}$. Hence $A = B$.

Finally, we prove that $\Psi$ is surjective. Indeed, for any $B \in \mathscr{M}, B = B_{11} + B_{12} + B_{21} + B_{22}$, where $B_{ij} = Q_i B Q_j \in \mathscr{N}_{ij}$. By Claim 8, for $B_{ii} \in \mathscr{N}_{ii}, i = 1,2$, there exists $A_{ii} \in \mathscr{M}_{ii}$ such that $\Phi(A_{ii}) = B_{ii} + f_i(A_{ii})I$ and then $B_{ii} = \Phi(A_{ii}) - f_i(A_{ii})I$. For $B_{ij}$, $1 \leqslant i \neq j \leqslant 2$, there exists $A_{ij} \in \mathscr{M}_{ij}$ such that $\Phi(A_{ij}) = B_{ij}$. Let $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathscr{M}$ by Claim 7. It follows that $\Psi(A) = (\Phi(A_{11}) - f_1(A_{11})I) + \Phi(A_{12}) + \Phi(A_{21}) + (\Phi(A_{22}) - f_2(A_{22})I) = B_{11} + B_{12} + B_{21} + B_{22} = B$.

For (6), we only prove the case $P \in \mathscr{M}_{11}$. The case $P \in \mathscr{M}_{22}$ is similar. For any projection $P \in \mathscr{M}_{11}$, $\Psi(P) = \Phi(P) - f_1(P)I = \hat{\Phi}(P) + \lambda I - f_1(P)I$ for some $\lambda \in \mathbb{C}$. Then $\Psi(P) = \hat{\Phi}(P) + \mu I$ for some $\mu \in \mathbb{C}$. We observe that $\Psi(P) \in \mathscr{N}_{11}$, thus $\hat{\Phi}(P)Q_2 = -\mu Q_2$. Noticing that $P \leqslant P_1$, by Claim 6, $\hat{\Phi}(P) \leqslant Q_1$, then $\hat{\Phi}(P)Q_2 = 0$. It follows that $\mu = 0$. Hence $\Psi(P) = \hat{\Phi}(P)$.

**Claim 10.** $\Psi$ *is linear.*

By Claim 9 (4) and (5), we only need to prove that $\Psi$ is additive on $\mathscr{M}_{ij}$ for $i,j = 1,2$. Let $A_{12}, B_{12} \in \mathscr{M}_{12}$. By Claim 1 and Claim 9(1),(2), we have that

$$\Psi(A_{12}+B_{12}) - \Psi(A_{12}) - \Psi(B_{12}) = \Phi(A_{12}+B_{12}) - \Phi(A_{12}) - \Phi(B_{12}) = \lambda I = C_{12}$$

for some $\lambda \in \mathbb{C}$ and $C_{12} \in \mathscr{M}_{12}$. This forces $\lambda = 0$, which implies $\Psi(A_{12}+B_{12}) = \Psi(A_{12}) + \Psi(B_{12})$. Hence $\Psi$ is additive on $\mathscr{M}_{12}$. Similarly, $\Psi$ is additive on $\mathscr{M}_{21}$. Let $A_{11}$ and $B_{11} \in \mathscr{M}_{11}$. By Claim 1 and Claim 9(2), we obtain that

$$\begin{aligned}&\Psi(A_{11}+B_{11}) - \Psi(A_{11}) - \Psi(B_{11})\\ =&\Phi(A_{11}+B_{11}) - f_1(A_{11}+B_{11})I - \Phi(A_{11}) + f_1(A_{11})I - \Phi(B_{11}) + f_1(B_{11})I = \lambda I = C_{11}\end{aligned}$$

for some $\lambda \in \mathbb{C}$ and $C_{11} \in \mathscr{M}_{11}$. This forces $\lambda = 0$, which implies $\Psi(A_{11}+B_{11}) = \Psi(A_{11}) + \Psi(B_{11})$. Hence $\Psi$ is additive on $\mathscr{M}_{11}$. Similarly, $\Psi$ is additive on $\mathscr{M}_{22}$.

**Claim 11.** $\Psi(A^*) = \Psi(A)^*$ *for any* $A \in \mathscr{M}$ *and* $\Psi$ *preserves the commutativity.*

First, we prove $\Psi(A^*) = \Psi(A)^*$ for any $A \in \mathscr{M}$.

Indeed, by Claim 10, we only need to prove that $\Psi(A_{ij}^*) = \Psi(A_{ij})^*$ for $i,j = 1,2$. Let $A_{12} \in \mathscr{M}_{12}$. By Lemma 2.3, $\Phi_{A_{12},A_{12}^*} = \pm\sigma + \tau$, where $\sigma$ commutes with $*$. Notice that $A_{12}$ is a commutator. Hence, we obtain that $\Psi(A_{12}^*) = \Phi(A_{12}^*) = \Phi_{A_{12},A_{12}^*}(A_{12}^*) = \sigma(A_{12}^*) = \sigma(A_{12})^* = \Phi_{A_{12},A_{12}^*}(A_{12})^* = \Phi(A_{12})^* = \Psi(A_{12})^*$. Similarly, $\Psi(A_{21}^*) = \Psi(A_{21})^*$.

Let $A_{11} \in \mathscr{M}_{11}$. By Lemma 2.3, $\Phi_{A_{11},A_{11}^*} = \pm\sigma + \tau$, where $\sigma$ commutes with $*$. Thus, $\Phi(A_{11}^*) - \Phi(A_{11})^* = \tau(A_{11}^*) - \tau(A_{11})^* \in \mathbb{C}I$. It follows that $\Psi(A_{11}^*) - \Psi(A_{11})^* = \tau(A_{11}^*) - \tau(A_{11})^* + \overline{f_1(A_{11})}I - f_1(A_{11}^*)I \in \mathbb{C}I$. Let $\Psi(A_{11}^*) - \Psi(A_{11})^* = \lambda I = C_{11}$ for some $\lambda \in \mathbb{C}$ and $C_{11} \in \mathscr{M}_{11}$, This forces $\lambda = 0$. Hence $\Psi(A_{11}^*) = \Psi(A_{11})^*$. Similarly, $\Psi(A_{22}^*) = \Psi(A_{22})^*$.

In the following, we prove that $\Psi$ preserves the commutativity.

Let $A, B \in \mathscr{M}$ with $AB = BA$. Since $\Psi(A^*) = \Psi(A)^*$ for any $A \in \mathscr{M}$, then

$$\begin{aligned}0 =&\Phi_{A,B^*}([A,B]) = [\Phi_{A,B^*}(A), \Phi_{A,B^*}(B^*)^*] = [\Phi(A), \Phi(B^*)^*] = [\Psi(A), \Psi(B^*)^*]\\ =&[\Psi(A), \Psi(B)]\end{aligned}$$

which implies $\Psi(A)\Psi(B) = \Psi(B)\Psi(A)$.

**Claim 12.** $\Psi$ *is a* $*$*-automorphism.*

Since $\Psi$ is a bijective $*$-linear map preserving the commutativity, it follows from [22, Theorem 2] that

$$\Psi = \alpha\phi + \tau_1,$$

where $\alpha \in \mathbb{C}\setminus\{0\}$, $\phi$ is an automorphism or anti-automorphism of $\mathscr{M}$, and $\tau_1$ is a linear map from $\mathscr{M}$ to $\mathbb{C}I$. Next, we shall prove the claim in three steps.

**Step 1.** We prove that $\alpha = 1$.

For $i = 1,2$, we have that

$$Q_i = \Psi(P_i) = \alpha\phi(P_i) + \beta_i I,$$

for some $\beta_i \in \mathbb{C}$. Noticing that $Q_i$ and $\phi(P_i)$ are idempotents, we have that

$$\alpha\phi(P_i) + \beta_i I = (\alpha^2 + 2\alpha\beta_i)\phi(P_i) + \beta_i^2 I.$$

Thus,

$$(\alpha - \alpha^2 - 2\alpha\beta_i)\phi(P_i) = (\beta_i^2 - \beta_i)I.$$

Noticing that $\phi(P_i) \in \mathscr{P}(\mathscr{M})$ and $\phi(P_i) \notin \mathbb{C}I$, we have $\alpha - \alpha^2 - 2\alpha\beta_i = 0$ and $\beta_i^2 - \beta_i = 0$. Hence, $\alpha = 1, \beta_i = 0, i = 1,2$ or $\alpha = -1, \beta_i = 1, i = 1,2$.

We claim that $\alpha = 1, \beta_i = 0, i = 1,2$. Otherwise $\alpha = -1, \beta_i = 1, i = 1,2$. Let $E \in \mathscr{P}(\mathscr{M})\setminus\{0,I\}$ with $E < P_2$. By Claim 6 and Claim 9(6), we obtain that

$$-\phi(E) + \gamma I = \Psi(E) = \hat{\Phi}(E) < \hat{\Phi}(P_2) = \Psi(P_2) = -\phi(P_2) + I,$$

for some $\gamma \in \mathbb{C}$. Noticing that $\hat{\Phi}(E), \hat{\Phi}(P_2) \in \mathscr{P}(\mathscr{M})\setminus\{0,I\}$, we have that

$$(-\phi(E) + \gamma I)(-\phi(P_2) + I) = -\phi(E) + \gamma I.$$

By the simple computation, we can get that $\phi(E) = \phi(\gamma P_2)$. Thus $E = \gamma P_2$, which is a contradiction. Hence $\alpha = 1, \beta_i = 0$ and $Q_i = \phi(P_i), i = 1,2$.

**Step 2.** We will prove that $\phi$ is an automorphism.

Otherwise, let $A_{12}$ be a non-zero element in $\mathscr{M}_{12}$. If $\phi$ is an anti-automorphism, then by Claim 9(2),

$$\Psi(A_{12}) = Q_1\Psi(A_{12})Q_2 = Q_1\phi(A_{12})Q_2 = \phi(P_1)\phi(A_{12})Q_2 = \phi(A_{12}P_1)Q_2 = 0.$$

Thus $A_{12} = 0$, which is a contradiction. Hence $\phi$ is an automorphism.

Up to now, we have proved that $\Psi = \phi + \tau_1$ and $\phi(P_i) = Q_i$, where $\phi$ is an automorphism of $\mathscr{M}$ and $\tau_1$ is a map from $\mathscr{M}$ to $\mathbb{C}I$.

**Step 3.** We will show that $\tau_1 \equiv 0$.

Indeed, $\phi(A_{ij}) = \phi(P_i)\phi(A_{ij})\phi(P_j) = Q_i\phi(A_{ij})Q_j \in \mathscr{N}_{ij}$ for all $A_{ij} \in \mathscr{M}_{ij}, i,j = 1,2$. On the other hand, $\Psi(A_{ij}) = \phi(A_{ij}) + \tau_1(A_{ij}) \in \mathscr{N}_{ij}, i,j = 1,2$. This forces that $\tau_1(A_{ij}) = 0$ for all $A_{ij} \in \mathscr{M}_{ij}, i,j = 1,2$. Then $\Psi(A) = \phi(A)$ for any $A \in \mathscr{M}$. It follows that $\tau_1 \equiv 0$.

Hence, $\Psi$ is a $*$-automorphism.

Finally, we shall prove Theorem 2.7 in two cases.

**Case 1.** Assume that $\Phi$ satisfies Claim 6(1).

For any $A \in \mathscr{M}$, we define $\tau(A) = \Phi(A) - \Psi(A)$. Then $\Phi = \Psi + \tau$. We get the homogeneity of $\tau$ directly by the the homogeneity of $\Phi$ and $\Psi$. Moreover, by the definition of $\Psi$, it follows that $\tau$ is a map from $\mathscr{M}$ into $\mathbb{C}I$. In the following, we will prove that $\tau$ vanishes on very sum of commutators.

Since $\mathscr{M}$ is a semi-finite von Neumann algebra factor on a complex Hilbert space $H$ with dimension greater than 3, then there exist three non-trivial projections $P_1, P_2, P_3 \in \mathscr{M}$ such that $P_1 + P_2 + P_3 = I$ and $P_1P_2 = P_1P_3 = P_2P_3 = 0$. Now let $P_0 = P_1 + 2P_2 + 4P_3$ and let $E \in \mathscr{M}$ be a sum of commutators. Then by the definition of 2-local $*$-Lie automorphism and Lemma 2.3, we have

$$\Phi(E) = \Phi_{P_0,E}(E) = \pi_{P_0,E}(E), \Phi(P_0) = \Phi_{P_0,E}(P_0) = \pi_{P_0,E}(P_0) + \lambda I$$

for some $\lambda \in \mathbb{C}$, where $\pi_{P_0,E}$ is a $*$-automorphism or the negative of a $*$-anti- -automorphism of $\mathscr{M}$. On the other hand,

$$\Phi(E) = \Psi(E) + \tau(E), \Phi(P_0) = \Psi(P_0) + \tau(P_0),$$

where $\Psi$ is a $*$-automorphism and $\tau$ is a homogeneous map from $\mathscr{M}$ into $\mathbb{C}I$. Thus,

$$\Psi(P_0) + \tau(P_0) = \Phi(P_0) = \pi_{P,E}(P_0) + \lambda I, \Psi(E) + \tau(E) = \Phi(E) = \pi_{P_0,E}(E).$$

If $\pi_{P_0,E}$ is the negative of an anti-automorphism, taking the spectrum, we have $\sigma(P_0) = -\sigma(P_0) + \mu$ for some $\mu \in \mathbb{C}$, that is, $\{1,2,4\} = \{-1+\mu, -2+\mu, -4+\mu\}$, which is a contradiction. So $\pi_{P_0,E}$ must be an automorphism. Moreover, $\sigma(E) + \nu = \sigma(E)$, where $\tau(E) = \nu I$, which implies that $\nu = 0$. Hence $\tau(E) = 0$.

Up to now, we proved that $\tau$ is a homogeneous map from $\mathscr{M}$ into $\mathbb{C}I$ vanishing on very sum of commutators.

**Case 2.** Assume that $\Phi$ satisfies Claim 6(2). Then $\Phi = -\Psi + \tau$, where $\Psi$ is a linear $*$-anti-automorphism of $\mathscr{M}$, and $\tau$ is a homogeneous map from $\mathscr{M}$ into $\mathbb{C}I$ vanishing on very sum of commutators.

Similar arguments to those given in the proof of Case 1 are valid for the proof of Case 2. Combining Case 1 with Case 2, we give the proof of Theorem 2.7.

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