Operators
and
Matrices
Volume 13, Number 3 (2019), 777–795
doi:10.7153/oam-2019-13-55

APPROXIMATE EQUIVALENCE OF REPRESENTATIONS OF AF ALGEBRAS INTO SEMIFINITE VON NEUMANN ALGEBRAS

SHILIN WEN, JUNSHENG FANG AND RUI SHI*

(*Communicated by H. Bercovici*)

Abstract. In this paper, we extend the "compact operator" part of Voiculescu's theorem on approximate equivalence of unital $*$-homomorphisms of an AF algebra when the range is in a countably decomposable, semifinite, properly infinite von Neumann factor. We also extend a result of Hadwin for approximate summands of representations into a finite von Neumann factor.

1. Introduction

For the past several decades, many contributions have been made in the field of operator theory **relative to general von Neumann algebras**. Among them, Hadwin, Zsidó, Kaftal, and many other people considered the Weyl-von Neumann theorem [12, 21, 8, 13, 14], and its non-commutative versions [6, 3] in general von Neumann algebras.

Inspired by the ten problems in Hilbert space proposed by Halmos [9], it is natural to ask whether reducible operators in a von Neumann algebra $\mathcal{R}$ form a norm-dense subset of $\mathcal{R}$? An operator $T \in \mathcal{R}$ is said to be *reducible* in $\mathcal{R}$, if there exists a non-trivial projection P in $\mathcal{R}$ such that $TP = PT$. In 1976, Voiculescu gave an affirmative answer to this question for the case $\mathcal{R} = \mathcal{B}(\mathcal{H})$, by proving the non-commutative Weyl-von Neumann theorem [18]. Enlightened by this, we obtained a series of results around the extended Weyl-von Neumann theorem in the setting of semifinite von Neumann algebras [8, 13, 14], recently. In the current paper, we continue to study the preceding Voiculescu's theorem (short for the Voiculescu's non-commutative Weyl-von Neumann theorem) for $*$-homomorphisms of AF algebras into countably decomposable, properly infinite, semifinite von Neumann algebras $\mathcal{R}$.

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of the bounded linear operators on $\mathcal{H}$. Recall that two representations ϕ and ψ of a C*-algebra $\mathcal{A}$ on

Mathematics subject classification (2010): 47C15.
Keywords and phrases: Weyl-von Neumann theorem, Voiculescu's theorem, AF algebras, countably decomposable semifinite von Neumann algebras.
Junsheng Fang was partly supported by NSFC(Grant No.11431011) and a Start-up funding for Hebei Normal University. The corresponding author Rui Shi was partly supported by NSFC(Grant No.11871130) and the Fundamental Research Funds for the Central Universities (Grant No.DUT18LK23).
Shilin Wen is a doctoral candidate supervised by both Junsheng Fang and Rui Shi.
* Corresponding author.

$\mathcal{H}$ are said to be *approximately* (*unitarily*) *equivalent*, denoted by $\phi \sim_a \psi$, if there exists a net $\{U_\lambda\}_{\lambda\in\Lambda}$ of unitary operators in $\mathcal{B}(\mathcal{H})$ such that the limit

$$\lim_{\lambda\in\Lambda} \left\| U_\lambda^* \phi(A) U_\lambda - \psi(A) \right\| = 0 \tag{1.1}$$

holds for every operator A in $\mathcal{A}$.

When $\mathcal{A}$ is separable, the net $\{U_\lambda\}_{\lambda\in\Lambda}$ can be chosen to be a sequence. Let $\mathcal{K}(\mathcal{H})$ denote the set of the compact operators in $\mathcal{B}(\mathcal{H})$. We say that two representations ϕ and ψ of a separable C*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ are approximately unitarily equivalent (relative to $\mathcal{K}(\mathcal{H})$), denoted by $\phi \sim_{\mathcal{A}} \psi \mod \mathcal{K}(\mathcal{H})$, if there exists a sequence $\{U_n\}_{n=1}^\infty$ of unitary operators in $\mathcal{B}(\mathcal{H})$ satisfying (1.1) and

$$U_n^* \phi(A) U_n - \psi(A) \in \mathcal{K}(\mathcal{H})$$

for every n and every $A \in \mathcal{A}$.

The Weyl-von Neumann theorem, due to Weyl [20] and von Neumann [15], states that a bounded self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ can be written as a diagonal operator, by adding a compact operator (in the proof of Weyl in 1909) or, for each $\varepsilon > 0$, there exists a Hilbert-Schmidt operator $H \in \mathcal{K}(\mathcal{H})$ with the Hilbert-Schmidt norm $\|H\|_2 < \varepsilon$, such that the difference $A - H$ is diagonal (in the proof of von Neumann in 1935). Later in 1971, Berg [2] extended the result for bounded normal operators up to compact perturbations. As a corollary of his main result in [19], Voiculescu proved that a bounded normal operator can be written as a diagonal operator up to an arbitrarily small Hilbert-Schmidt perturbation.

In 1976, not only as an important technique for [19] but also for the Brown-Douglas-Fillmore theory, Voiculescu [18] proved a non-commutative version of the Weyl-von Neumann theorem characterizing approximate equivalence of two unital representations $\phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ is a separable unital C*-algebra and $\mathcal{H}$ is a complex separable Hilbert space. Precisely, the theorem is as follows:

THEOREM 1.1. *Suppose $\mathcal{A}$ is a separable unital C*-algebra, $\mathcal{H}$ is a separable Hilbert space and $\phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ are unital $*$-homomorphisms. The following are equivalent:*

1. $\phi \sim_a \psi$;
2. $\phi \sim_{\mathcal{A}} \psi \mod \mathcal{K}(\mathcal{H})$;
3. $\ker\phi = \ker\psi$, $\phi^{-1}(\mathcal{K}(\mathcal{H})) = \psi^{-1}(\mathcal{K}(\mathcal{H}))$, *and the nonzero parts of the restrictions* $\phi|_{\phi^{-1}(\mathcal{K}(\mathcal{H}))}$ *and* $\psi|_{\psi^{-1}(\mathcal{K}(\mathcal{H}))}$ *are unitarily equivalent.*

In 1977, Arveson [1] introduced quasicentral approximate units for C*-algebras and gave a different proof of the Voiculescu's theorem. Later in [7], Hadwin gave a different characterization of approximate equivalence of $*$-homomorphisms. In the same paper, Hadwin (Lemma 2.3 of [7]) proved an analogue for *approximate summands*. Recall that, for $T \in \mathcal{B}(\mathcal{H})$, let $\operatorname{rank}(T)$ denote the Hilbert-space dimension of the closure of the range, $\operatorname{ran}(T)$, of T.

THEOREM 1.2. (Lemma 2.3 of [7]) *Suppose $\mathcal{A}$ is a separable unital C*-algebra, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ are unital representations. The following are equivalent:*

1. *There is a representation $\gamma : \mathcal{A} \to \mathcal{B}(\mathcal{K}_1)$ for some Hilbert space $\mathcal{K}_1$ such that*

$$\psi \oplus \gamma \sim_a \phi\,;$$

2. *For every $A \in \mathcal{A}$,*

$$\operatorname{rank}(\psi(A)) \leqslant \operatorname{rank}(\phi(A)).$$

In her 1994 doctoral dissertation (see also [6]), Huiru Ding extended some of the results in [7] to the case in which $\mathcal{B}(\mathcal{H})$ is replaced by a von Neumann algebra $\mathcal{R}$. The following are some terms adopted in the current paper.

Suppose that $\mathcal{R}$ is a von Neumann algebra. Let S and T be two operators in $\mathcal{R}$. We define the $\mathcal{R}$-rank of T (denoted by $\mathcal{R}\text{-rank}(T)$) to be the *Murray-von Neumann equivalence class* of the projection onto the closure of $\operatorname{ran}(T)$. If there exists a projection E in $\mathcal{R}\text{-rank}(T)$ and a projection F in $\mathcal{R}\text{-rank}(S)$ such that E is a subprojection of F, denoted by $E \leqslant F$, then define

$$\mathcal{R}\text{-rank}(T) \leqslant \mathcal{R}\text{-rank}(S). \tag{1.2}$$

In Chapter 6 of [11], the relation "$\leqslant$" in (1.2) is verifed to be a partial order. Suppose that $\mathcal{A}$ is a unital C*-algebra. Let ϕ and ψ be unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{R}$. The homomorphisms ϕ and ψ are said to be *approximately equivalent in* $\mathcal{R}$ (denoted by $\phi \sim_a \psi$ in $\mathcal{R}$), if there exists a net $\{U_\lambda\}_{\lambda\in\Lambda}$ of unitary operators **in** $\mathcal{R}$ such that, for every $A \in \mathcal{A}$,

$$\lim_{\lambda\in\Lambda} \left\| U_\lambda^* \phi(A) U_\lambda - \psi(A) \right\| = 0.$$

THEOREM 1.3. (Corollary 3 of [6]) *Suppose that $\mathcal{A}$ is a unital C*-algebra that is a direct limit of finite direct sums of commutative C*-algebras tensored with matrix algebras, and $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space. If $\phi, \psi : \mathcal{A} \to \mathcal{R}$ are unital $*$-homomorphisms, then the following are equivalent:*

1. *$\phi \sim_a \psi$ in $\mathcal{R}$;*

2. *For every $A \in \mathcal{A}$,*

$$\mathcal{R}\text{-rank}(\phi(A)) = \mathcal{R}\text{-rank}(\psi(A)).$$

Another interesting result based on generalizations of the Voiculescu's theorem is proved in [3], where the authors characterized properly infinite injective von Neumann algebras and nuclear C*-algebras by using a uniqueness theorem.

Enlightened by the preceding results, in the current paper, we concentrate on the non-commutative Weyl-von Neumann theorem for representations of AF algebras into countably decomposable, semifinite, infinite von Neumann factors. In Section 2, we extend the concept of approximate equivalence modulo the "compact" operators in the setting of semifinite von Neumann algebras. In Section 3, relative to finite von Neumann algebras, we characterize the approximate summands of a $*$-homomorphism of an AF algebra. Note that, if $\mathcal{R}$ mentioned in Theorem 1.3 is a von Neumann factor with a faithful normal tracial weight τ, then, for every $A \in \mathcal{A}$, it follows that:

$$\mathcal{R}\text{-rank}\,(\phi(A)) \leqslant \mathcal{R}\text{-rank}\,(\psi(A)) \quad \Leftrightarrow \quad \tau(R(\phi(A))) \leqslant \tau(R(\psi(A))). \tag{1.3}$$

Thus, the main theorem in Section 3 is as follows.

THEOREM 1.4. *Let $\mathcal{A}$ be a unital AF algebra and $\mathcal{R}$ be a type II_1 factor with a faithful normal tracial state τ. If P is a projection in $\mathcal{R}$, $\pi : \mathcal{A} \to \mathcal{R}$ is a unital $*$-homomorphism and $\rho : \mathcal{A} \to P\mathcal{R}P$ is a unital $*$-homomorphism such that*

$$\tau(R(\rho(X))) \leqslant \tau(R(\pi(X))), \quad \forall X \in \mathcal{A},$$

then there exists a unital $*$-homomorphism $\gamma : \mathcal{A} \to P^{\perp}\mathcal{R}P^{\perp}$ such that

$$\gamma \oplus \rho \sim_a \pi \ \text{ in } \mathcal{R}.$$

In Section 4, for two $*$-homomorphisms ϕ and ψ of an AF algebra into a countable decomposable, semifinite, infinite von Neumann factor $\mathcal{R}$ with a faithful normal semifinite tracial weight τ, the main theorem states that the approximately unitary equivalence of ϕ and ψ in $\mathcal{R}$ implies that these two $*$-homomorphisms are approximately equivalent modulo the "compact" ideal $\mathcal{K}(\mathcal{R},\tau)$. Precisely, Theorem 4.10 is as follows.

THEOREM 1.5. *Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is an AF subalgebra of $\mathcal{R}$ with an identity $I_{\mathcal{A}}$.*

If ϕ and ψ are unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{R}$, then the following are equivalent:

(i) $\phi \sim_a \psi$ in $\mathcal{R}$, namely, ϕ and ψ are approximately unitarily equivalent in $\mathcal{R}$;

(ii) $\phi \sim_{\mathcal{A}} \psi \mod \mathcal{K}(\mathcal{R},\tau)$, namely, ϕ and ψ are strongly-approximately-unitarily-equivalent over $\mathcal{A}$, (based on Definition 2.4, which comes later in Section 2).

Since a separable C*-algebra of "compact" operators in a semifinite von Neumann algebra may contain no minimal projection, we can not apply some classical results about compact operators of $\mathcal{B}(\mathcal{H})$ to general cases in the setting of semifinite von Neumann algebras. Therefore, we develop a series of new techniques in Section 4.

2. Preliminaries

In the setting of von Neumann algebras. The compact ideal $\mathcal{K}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ can be extended in the following way. In this section, we let $\mathcal{R}$ be a countably decomposable, properly infinite, semifinite von Neumann algebra with a faithful normal semifinite tracial weight τ. Let

$$\begin{aligned}\mathcal{PF}(\mathcal{R},\tau) &= \{P \ : \ P = P^* = P^2 \in \mathcal{R} \text{ and } \tau(P) < \infty\},\\ \mathcal{F}(\mathcal{R},\tau) &= \{XPY \ : \ P \in \mathcal{PF}(\mathcal{R},\tau) \text{ and } X,Y \in \mathcal{R}\},\\ \mathcal{K}(\mathcal{R},\tau) &= \|\cdot\|\text{-norm closure of } \mathcal{F}(\mathcal{R},\tau) \text{ in } \mathcal{R},\end{aligned} \tag{2.1}$$

be the sets of finite rank projections, finite rank operators, and compact operators in $(\mathcal{R},\tau)$, respectively.

For a von Neumann algebra $\mathcal{R}$, we denote by $\mathcal{K}(\mathcal{R})$ the $\|\cdot\|$-norm closed ideal generated by finite projections in $\mathcal{R}$. In general, $\mathcal{K}(\mathcal{R},\tau)$ is a subset of $\mathcal{K}(\mathcal{R})$. That is because a finite projection might not be a finite rank projection with respect to τ. However, if $\mathcal{R}$ is a factor with a faithful, normal, semifinite tracial weight τ, then Proposition 8.5.2 of [11] entails the equality

$$\mathcal{K}(\mathcal{R},\tau) = \mathcal{K}(\mathcal{R}).$$

To extend the definition of approximate equivalence of two unital $*$-homomorphisms of a separable C^*-algebra $\mathcal{A}$ into $\mathcal{R}$ (**relative to** $\mathcal{K}(\mathcal{R},\tau)$), we need to develop the following notation and definitions.

Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Suppose that $\{E_{i,j}\}_{i,j=1}^{\infty}$ is a system of matrix units of $\mathcal{B}(\mathcal{H})$.

For a countably decomposable, properly infinite von Neumann algebra $\mathcal{R}$ with a faithful normal semifinite tracial weight τ, there exists a sequence $\{V_i\}_{i=1}^{\infty}$ of partial isometries in $\mathcal{R}$ such that

$$V_iV_i^* = I_{\mathcal{R}}, \quad \sum_{i=1}^{\infty} V_i^*V_i = I_{\mathcal{R}}, \quad \text{and} \quad V_jV_i^* = 0 \text{ when } i \neq j.$$

DEFINITION 2.1. (Definition 2.2.1 of [13]) Let $\mathcal{R}\otimes\mathcal{B}(\mathcal{H})$ be a von Neumann algebra tensor product of $\mathcal{R}$ and $\mathcal{B}(\mathcal{H})$. For all $X \in \mathcal{R}$ and all $\sum_{i,j=1}^{\infty} X_{i,j}\otimes E_{i,j} \in \mathcal{R}\otimes\mathcal{B}(\mathcal{H})$, define

$$\phi : \mathcal{R} \to \mathcal{R}\otimes\mathcal{B}(\mathcal{H}) \quad \text{and} \quad \psi : \mathcal{R}\otimes\mathcal{B}(\mathcal{H}) \to \mathcal{R}$$

by

$$\phi(X) = \sum_{i,j=1}^{\infty} (V_iXV_j^*)\otimes E_{i,j} \quad \text{and} \quad \psi(\sum_{i,j=1}^{\infty} X_{i,j}\otimes E_{i,j}) = \sum_{i,j=1}^{\infty} V_i^*X_{i,j}V_j.$$

By Lemma 2.2.2 of [13], both ϕ and ψ are normal $*$-homomorphisms satisfying

$$\psi\circ\phi = id_{\mathcal{R}} \qquad \text{and} \qquad \phi\circ\psi = id_{\mathcal{R}\otimes\mathcal{B}(\mathcal{H})}.$$

DEFINITION 2.2. (Definition 2.2.3 of [13]) Define a mapping $\tilde{\tau} : (\mathcal{R}\otimes\mathcal{B}(\mathcal{H}))_+ \to [0,\infty]$ to be

$$\tilde{\tau}(y) = \tau(\psi(y)), \qquad \forall\, y \in (\mathcal{R}\otimes\mathcal{B}(\mathcal{H}))_+.$$

By the above definitions, the following are proved in Lemma 2.2.4 of [13]:

(i) $\tilde{\tau}$ is a faithful, normal, semifinite tracial weight of $\mathcal{R}\otimes\mathcal{B}(\mathcal{H})$;

(ii) $\tilde{\tau}(\sum_{i,j=1}^{\infty} X_{i,j}\otimes E_{i,j}) = \sum_{i=1}^{\infty}\tau(X_{i,i})$ for all $\sum_{i,j=1}^{\infty} X_{i,j}\otimes E_{i,j} \in (\mathcal{R}\otimes\mathcal{B}(\mathcal{H}))_+$;

(iii)
$$\begin{aligned}\mathcal{PF}(\mathcal{R}\otimes\mathcal{B}(\mathcal{H}),\tilde{\tau}) &= \phi(\mathcal{PF}(\mathcal{R},\tau)),\\ \mathcal{F}(\mathcal{R}\otimes\mathcal{B}(\mathcal{H}),\tilde{\tau}) &= \phi(\mathcal{F}(\mathcal{R},\tau)),\\ \mathcal{K}(\mathcal{R}\otimes\mathcal{B}(\mathcal{H}),\tilde{\tau}) &= \phi(\mathcal{K}(\mathcal{R},\tau)).\end{aligned}$$

REMARK 2.3. It shows that $\tilde{\tau}$ is a natural extension of τ from $\mathcal{R}$ to $\mathcal{R}\otimes\mathcal{B}(\mathcal{H})$. If no confusion arises, $\tilde{\tau}$ will be also denoted by τ. By Proposition 2.2.9 of [13], the ideal $\mathcal{K}(\mathcal{R}\otimes\mathcal{B}(\mathcal{H}),\tilde{\tau})$ is independent of the choice of the system of matrix units $\{E_{i,j}\}_{i,j=1}^{\infty}$ of $\mathcal{B}(\mathcal{H})$ and the choice of the family $\{V_i\}_{i=1}^{\infty}$ of partial isometries in $\mathcal{R}$.

Let $\mathcal{A}$ be a separable C*-subalgebra of $\mathcal{R}$ with an identity $I_{\mathcal{A}}$. Suppose that ψ is a positive mapping from $\mathcal{A}$ into $\mathcal{R}$ such that $\psi(I_{\mathcal{A}})$ is a projection in $\mathcal{R}$. Then for all $0\leqslant X\in\mathcal{A}$, we have $0\leqslant\psi(X)\leqslant\|X\|\psi(I_{\mathcal{A}})$. Therefore, it follows that

$$\psi(X)\psi(I_{\mathcal{A}}) = \psi(I_{\mathcal{A}})\psi(X) = \psi(X)$$

for all positive $X\in\mathcal{A}$. In other words, $\psi(I_{\mathcal{A}})$ can be viewed as an identity of $\psi(\mathcal{A})$. Or, $\psi(\mathcal{A})\subseteq\psi(I_{\mathcal{A}})\mathcal{R}\psi(I_{\mathcal{A}})$.

DEFINITION 2.4. (Definition 2.3.1 of [13]) Suppose that $\{E_{i,j}\}_{i,j\geqslant 1}$ is a system of matrix units of $\mathcal{B}(\mathcal{H})$. Let $M,N\in\mathbb{N}\cup\{\infty\}$. Suppose that $\psi_1,\ldots,\psi_M$ and $\phi_1,\ldots,\phi_N$ are positive mappings from $\mathcal{A}$ into $\mathcal{R}$ such that $\psi_1(I_{\mathcal{A}}),\ldots,\psi_M(I_{\mathcal{A}})$, $\phi_1(I_{\mathcal{A}}),\ldots,\phi_N(I_{\mathcal{A}})$ are projections in $\mathcal{R}$.

(a) Let $\mathcal{F}\subseteq\mathcal{A}$ be a finite subset and $\varepsilon>0$. Say $\psi_1\oplus\cdots\oplus\psi_M$ is $(\mathcal{F},\varepsilon)$*-strongly--approximately-unitarily-equivalent* to $\phi_1\oplus\cdots\oplus\phi_N$ over $\mathcal{A}$, denoted by

$$\psi_1\oplus\psi_2\oplus\cdots\oplus\psi_M \sim_{\mathcal{A}}^{(\mathcal{F},\varepsilon)} \phi_1\oplus\phi_2\oplus\cdots\oplus\phi_N \qquad \mod \mathcal{K}(\mathcal{R},\tau)$$

if there exists a partial isometry V in $\mathcal{R}\otimes\mathcal{B}(\mathcal{H})$ such that:

(i) $V^*V = \sum_{i=1}^{M}\psi_i(I_{\mathcal{A}})\otimes E_{i,i}$ and $VV^* = \sum_{i=1}^{N}\phi_i(I_{\mathcal{A}})\otimes E_{i,i}$;

(ii) $\sum_{i=1}^{M}\psi_i(X)\otimes E_{i,i} - V^*\left(\sum_{i=1}^{N}\phi_i(X)\otimes E_{i,i}\right)V \in \mathcal{K}(\mathcal{R}\otimes\mathcal{B}(\mathcal{H}),\tau)$ for all $X\in\mathcal{A}$;

(iii) $\|\sum_{i=1}^{M}\psi_i(X)\otimes E_{i,i} - V^*\left(\sum_{i=1}^{N}\phi_i(X)\otimes E_{i,i}\right)V\| < \varepsilon$ for all $X\in\mathcal{F}$.

(b) Say $\psi_1 \oplus \cdots \oplus \psi_M$ is *strongly-approximately-unitarily-equivalent* to $\phi_1 \oplus \cdots \oplus \phi_N$ over $\mathcal{A}$, denoted by

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathcal{A}} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N \qquad \mod \mathcal{K}(\mathcal{R}, \tau)$$

if, for any finite subset $\mathcal{F} \subseteq \mathcal{A}$ and $\varepsilon > 0$,

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathcal{A}}^{(\mathcal{F}, \varepsilon)} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N \qquad \mod \mathcal{K}(\mathcal{R}, \tau).$$

In the above definitions, if $\mathcal{R} = \mathcal{B}(\mathcal{H})$, then the strongly-approximately-unitary--equivalence of representations ϕ and ψ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ coincides with the approximately-unitarily-equivalence of representations ϕ and ψ relative to $\mathcal{K}(\mathcal{H})$.

REMARK 2.5. Recall that a $(separable)$ C^*-algebra $\mathcal{A}$ is **approximately finite dimensional** or **AF** if it is the norm-closure of an increasing union of finite dimensional subalgebras $\mathcal{A}_n$. If $\mathcal{A}$ is unital, then we assume that each $\mathcal{A}_n$ contains the identity of $\mathcal{A}$.

AF algebras are an important collection of $(separable)$ C^*-algebras. People developed rich results about AF algebras in the past several decades. In the current paper, we mainly consider AF subalgebras of a semifinite von Neumann factor with separable predual. The reader is referred to Chapter III of [4] for the definition of an AF algebra. For more details about inductive limit C^*-algebras, the reader is referred to Chapter 6 of [16].

3. Representations relative to finite von Neumann algebras

In this section, $(\mathcal{R}, \tau)$ is always assumed to be a type II_1 factor with separable predual, where τ denotes the faithful normal tracial state. Recall that two $*$-homomorphisms ρ and π of a C^*-algebra $\mathcal{A}$ into $\mathcal{R}$ are said to be *unitarily equivalent*, denoted by $\rho \simeq \pi$ in $\mathcal{R}$, if there exists a unitary operator U in $\mathcal{R}$ such that the equality $U^*\rho(A)U = \pi(A)$ holds for every A in $\mathcal{A}$.

The main theorem in this section is to express a $*$-homomorphism π of an AF algebra $\mathcal{A}$ into $\mathcal{R}$ as an "approximate direct sum" by a natural hypothesis. For an AF algebra $\mathcal{A}$, it is convenient to assume that there exists an increasing sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of finite-dimensional C^*-subalgebras such that

$$\mathcal{A} = \overline{\cup_{n=1}^{\infty} \mathcal{A}_n}^{\|\cdot\|}. \tag{3.1}$$

LEMMA 3.1. *Let $\mathcal{A}$ be a unital finite-dimensional C^*-algebra and $(\mathcal{R}, \tau)$ be a type II_1 factor. Suppose that P is a projection in $\mathcal{R}$.*

If $\pi : \mathcal{A} \to \mathcal{R}$ is a unital $*$-homomorphism and $\rho : \mathcal{A} \to P\mathcal{R}P$ is a unital $*$-homomorphism such that:

$$\tau(R(\rho(X))) \leqslant \tau(R(\pi(X))), \quad \forall X \in \mathcal{A},$$

then there exists a unital $*$-homomorphism $\gamma : \mathcal{A} \to P^{\perp}\mathcal{R}P^{\perp}$ such that

$$\gamma \oplus \rho \simeq \pi \ \textit{ in } \mathcal{R}.$$

Furthermore, if $\gamma' : \mathcal{A} \to P^{\perp}RP^{\perp}$ is another $*$-homomorphism satisfying

$$\gamma' \oplus \rho \simeq \pi \ \textit{ in } \mathcal{R},$$

then $\gamma' \simeq \gamma$ in $P^{\perp}\mathcal{R}P^{\perp}$.

Proof. Let $I_{\mathcal{R}}$ be the unit of $\mathcal{R}$. We may assume that the projection P is nontrivial. Let $\mathcal{A}$ be in the form

$$\mathcal{A} = \oplus_{l=1}^{n} M_{k_l}(\mathbb{C}),$$

where l, n, k_l are positive integers. For each $l \geqslant 1$, let $\{E_{ij}^{l}\}_{1 \leqslant i,j \leqslant k_l}$ be the canonical system of matrix units for $M_{k_l}(\mathbb{C})$, i.e., E_{ij}^{l} is a k_l-by-k_l matrix with the (i,j)-th entry 1 and others 0. Denote by

$$B_{ij}^{l} = \rho(E_{ij}^{l}), \quad A_{ij}^{l} = \pi(E_{ij}^{l}), \qquad \text{for } 1 \leqslant i,j \leqslant k_l.$$

Hence, we have

$$\sum_{l=1}^{n} \sum_{i=1}^{k_l} B_{ii}^{l} = P, \quad \sum_{l=1}^{n} \sum_{i=1}^{k_l} A_{ii}^{l} = I_{\mathcal{R}}.$$

By the hypothesis, we have that

$$\tau(B_{ii}^{l}) \leqslant \tau(A_{ii}^{l}) \quad \text{and} \quad \tau(B_{ij}^{l}) = \tau(A_{ij}^{l}) = 0, \quad \text{for } i \neq j,\ 1 \leqslant i,j \leqslant k_l,\ 1 \leqslant l \leqslant n.$$

Thus, there exists a system of matrix units $\{F_{ij}^{l}\}_{1 \leqslant i,j \leqslant k_l}$ of $P^{\perp}\mathcal{R}P^{\perp}$ for $1 \leqslant l \leqslant n$, such that

$$\sum_{l=1}^{n} \sum_{i=1}^{k_l} F_{ii}^{l} = P^{\perp}, \quad \tau(B_{ii}^{l}) + \tau(F_{ii}^{l}) = \tau(A_{ii}^{l}), \quad \text{and} \quad \tau(F_{ij}^{l}) = 0,$$

for $i \neq j$, $1 \leqslant i,j \leqslant k_l$, $1 \leqslant l \leqslant n$.

Define a linear mapping γ by

$$\gamma : \mathcal{A} \to P^{\perp}\mathcal{R}P^{\perp}, \quad \gamma(E_{ij}^{l}) = F_{ij}^{l}, \quad \text{for } 1 \leqslant i,j \leqslant k_l, \quad 1 \leqslant l \leqslant n.$$

It is routine to verify that γ is a $*$-homomorphism. The equality $\tau(\rho(A) + \gamma(A)) = \tau(\pi(A))$ holds for every $A \in \mathcal{A}$. Thus, we obtain that $\gamma \oplus \rho \simeq \pi$ in $\mathcal{R}$.

If $\gamma' : \mathcal{A} \to P^{\perp}\mathcal{R}P^{\perp}$ is another $*$-homomorphism satisfying $\gamma' \oplus \rho \simeq \pi$ in $\mathcal{R}$, then, the equality

$$\tau(R(\gamma(A))) = \tau(R(\gamma'(A)))$$

holds for every $A \in \mathcal{A}$. This implies that $\gamma' \simeq \gamma$ in $P^{\perp}\mathcal{R}P^{\perp}$. $\square$

THEOREM 3.2. *Let $\mathcal{A}$ be a unital AF algebra and $(\mathcal{R},\tau)$ be a type II_1 factor. If P is a projection in $\mathcal{R}$, $\pi:\mathcal{A}\to\mathcal{R}$ is a unital $*$-homomorphism, and $\rho:\mathcal{A}\to P\mathcal{R}P$ is a unital $*$-homomorphism such that*

$$\tau(R(\rho(X)))\leqslant\tau(R(\pi(X))),\quad \forall X\in\mathcal{A},$$

then there exists a unital $*$-homomorphism $\gamma:\mathcal{A}\to P^{\perp}\mathcal{R}P^{\perp}$ such that

$$\gamma\oplus\rho\sim_a\pi \ \text{ in } \mathcal{R}.$$

Proof. Since $\mathcal{A}$ is AF, as in (3.1), $\mathcal{A}$ can be written in the form

$$\mathcal{A}=\overline{\cup_{n=1}^{\infty}\mathcal{A}_n}^{\|\cdot\|}.$$

Consider the restrictions of π and ρ on $\mathcal{A}_n$, denoted by π_n and ρ_n, respectively. By Lemma 3.1, there exists a unital $*$-homomorphism $\varphi_n:\mathcal{A}_n\to P^{\perp}\mathcal{R}P^{\perp}$ such that

$$\varphi_n\oplus\rho_n\simeq\pi_n \ \text{ in } \mathcal{R},\quad \forall n\geqslant 1. \tag{3.2}$$

Define $\gamma_1=\varphi_1$ on $\mathcal{A}_1$. By applying (3.2), we have

$$\varphi_2|_{\mathcal{A}_1}\oplus\rho_1\simeq\pi_1 \ \text{ in } \mathcal{R},\quad \text{ on } \mathcal{A}_1.$$

Thus, by using Lemma 3.1, there exists a unitary operator $U_2\in P^{\perp}\mathcal{R}P^{\perp}$ such that

$$U_2(\varphi_2|_{\mathcal{A}_1})U_2^*=\gamma_1.$$

Note that

$$U_2\varphi_2U_2^*\oplus\rho_2\simeq\varphi_2\oplus\rho_2\simeq\pi_2 \ \text{ in } \mathcal{R},\quad \text{ on } \mathcal{A}_2.$$

Define $\gamma_2=U_2\varphi_2U_2^*$ on $\mathcal{A}_2$. Then $\gamma_2\oplus\rho_2\simeq\pi_2$ in $\mathcal{R}$, on $\mathcal{A}_2$, and $\gamma_2|_{\mathcal{A}_1}=\gamma_1$ on $\mathcal{A}_1$. Similarly, for every $n\geqslant 2$, we can define a unital $*$-homomorphism $\gamma_n:\mathcal{A}_n\to P^{\perp}\mathcal{R}P^{\perp}$ such that:

1. the relation $\gamma_n\oplus\rho_n\simeq\pi_n$ in $\mathcal{R}$ holds for every operator in $\mathcal{A}_n$;
2. $\gamma_n|_{\mathcal{A}_{n-1}}=\gamma_{n-1}:\mathcal{A}_{n-1}\to P^{\perp}\mathcal{R}P^{\perp}$.

For every X in $\mathcal{A}$, the definition for AF algebras as in (3.1) guarantees that there exists a sequence $\{X_n:X_n\in\mathcal{A}_n\}_{n\geqslant 1}$ of operators in $\mathcal{A}$ such that

$$\lim_{n\to\infty}\|X-X_n\|=0. \tag{3.3}$$

Note that $\gamma_m(X_n)=\gamma_n(X_n)$ for $m\geqslant n$. Thus the inequality

$$\|\gamma_{n+k}(X_{n+k})-\gamma_n(X_n)\|=\|\gamma_{n+k}(X_{n+k})-\gamma_{n+k}(X_n)\|\leqslant\|X_{n+k}-X_n\| \tag{3.4}$$

ensures that $\{\gamma_n(X_n)\}_{n\geqslant 1}$ is a Cauchy sequence. Hence, by using (3.3) and (3.4), we can define a unital $*$-homomorphism $\gamma:\mathcal{A}\to P^{\perp}\mathcal{R}P^{\perp}$ by

$$\gamma(X)=\lim_{n\to\infty}\gamma_n(X_n),\quad \forall X\in\mathcal{A}, \tag{3.5}$$

where $\{X_n : X_n \in \mathcal{A}_n\}_{n\geqslant 1}$ is as in (3.3). This γ is well-defined, since $\gamma(X)$ is independent of the Cauchy sequence $\{\gamma_n(X_n)\}_{n\geqslant 1}$. It follows that $\gamma|_{\mathcal{A}_n} = \gamma_n$ and $\gamma|_{\mathcal{A}_n} \oplus \rho_n \simeq \pi_n$ in $\mathcal{R}$, on $\mathcal{A}_n$.

Since $\mathcal{A}$ is AF, there exists an increasing finite subsets $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ of $\mathcal{A}$ such that $\cup_{k\geqslant 1}\mathcal{F}_k$ is dense in $\mathcal{A}$.

For every $n \geqslant 1$, the relation $\gamma_n \oplus \rho_n \simeq \pi_n$ in $\mathcal{R}$ implies that there exists a unitary operator V_n in $\mathcal{R}$ such that

$$V_n(\gamma_n(A) \oplus \rho_n(A))V_n^* = \pi_n(A), \quad \forall A \in \mathcal{A}_n. \tag{3.6}$$

Given $\mathcal{F}_k$ and $1/k$, there exists an $\mathcal{A}_{n_k}$ such that for every A in $\mathcal{F}_k$ there exists a B in $\mathcal{A}_{n_k}$ with $\|A - B\| < 1/k$. Thus, for every A in $\mathcal{F}_k$, equalities (3.5) and (3.6) yield that

$$\|\pi(A) - V_{n_k}(\gamma(A) \oplus \rho(A))V_{n_k}^*\| \leqslant 2/k + \|\pi_{n_k}(B) - V_{n_k}(\gamma_{n_k}(B) \oplus \rho_{n_k}(B))V_{n_k}^*\| = 2/k.$$

Hence, the mapping $\gamma : \mathcal{A} \to P^\perp \mathcal{R} P^\perp$ is the required unital $*$-homomorphism such that

$$\gamma \oplus \rho \sim_a \pi \ \text{ in } \mathcal{R}.$$

This completes the proof. $\square$

4. Representations relative to semifinite infinite von Neumann algebras

In this section, suppose that $\mathcal{R}$ is a countably decomposable, infinite, semifinite von Neumann factor with a faithful, normal, semifinite tracial weight τ. Recall that the notation $\mathcal{PF}(\mathcal{R},\tau)$, $\mathcal{F}(\mathcal{R},\tau)$, and $\mathcal{K}(\mathcal{R},\tau)$ are the sets of finite rank projections, finite rank operators, and compact operators **in** $(\mathcal{R},\tau)$, respectively, which are introduced in (2.1). For each $T \in \mathcal{R}$, denote by $R(T)$ the range projection onto the closure of the range of T. By Theorem 6.8.3 of [11], the norm-closed two-sided ideal $\mathcal{K}(\mathcal{R},\tau)$ introduced in (2.1) can be also viewed in the following way:

$$\mathcal{K}(\mathcal{R},\tau) = \|\cdot\|\text{-norm closure of } \{T \in \mathcal{R} : \tau(R(T)) < \infty\}. \tag{4.1}$$

The following two lemmas from [8] are useful in the proof of the main theorem in this section.

LEMMA 4.1. (Lemma 3.1 of [8]) *For an operator A in $\mathcal{R}$, the following are equivalent:*

1. *A is in $\mathcal{K}(\mathcal{R},\tau)$;*
2. *$|A|$ is in $\mathcal{K}(\mathcal{R},\tau)$;*
3. *for every $\varepsilon > 0$, $\tau(\chi_{[0,\varepsilon)}(|A|)) = \infty$ and $\tau(\chi_{[\varepsilon,\infty)}(|A|)) < \infty$;*
4. *for every $\varepsilon > 0$, $\tau(\chi_{[0,\varepsilon]}(|A|)) = \infty$ and $\tau(\chi_{(\varepsilon,\infty)}(|A|)) < \infty$.*

LEMMA 4.2. (Lemma 3.2 of [8]) *Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is a unital C^*-algebra. If ϕ and $\psi : \mathcal{A} \to \mathcal{R}$ are two unital $*$-homomorphisms such that*

$$\phi \sim_a \psi \ \text{ in } \mathcal{R},$$

then it follows that

$$\phi(A) \in \mathcal{K}(\mathcal{R},\tau) \quad \Leftrightarrow \quad \psi(A) \in \mathcal{K}(\mathcal{R},\tau), \quad \forall\, A \in \mathcal{A}.$$

REMARK 4.3. Recall that a separable C^*-algebra $\mathcal{A}$ is "AF" (short for approximately finite-dimensional), if $\mathcal{A}$ is an inductive limit of an increasing sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of finite-dimensional C^*-algebras with respect to the norm topology. In the rest of this paper, for a $*$-algebra $\mathcal{B}$, denote the closure of $\mathcal{B}$ in the operator norm by $\overline{\mathcal{B}}^{\|\cdot\|}$.

For an AF subalgebra $\mathcal{A}$ of $(\mathcal{R},\tau)$, it is convenient to assume that there exists an increasing sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of finite-dimensional C^*-subalgebras such that

$$\mathcal{A} = \overline{\cup_{n=1}^{\infty}\mathcal{A}_n}^{\|\cdot\|}. \tag{4.2}$$

By applying Lemma 3.4.1 of [4], we have

$$\mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau) = \overline{\cup_{n=1}^{\infty}(\mathcal{A}_n \cap \mathcal{K}(\mathcal{R},\tau))}^{\|\cdot\|}. \tag{4.3}$$

Note that each $\mathcal{A}_n \cap \mathcal{K}(\mathcal{R},\tau)$ is $*$-isomorphic to a finite-dimensional C^*-algebra. Hence, for each positive operator F in the unit ball of $\mathcal{A}_n \cap \mathcal{K}(\mathcal{R},\tau)$, the spectrum of F is a finite subset $\{\lambda_1,\ldots,\lambda_k\}$ in $[0,1]$. Lemma 4.1 entails that F belongs to $\mathcal{A}_n \cap \mathcal{F}(\mathcal{R},\tau)$. Thus, we have

$$\mathcal{A}_n \cap \mathcal{F}(\mathcal{R},\tau) = \mathcal{A}_n \cap \mathcal{K}(\mathcal{R},\tau).$$

It follows that the increasing sequence $\{F^{1/m}\}_{m\geqslant 1}$ converges in the norm topology. By applying Lemma 5.1.5 of [10], the uniqueness of the limit implies that $R(F)$ is the limit of $\{F^{1/m}\}_{m\geqslant 1}$. Therefore, $R(F)$ also belongs to the finite-dimensional C^*-algebra $\mathcal{A}_n \cap \mathcal{K}(\mathcal{R},\tau)$. Moreover, there exists a sequence $\{K_n\}_{n\geqslant 1}$ of finite rank operators in the unit ball of $\mathcal{A} \cap \mathcal{F}(\mathcal{R},\tau)$, which is norm-dense in $\mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau)$.

DEFINITION 4.4. Let $P_{\mathcal{K}(\mathcal{A},\tau)}$ be a projection defined as

$$P_{\mathcal{K}(\mathcal{A},\tau)} := \bigvee\nolimits_{K \in \mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau)} R(K),$$

where $\mathcal{A}$ is an AF subalgebra of $(\mathcal{R},\tau)$.

LEMMA 4.5. *Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is an AF subalgebra of $\mathcal{R}$. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of positive finite rank operators in $\mathcal{A}_+ \cap \mathcal{F}(\mathcal{R},\tau)$, which is norm-dense in $\mathcal{A}_+ \cap \mathcal{K}(\mathcal{R},\tau)$ and $P := \vee_{n\geqslant 1} R(K_n)$. Then the following are true:*

1. $P = P_{\mathcal{K}(\mathcal{A},\tau)}$;
2. *the equality* $PX = XP$ *holds for every* $X \in \mathcal{A}$.

Proof. Assume that $(\mathcal{R},\tau)$ acts on a complex separable Hilbert space $\mathcal{H}$. Remark 4.3 guarantees the existence of a norm-dense subset of finite rank operators in $\mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau)$. For each K in $\mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau)$, there exists a subsequence $\{K_{n_i}\}_{i=1}^{\infty}$ of $\{K_n\}_{n=1}^{\infty}$ such that

$$\lim_{i\to\infty} \||K^*| - K_{n_i}\| = 0.$$

Let $K = |K^*|V$ be the polar decomposition of K, by Theorem 6.1.2 of [11]. Then for each x in $\mathcal{H}$, it follows that

$$\|Kx - K_{n_i}Vx\| \leqslant \||K^*| - K_{n_i}\|\|x\| \to 0, \quad (\text{as } i \to \infty).$$

Since $PK_{n_i}Vx = K_{n_i}Vx$, it follows that $PK = K$ holds for every K in $\mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau)$. This means that $P = P_{\mathcal{K}(\mathcal{A},\tau)}$. Specially, if P is trivial in $\mathcal{R}$, then it is evident that P reduces $\mathcal{A}$.

In the following, assume contrarily that there exists an operator A in $\mathcal{A}$ such that $P^{\perp}AP \neq 0$. Then there exists a vector $x \in \operatorname{ran} P$ such that $P^{\perp}Ax \neq 0$. There is also a sequence of vectors $\{J_n x_n\}_{n\geqslant 1} \subset \operatorname{ran} P$ with compact operators J_n's in $\mathcal{A} \cap \mathcal{K}(\mathcal{R},\tau)$ such that

$$\lim_{n\to\infty} \|J_n x_n - x\| = 0.$$

Since each $P^{\perp}AJ_n x_n = 0$, we obtain $P^{\perp}Ax = 0$. This is a contradiction. Thus for each A in $\mathcal{A}$, we have $P^{\perp}AP = 0$. It follows that $PAP^{\perp} = 0$ for each A in $\mathcal{A}$. Therefore, P reduces $\mathcal{A}$. This completes the proof. $\square$

DEFINITION 4.6. Let $\mathcal{R}$ be a countably decomposable, properly infinite, semifinite von Neumann algebra with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is an AF subalgebra of $\mathcal{R}$. By Lemma 4.5, the projection $P_{\mathcal{K}(\mathcal{A},\tau)}$ reduces $\mathcal{A}$. Define

$$id_0(A) := AP_{\mathcal{K}(\mathcal{A},\tau)} \quad \text{and} \quad id_e(A) := AP^{\perp}_{\mathcal{K}(\mathcal{A},\tau)} \quad \forall A \in \mathcal{A}. \tag{4.4}$$

Then id_0 and id_e are well-defined $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{A}P_{\mathcal{K}(\mathcal{A},\tau)}$ and $\mathcal{A}P^{\perp}_{\mathcal{K}(\mathcal{A},\tau)}$, respectively.

Let ρ be a unital $*$-isomorphism of $\mathcal{A}$ into $\mathcal{R}$. Define

$$\rho_0(A) := id_0(\rho(A)) \quad \text{and} \quad \rho_e(A) := id_e(\rho(A)) \quad \forall A \in \mathcal{A}. \tag{4.5}$$

Then ρ_0 and ρ_e are well-defined $*$-homomorphisms of $\mathcal{A}$ into $\rho(\mathcal{A})P_{\mathcal{K}(\rho(\mathcal{A}),\tau)}$ and $\rho(\mathcal{A})P^{\perp}_{\mathcal{K}(\rho(\mathcal{A}),\tau)}$, respectively.

In terms of the classical method to prove the Voiculescu's Theorem, if id and ρ are approximately unitarily equivalent $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{R}$, then we first concern the strongly-approximate-equivalence of id_0 and ρ_0 relative to $\mathcal{K}(\mathcal{R},\tau)$, as in Definition 2.4. Next, we apply Theorem 5.3.1 of [13] to complete the proof of the main result.

THEOREM 4.7. *Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is an AF subalgebra of $(\mathcal{R},\tau)$. Let id and ρ be approximately unitarily equivalent $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{R}$. Then the $*$-homomorphisms id_0 and ρ_0 (as in Definition 4.6) are strongly-approximately-unitarily-equivalent over $\mathcal{A}$ (see Definition 2.4), i.e.*

$$id_0 \sim_{\mathcal{A}} \rho_0 \mod \mathcal{K}(\mathcal{R},\tau). \tag{4.6}$$

Proof. For the AF algebra $\mathcal{A}$, the existence of a norm-dense subset of positive finite rank operators in the unit ball of $\mathcal{A}_+\cap\mathcal{K}(\mathcal{R},\tau)$, is verified in Remark 4.3. Furthermore, Lemma 4.5 guarantees that id_0 and ρ_0 are $*$-homomorphisms of $\mathcal{A}$ into $P\mathcal{R}P$ and $Q\mathcal{R}Q$, respectively. Let $\{K_n\}_{n\geqslant 1}$ be a sequence of positive finite rank operators in the unit ball of $(\cup_{n\geqslant 1}\mathcal{A}_n)\cap\mathcal{F}(\mathcal{R},\tau)$, which is norm-dense in the unit ball of $\mathcal{A}_+\cap\mathcal{K}(\mathcal{R},\tau)$. Define projections P and Q as

$$P := \vee_{n\geqslant 1}R(K_n) \quad \text{and} \quad Q := \vee_{n\geqslant 1}R(\rho(K_n)). \tag{4.7}$$

Then, by applying Lemma 4.2 and Lemma 4.5, we have

$$P = P_{\mathcal{K}(\mathcal{A},\tau)} \quad \text{and} \quad Q = P_{\mathcal{K}(\rho(\mathcal{A}),\tau)}.$$

Let $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\cdots$ be a monotone increasing sequence of finite subsets of the unit ball of $\cup_{n\geqslant 1}\mathcal{A}_n$ such that $\cup_{n\geqslant 1}\mathcal{F}_n$ is norm dense in the unit ball of $\mathcal{A}$. Define $\mathcal{F}_0 := \mathcal{F}_1$. Note that the approximate equivalence of id and ρ also implies that they are both injective.

In the following, we construct at most countably many, mutually orthogonal, finite rank projections E_i's in $\cup_{n\geqslant 1}\mathcal{A}_n$ such that $\sum_{i\geqslant 1}E_i = P$.

Choose $n_1\geqslant 1$ such that $\mathcal{F}_1\cup\{K_1\}\subset\mathcal{A}_{n_1}$. Let P_1 be the central support of K_1 in the center $Z(\mathcal{A}_{n_1})$ of $\mathcal{A}_{n_1}$. Thus, $AP_1 = P_1A$ for each A in $\mathcal{F}_1$. Since $\mathcal{A}_{n_1}$ is finite-dimensional, we have that $P_1\in\mathcal{A}_{n_1}$. Moreover, we obtain that

$$\tau(R(K_1))<\infty \quad\Rightarrow\quad \tau(P_1)<\infty.$$

Note that $P_1\leqslant P$. If $P_1 = P$, then $P\in\mathcal{A}_+\cap\mathcal{K}(\mathcal{R},\tau)$ and we complete the construction. Otherwise, $P_1<P$. In this case, we let $J_1 = K_1$ and J_2 be the first element after J_1 in $\{K_n\}_{n\geqslant 1}$ such that $(P-P_1)J_2\neq 0$. Choose $n_2\geqslant n_1$ such that $\mathcal{F}_2\cup\{J_1,J_2\}\subset\mathcal{A}_{n_2}$. Note that P_1 is also in $\mathcal{A}_{n_2}$. Let P_2 be the central support of $P_1\vee R(J_2)$ in the center $Z(\mathcal{A}_{n_2})$ of $\mathcal{A}_{n_2}$. Thus, $AP_2 = P_2A$ for each A in $\mathcal{F}_2$. We define

$$E_1 := P_1 \quad \text{and} \quad E_2 := P_2 - P_1.$$

Since $P_2>P_1$, E_2 is also a projection. Furthermore, $AE_i = E_iA$ for each A in $\mathcal{F}_1$ and $i = 1,2$. Since $\mathcal{A}_{n_2}$ is finite dimensional, it follows that

$$\tau(R(J_2)\vee P_1)<\infty \quad\Rightarrow\quad \tau(P_2)<\infty.$$

If $P_2 = P$, then we complete the construction. Otherwise, $P_2<P$ and we continue the construction.

In general, assume that we obtain J_k and P_k for a certain $k \geqslant 1$. If $P_k = P$,then we complete the construction. Otherwise, $P_k < P$ and we let J_{k+1} be the first element after J_k in $\{K_n\}_{n\geqslant 1}$ such that $(P-P_k)J_{k+1} \neq 0$. Choose $n_{k+1} \geqslant n_k$ such that $\mathcal{F}_{k+1} \cup \{J_1,\ldots,J_{k+1}\} \subset \mathcal{A}_{n_{k+1}}$. Note that $P_1,\ldots,P_k$ are also in $\mathcal{A}_{n_{k+1}}$. Let P_{k+1} be the central support of $P_k \vee R(J_{k+1})$ in the center $Z(\mathcal{A}_{n_{k+1}})$ of $\mathcal{A}_{n_{k+1}}$. Then $AP_{k+1} = P_{k+1}A$ for each A in $\mathcal{F}_{k+1}$. Also, $\tau(P_{k+1}) < \infty$. Define

$$E_{k+1} := P_{k+1} - P_k.$$

It follows that

$$AE_i = E_iA \tag{4.8}$$

for each A in $\mathcal{F}_j$ and $i = j+1,\ldots,k+1$, where $j \geqslant 0$. If $P_{k+1} = P$, then we complete the construction. Otherwise, we continue to construct P_{k+2}.

By this inductive construction, we obtain at most countably many, mutually orthogonal, finite rank projections $\{E_i\}_{1\leqslant i\leqslant N} \subset (\cup_{n\geqslant 1}\mathcal{A}_n) \cap \mathcal{F}(\mathcal{R},\tau)$ such that $\sum_{1\leqslant i\leqslant N} E_i = P$, where $N \in \mathbb{N}\cup\{\infty\}$.

Note that $E_i\mathcal{A}_{n_i}E_i$ is also a finite-dimensional $*$-subalgebra, for $1 \leqslant i \leqslant N$. The approximate equivalence of id and ρ guarantees the equality

$$\tau(E) = \tau(\rho(E)) < \infty \tag{4.9}$$

for each projection E in $E_i\mathcal{A}_{n_i}E_i$. As a similar technique applied in Lemma 3.1, for $1 \leqslant i \leqslant N$, there exists a partial isometry W_i in $\mathcal{R}$ such that

$$E_i = W_i^*W_i, \quad \rho(E_i) = W_iW_i^* \quad \text{and} \quad \rho(A) = W_iAW_i^*, \quad \forall A \in E_i\mathcal{A}_{n_i}E_i. \tag{4.10}$$

Another quick application of (4.9) implies that $\sum_{1\leqslant i\leqslant N}\rho(E_i) = Q$.

For each vector x in the underlying Hilbert space $\mathcal{H}$, we have that

$$\|\sum_{1\leqslant i\leqslant N} W_ix\|^2 = \sum_{1\leqslant i\leqslant N}\|W_ix\|^2 = \sum_{1\leqslant i\leqslant N}(E_ix,x) = \|Px\|^2.$$

It follows that $W := \sum_{1\leqslant i\leqslant N} W_i$ is a well-defined partial isometry in $\mathcal{R}$ such that

$$W^*W = P \quad \text{and} \quad WW^* = Q. \tag{4.11}$$

Note that if j and k are two different integers strictly greater than i, then by applying (4.8), for each A in $\mathcal{F}_i$, we have

$$W_jAW_k^* = W_jE_jAE_kW_k^* = W_jE_jE_kAW_k^* = 0. \tag{4.12}$$

By (4.10), (4.11) and (4.12), we have:

1. for each A in $\mathcal{F}_0$,

$$\begin{aligned} Wid_0(A)W^* = WAPW^* = (\sum_{i\geqslant 1} W_i)A(\sum_{i\geqslant 1} W_i)^* &= \sum_{i\geqslant 1} W_iAW_i^* + \sum_{j\neq k} W_jAW_k^* \\ &= \sum_{i\geqslant 1} W_iAW_i^* = \sum_{i\geqslant 1}\rho(AE_i) = \rho(A)Q = \rho_0(A); \end{aligned}$$

2. in the case $N < \infty$, we have $W \in \mathcal{F}(\mathcal{R},\tau)$. Thus, for each $i \geqslant 0$ and each A in $\mathcal{F}_i$, we obtain that
$$WAW^* - \rho(A)Q \in \mathcal{F}(\mathcal{R},\tau);$$

3. in the case $N = \infty$, for each $i \geqslant 0$ and each A in $\mathcal{F}_i$,
$$\begin{aligned}
WAW^* - \rho(A)Q =& (\sum_{j=1}^{i} W_j)A(\sum_{j=1}^{i} W_j)^* + \sum_{j>i} W_jAW_j^* + \sum_{\substack{k,l>i\\ k\neq l}} W_kAW_l^* \\
&+ \sum_{k>i}\left(W_kA(\sum_{j=1}^{i} W_j^*)\right) + \sum_{l>i}\left((\sum_{j=1}^{i} W_j)AW_l^*\right) - \rho(AP_i) \\
&- \sum_{j>i}\rho(AE_j) \\
=& (\sum_{j=1}^{i} W_j)A(\sum_{j=1}^{i} W_j)^* + \sum_{j>i} W_jAW_j^* - \rho(AP_i) - \sum_{j>i}\rho(AE_j) \\
=& (\sum_{j=1}^{i} W_j)A(\sum_{j=1}^{i} W_j)^* - \rho(AP_i) \in \mathcal{F}(\mathcal{R},\tau).
\end{aligned}$$

For each fixed $j \geqslant 1$, write $\{\mathcal{E}_i = \mathcal{F}_{i+j-1}\}_{i\geqslant 1}$. We can iterate the preceding arguments to construct a partial isometry V_j in $\mathcal{R}$ with respect to $\{\mathcal{E}_i\}_{i\geqslant 1}$ such that:

1. for every A in $\cup_{i\geqslant 1}\mathcal{F}_i$ and $j \geqslant 1$,
$$V_j^* id_0(A)V_j - \rho_0(A) \in \mathcal{F}(\mathcal{R},\tau);$$

2. for each A in $\mathcal{E}_1 (= \mathcal{F}_j)$,
$$\|V_j^* id_0(A)V_j - \rho_0(A)\| = 0.$$

Note that $\cup_{i\geqslant 1}\mathcal{F}_i$ is norm-dense in the unit ball of $\mathcal{A}$. Thus, for each A in $\mathcal{A}$, we obtain that
$$\lim_{j\to\infty} \|V_j^* id_0(A)V_j - \rho_0(A)\| = 0 \quad \text{and} \quad V_j^* id_0(A)V_j - \rho_0(A) \in \mathcal{K}(\mathcal{R},\tau).$$

This completes the proof of (4.6). $\square$

REMARK 4.8. Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ. The following facts are useful:

1. Suppose that $\mathcal{A}$ is an AF subalgebra of $\mathcal{R}$. For each $*$-homomorphism ρ of $\mathcal{A}$ into $\mathcal{R}$, it follows that $\rho(\mathcal{A})$ is also AF;

2. Suppose that ϕ and ψ are approximately equivalent $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{R}$. Thus, we have $\ker\phi = \ker\psi$. Define a mapping ρ of $\phi(\mathcal{A})$ onto $\psi(\mathcal{A})$ by
$$\rho(T) := \psi\circ\phi^{-1}(T), \quad \forall T \in \phi(\mathcal{A}).$$
Since $\ker\phi = \ker\psi$, we obtain that ρ is well-defined and injective. Hence this follows that ρ is a $*$-isomorphism of $\phi(\mathcal{A})$ onto $\psi(\mathcal{A})$. Therefore, ϕ and ψ are approximately equivalent if and only if ρ and $id_{\phi(\mathcal{A})}$ are approximately equivalent.

As another useful tool, we cite Theorem 5.3.1 of [13] as follows. Note that, in the remainder, the symbol "$\sim_{\mathcal{A}}$" follows from Definition 2.4.

THEOREM 4.9. *Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ, and let $\mathcal{K}(\mathcal{R},\tau)$ be the set of compact operators in $(\mathcal{R},\tau)$. Suppose that $\mathcal{A}$ is a separable nuclear C^*-subalgebra of $\mathcal{R}$ with an identity $I_{\mathcal{A}}$.*

If $\rho : \mathcal{A} \to \mathcal{R}$ is a $*$-homomorphism satisfying $\rho(\mathcal{A}\cap\mathcal{K}(\mathcal{R},\tau)) = \mathbf{0}$, then
$$id_{\mathcal{A}} \sim_{\mathcal{A}} id_{\mathcal{A}} \oplus \rho \qquad \mathrm{mod} \ \ \mathcal{K}(\mathcal{R},\tau).$$

We are ready for our main theorem.

THEOREM 4.10. *Let $\mathcal{R}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ, and let $\mathcal{K}(\mathcal{R},\tau)$ be the set of compact operators in $(\mathcal{R},\tau)$. Suppose that $\mathcal{A}$ is an AF subalgebra of $\mathcal{R}$ with an identity $I_{\mathcal{A}}$.*

If ϕ and ψ are unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{R}$, then the following are equivalent:

(i) ϕ and ψ are approximately unitarily equivalent in $\mathcal{R}$;

(ii) ϕ and ψ are strongly-approximately-unitarily-equivalent over $\mathcal{A}$, i.e.
$$\phi \sim_{\mathcal{A}} \psi \quad \mathrm{mod} \ \ \mathcal{K}(\mathcal{R},\tau).$$

Proof. Note that the direction $(ii) \Rightarrow (i)$ is easy by Definition 2.4. Thus, we only need to prove that $(i) \Rightarrow (ii)$.

Remark 4.8 entails that both $\phi(\mathcal{A})$ and $\psi(\mathcal{A})$ are AF, and the $*$-isomorphism
$$\rho : \phi(\mathcal{A}) \to \psi(\mathcal{A}), \qquad \text{defined by} \qquad \rho(B) := \psi(\phi^{-1}(B)), \ \forall\, B \in \phi(\mathcal{A})$$
is well-defined. Moreover, the following are equivalent:

1. $\phi \sim_{\mathcal{A}} \psi \ \ \mathrm{mod} \ \ \mathcal{K}(\mathcal{R},\tau)$;

2. $id_{\phi(\mathcal{A})} \sim_{\phi(\mathcal{A})} \rho \ \ \mathrm{mod} \ \ \mathcal{K}(\mathcal{R},\tau)$.

Since $\phi(\mathcal{A})$ is AF, as in Remark 4.3, there exists a sequence $\{K_n\}_{n\geqslant 1}$ of positive finite rank operators in the unit ball of $\phi(\mathcal{A})_+\cap\mathcal{F}(\mathcal{R},\tau)$, norm-dense in the unit ball of $\phi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau)$. Similarly, as in (4.7), define P and Q as

$$P:=\vee_{n\geqslant 1}R(K_n) \quad \text{and} \quad Q:=\vee_{n\geqslant 1}R(\rho(K_n)). \tag{4.13}$$

By Lemma 4.5, P equals the union of the range projections of operators in $\phi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau)$, and $\phi(\mathcal{A})$ is reduced by P. Thus, the identity mapping id on $\phi(\mathcal{A})$ can be decomposed in the form

$$id=id_0\oplus id_e, \tag{4.14}$$

where id_0 is the restriction of $id(\cdot)P$ on ran$P$, and id_e is the restriction of $id(\cdot)P^{\perp}$ on ran$P^{\perp}$. We also write that

$$id_0(\phi(\mathcal{A}))=\phi_0(\mathcal{A}) \quad \text{and} \quad id_e(\phi(\mathcal{A}))=\phi_e(\mathcal{A}).$$

It is easy to verify that $id_e(\phi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau))=\mathbf{0}$.

On the other hand, we have that Q equals the union of the range projections of operators in $\psi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau)$, and $\psi(\mathcal{A})$ is reduced by Q. Thus, the $*$-isomorphism ρ of $\phi(\mathcal{A})$ can be decomposed in the form

$$\rho=\rho_0\oplus\rho_e, \tag{4.15}$$

where $\rho_0(A)=\rho(A)Q|_{\mathrm{ran}Q}$ and $\rho_e(A)=\rho(A)Q^{\perp}|_{\mathrm{ran}Q^{\perp}}$ for every A in $\phi(\mathcal{A})$. We also write that

$$\rho_0(\phi(\mathcal{A}))=\psi_0(\mathcal{A}) \quad \text{and} \quad \rho_e(\phi(\mathcal{A}))=\psi_e(\mathcal{A}).$$

Note that $\rho_e(\phi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau))=\mathbf{0}$. Furthermore, by applying Theorem 4.7, there exists a partial isometry W in $\mathcal{R}$ such that $P=W^*W$ and $Q=WW^*$.

It is worth pointing out that operators in $\phi_0(\mathcal{A})$ might not belong to $\phi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau)$ in general. Moreover, for a positive operator $\phi(A)$ in $\phi(\mathcal{A})\cap\mathcal{K}(\mathcal{R},\tau)$, the weight $\tau(R(\phi(A)))$ might be arbitrarily small. Then, we can not apply the classical tools developed in $\mathcal{B}(\mathcal{H})$ directly. This is the motivation to develop Theorem 4.7.

By (4.2), there exists a monotone increasing sequence $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\cdots$ of finite subsets of the unit ball of $\cup_{k\geqslant 1}\mathcal{A}_k$ such that $\cup_{k\geqslant 1}\mathcal{F}_k$ is norm-dense in the unit ball of $\mathcal{A}$. Likewise, the union $\cup_{k\geqslant 1}\phi(\mathcal{F}_k)$ (resp. $\cup_{k\geqslant 1}\psi(\mathcal{F}_k)$) is norm-dense in the unit ball of $\phi(\mathcal{A})$ (resp. $\psi(\mathcal{A})$). By applying Theorem 4.7, there exists a partial isometry V_k in $(\mathcal{R},\tau)$ such that the inequality

$$\|V_k^*\phi_0(A)V_k-\psi_0(A)\|<\frac{1}{2^k}$$

holds for every A in $\mathcal{F}_k$.

Furthermore, for every A in $\mathcal{A}$, we have that $V_k^*\phi_0(A)V_k-\psi_0(A)$ belongs to the ideal $\mathcal{K}(\mathcal{R},\tau)$. Therefore, there exists a sequence $\{V_k\}_{k\geqslant 1}$ of partial isometries in $\mathcal{R}$ such that:

1. $\lim_{k\to\infty}\|V_k^*\phi_0(A)V_k-\psi_0(A)\|=0$, for every A in $\mathcal{A}$;

2. $V_k^* \phi_0(A) V_k - \psi_0(A)$ belongs to $\mathcal{K}(\mathcal{R}, \tau)$ for every A in $\mathcal{A}$ and $k \geqslant 1$.

Notice that
$$id_e(\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{R}, \tau)) = \rho_e(\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{R}, \tau)) = \mathbf{0}.$$

Thus, by applying Theorem 4.9, Theorem 4.7, and the decompositions in (4.14) and (4.15), it follows that

$$\begin{aligned}
\phi = (id_0 \circ \phi) \oplus (id_e \circ \phi) &\sim_{\mathcal{A}} (id_0 \circ \phi) \oplus (id_e \circ \phi) \oplus (\rho_e \circ \phi) && \mod \mathcal{K}(\mathcal{R}, \tau) \\
&= \phi_0 \oplus \phi_e \oplus \psi_e \\
&\sim_{\mathcal{A}} \psi_0 \oplus \psi_e \oplus \phi_e && \mod \mathcal{K}(\mathcal{R}, \tau) \\
&= (\rho_0 \circ \phi) \oplus (\rho_e \circ \phi) \oplus (id_e \circ \phi) \\
&= (\rho \circ \phi) \oplus (id_e \circ \phi) \sim_{\mathcal{A}} (\rho \circ \phi) = \psi && \mod \mathcal{K}(\mathcal{R}, \tau)
\end{aligned}$$

This completes the proof. □

Acknowledgement. The corresponding author Rui Shi is grateful to Don Hadwin and Junhao Shen for many helpful discussions while visiting University of New Hampshire in 2016. The joint papers [8, 13, 14] were written based on those discussions and provided us with important methods in the current paper and the papers in the sequel.

REFERENCES

[1] WILLIAM ARVESON, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), no. 2, 329–355.
[2] DAVID BERG, *An extension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc. **160** (1971), 365–371.
[3] ALIN CIUPERCA, THIERRY GIORDANO, PING WONG NG AND ZHUANG NIU, *Amenability and uniqueness*, Adv. Math. **240** (2013), 325–345.
[4] KENNETH DAVIDSON, *C^*-algebras by example*, Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.
[5] KENNETH DAVIDSON, *Normal operators are diagonal plus Hilbert-Schmidt*, J. Operator Theory **20** (1988), no. 2, 241–249.
[6] HUIRU DING AND DON HADWIN, *Approximate equivalence in von Neumann algebras*, Sci. China Ser. A **48** (2005), no. 2, 239–247.
[7] DONALD HADWIN, *Nonseparable approximate equivalence*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 203–231.
[8] DONALD HADWIN AND RUI SHI, *A note on representations of commutative C^*-algebras in semifinite von Neumann algebras*, Oper. and Matrices , **12** (2018), no. 4, 1129–1144.
[9] PAUL HALMOS, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
[10] RICHARD KADISON AND JOHN RINGROSE, *Fundamentals of the theory of operator algebras. Vol. I. Elementary theory*, Reprint of the 1983 original, Graduate Studies in Mathematics, 15. American Mathematical Society, Providence, RI, 1997.
[11] RICHARD KADISON AND JOHN RINGROSE, *Fundamentals of the theory of operator algebras. Vol. II. Advanced theory*, Corrected reprint of the 1986 original, Graduate Studies in Mathematics, 16. American Mathematical Society, Providence, RI, 1997.
[12] VICTOR KAFTAL, *On the theory of compact operators in von Neumann algebras. II*, Pacific J. Math. **79** (1978), no. 1, 129–137.
[13] QIHUI LI, JUNHAO SHEN, RUI SHI, *A generalization of the Voiculescu theorem for normal operators in semifinite von Neumann algebras*, arXiv:1706.09522 [math.OA].
[14] QIHUI LI, JUNHAO SHEN, RUI SHI, LIGUANG WANG, *Perturbations of self-adjoint operators in semifinite von Neumann algebras: Kato-Rosenblum theorem*, J. Funct. Anal. **275** (2018), no. 2, 259–287.

[15] JOHN VON NEUMANN, *Charakterisierung des Spektrums eines Integraloperators*, Actualits Sci. Indust. **229**, Hermann, Paris, 1935.
[16] MIKAEL RØRDAM, FLEMMING LARSEN AND NIELS JAKOB LAUSTSEN, *An introduction to K-theory for* C^* *-algebras*, London Mathematical Society Student Texts, 49. Cambridge University Press, Cambridge, 2000.
[17] MASAMICHI TAKESAKI, *Theory of operator algebras. I.* Reprint of the first (1979) edition, Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, 2002.
[18] DAN VOICULESCU, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113.
[19] DAN VOICULESCU, *Some results on norm-ideal perturbations of Hilbert space operators*, J. Operator Theory **2** (1979), no. 1, 3–37.
[20] HERMANN WEYL, *Über beschränkte quadratische formen, deren differenz vollstetig ist*, Rend. Circ. Mat. Palermo **27** (1) (1909), 373–392.
[21] LÁSZLÓ ZSIDÓ, *The Weyl-von Neumann theorem in semifinite factors*, J. Funct. Anal. **18** (1975), 60–72.

(Received November 3, 2018)

Junsheng Fang
School of Mathematical Sciences
Dalian University of Technology
Dalian, 116024, China
Department of Mathematics
Hebei Normal University
Shijiazhuang, 050024, China
e-mail: junshengfang@hotmail.com

Rui Shi
School of Mathematical Sciences
Dalian University of Technology
Dalian, 116024, China
e-mail: ruishi@dlut.edu.cn, ruishi.math@gmail.com

Shilin Wen
School of Mathematical Sciences
Dalian University of Technology
Dalian, 116024, China
e-mail: shilinwen127@gmail.com