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ON THE STABILITY OF LEFT δ–CENTRALIZERS ON BANACH LIE TRIPLE SYSTEMS

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Abstract. In this paper under a condition, we prove that every Jordan left δ-centralizer on a Lie triple system is a left δ-centralizer. Moreover, we use a fixed point method to prove the generalized Hyers-Ulam-Rassias stability associated with the Pexiderized Cauchy-Jensen type functional equation

$$rf\left(\frac{x+y}{r}\right)+sg\left(\frac{x-y}{s}\right)=2h(x),$$

for $r,s\in\mathbb{R}\setminus\{0\}$ in Banach Lie triple systems.

1. Introduction

The notion of Lie triple system was first introduced by N. Jacobson ([2, 13]). We recall that a Lie triple system is a vector space $\mathscr{A}$ over a field $\mathbb{F}$, equipped with a trilinear mapping $(a,b,c)\mapsto[a,b,c]$ of $\mathscr{A}\times\mathscr{A}\times\mathscr{A}$ into $\mathscr{A}$ satisfying the following axioms

(i) $[a,b,c]=-[b,a,c]$,
(ii) $[a,b,c]+[b,c,a]+[c,a,b]=0$,
(iii) $[x,y,[a,b,c]]=[[x,y,a],b,c]+[a,[x,y,b],c]+[a,b,[x,y,c]]$,

for all $x,y,a,b,c\in\mathscr{A}$.

A normed (Banach) Lie triple system is a normed (Banach) space $(\mathscr{A},\|.\|)$ with a trilinear mapping $(a,b,c)\mapsto[a,b,c]$ of $\mathscr{A}\times\mathscr{A}\times\mathscr{A}$ into $\mathscr{A}$ satisfying $(i),(ii),(iii)$ and $\|[a,b,c]\|\leqslant\|a\|\|b\|\|c\|$ for all $a,b,c\in\mathscr{A}$. It is clear that every Lie algebra with product $[.,.]$ is a Lie triple system with respect to $[x,y,z]:=[[x,y],z]$. Conversely, any Lie triple system $\mathscr{A}$ can be considered as a subspace of a Lie algebra ([14, 15, 17]).

DEFINITION 1. Let $\mathbb{C}$ be the complex filed and $\mathscr{A}$ be a Lie triple system over $\mathbb{C}$. Let $\delta:\mathscr{A}\to\mathscr{A}$ be a $\mathbb{C}$-linear mapping. We say that a $\mathbb{C}$-linear mapping $T:\mathscr{A}\to\mathscr{A}$ is called a left δ-centralizer on $\mathscr{A}$ if

$$T([a,b,c])=[T(a),\delta(b),\delta(c)],$$

for all $a,b,c\in\mathscr{A}$. If $\delta=I_{\mathscr{A}}$, a left δ-centralizer is called a left centralizer.

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DEFINITION 2. Let $\mathbb{C}$ be the complex filed and $\mathscr{A}$ be a Lie triple system over $\mathbb{C}$. Let $\delta : \mathscr{A} \to \mathscr{A}$ be a $\mathbb{C}$-linear mapping. We say that a $\mathbb{C}$-linear mapping $T : \mathscr{A} \to \mathscr{A}$ is called a Jordan left δ-centralizer on $\mathscr{A}$ if

$$T([a,b,a]) = [T(a), \delta(b), \delta(a)],$$

for all $a, b \in \mathscr{A}$. If $\delta = I_{\mathscr{A}}$, a Jordan left δ-centralizer is called a Jordan left centralizer.

The stability problem of functional equations originated from a question of Ulam [27] in 1940, concerning the stability of group homomorphisms. Let $(\mathscr{G}_1, .)$ be a group and let $(\mathscr{G}_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, dose there exist a $\delta > 0$, such that if a mapping $h : \mathscr{G}_1 \longrightarrow \mathscr{G}_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in \mathscr{G}_1$, then there exists a homomorphism $H : \mathscr{G}_1 \longrightarrow \mathscr{G}_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in \mathscr{G}_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [11] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : \mathscr{E} \to \mathscr{E}'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta,$$

for all $x, y \in \mathscr{E}$, and some $\delta > 0$. Then there exists a unique additive mapping $g : \mathscr{E} \to \mathscr{E}'$ such that

$$\|f(x) - g(x)\| \leqslant \delta,$$

for all $x \in \mathscr{E}$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in \mathscr{E}$, then g is linear. In 1950, T. Aoki [1] was the second author to treat this problem for additive mappings. Finally in 1978, Th. M. Rassias [23] proved the following theorem:

Theorem (Th. M. Rassias). Let $f : \mathscr{E} \to \mathscr{E}'$ be a mapping from a norm vector space $\mathscr{E}$ into a Banach space $\mathscr{E}'$ subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in \mathscr{E}$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $g : \mathscr{E} \to \mathscr{E}'$ such that

$$\|f(x) - g(x)\| \leqslant \frac{2\varepsilon}{2 - 2^p}\|x\|^p,$$

for all $x \in \mathscr{E}$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into $\mathscr{E}'$ is continuous for each fixed $x \in \mathscr{E}$, then g is linear.

This stability phenomenon of this kind is called the Hyers-Ulam-Rassias stability. In 1991, Z. Gajda [10] answered the question for the case $p > 1$, which was rased by Rassias. In 1994, a generalization of the Rassias' theorem was obtained by Găvruta as follows [9]. We refer the readers to [3]–[8], [12], [18], [20]–[22],[24]–[26] and references therein for more detailed results on the stability problems of various functional equations and mappings and their Pexider types.

In this paper under a condition, we prove that every Jordan left δ-centralizer on a Lie triple system is a left δ-centralizer. Moreover, some results concerning the stability of the left δ-centralizers on Banach Lie triple systems are presented.

2. Left δ-centralizers

Throughout this section, let $\mathbb{C}$ be the complex filed and $\mathscr{A}$ be a Lie triple system over $\mathbb{C}$. It is clear that every left δ-centralizer on a Lie triple system $\mathscr{A}$ is a Jordan left δ-centralizer. In this section, under a condition, we prove that every Jordan left δ-centralizer on a Lie triple system $\mathscr{A}$ is a left δ-centralizer. So we conclude that every Jordan left centralizer on $\mathscr{A}$ is a left centralizer.

THEOREM 1. *Let $T:\mathscr{A}\to\mathscr{A}$ be a Jordan left δ-centralizer such that*

$$[T(a),\delta(b),\delta(c)] = [T(a),\delta(c),\delta(b)] + [T(c),\delta(b),\delta(a)], \tag{1}$$

for all $a,b,c\in\mathscr{A}$. Then T is a left δ-centralizer.

Proof. Since $T:\mathscr{A}\to\mathscr{A}$ is a Jordan left δ-centralizer,

$$T([a+c,b,a+c]) = T([a,b,a]) + T([c,b,c]) + [T(a),\delta(b),\delta(c)] + [T(c),\delta(b),\delta(a)], \tag{2}$$

for all $a,b,c\in\mathscr{A}$. On the other hand, we have

$$[a+c,b,a+c] = [a,b,a] + [c,b,c] + [a,b,c] + [c,b,a],$$

for all $a,b,c\in\mathscr{A}$. Hence

$$T([a+c,b,a+c]) = T([a,b,a]) + T([c,b,c]) + T([a,b,c]) + T([c,b,a]), \tag{3}$$

for all $a,b,c\in\mathscr{A}$. It follows from (2) and (3) that

$$T([a,b,c]) + T([c,b,a]) = [T(a),\delta(b),\delta(c)] + [T(c),\delta(b),\delta(a)], \tag{4}$$

for all $a,b,c\in\mathscr{A}$. Since $[a,b,c]+[b,c,a]+[c,a,b]=0,$ we get that

$$T([a,b,c]) + T([c,b,a]) = 2T([a,b,c]) - T([a,c,b]), \tag{5}$$

for all $a,b,c\in\mathscr{A}$. By (1), we have

$$[T(a),\delta(b),\delta(c)] + [T(c),\delta(b),\delta(a)] = 2[T(a),\delta(b),\delta(c)] - [T(a),\delta(c),\delta(b)], \tag{6}$$

for all $a,b,c\in\mathscr{A}$. By utilizing the equations (4), (5) and (6), we arrive at

$$2T([a,b,c]) - T([a,c,b]) = 2[T(a),\delta(b),\delta(c)] - [T(a),\delta(c),\delta(b)], \tag{7}$$

for all $a,b,c\in\mathscr{A}$. Setting $b=c$ in (7), we get

$$2T([a,b,b]) - T([a,b,b]) = 2[T(a),\delta(b),\delta(b)] - [T(a),\delta(b),\delta(b)],$$

for all $a,b,c\in\mathscr{A}$. Therefore, $T([a,b,b] = [T(a),\delta(b),\delta(b)]$ for all $a,b\in\mathscr{A}$. Since $T([a,b+c,b+c] = [T(a),\delta(b+c),\delta(b+c)]$ and $[.,.,.]$ is trilinear, we have

$$T([a,b,c]) + T([a,c,b]) = [T(a),\delta(b),\delta(c)] + [T(a),\delta(c),\delta(b)], \tag{8}$$

for all $a,b,c\in\mathscr{A}$. If we add (7) to (8), we have

$$T([a,b,c]) = [T(a),\delta(b),\delta(c)],$$

for all $a,b,c\in\mathscr{A}$. So the proof is completed. $\square$

COROLLARY 1. *Every Jordan left centralizer T on a Lie triple system $\mathscr{A}$ is a left centralizer if*

$$[T(a),b,c] = [T(a),c,b] + [T(c),b,a],$$

for all $a,b,c \in \mathscr{A}$.

3. Stability of left δ-centralizers

Throughout this section, suppose that $\mathscr{A}$ is a Banach Lie triple system. In this section, using the fixed point method, we prove the stability of left δ-centralizers associated to the Pexiderized Cauchy- Jensen type functional equation

$$rf(\frac{a+b}{r}) + sg(\frac{a-b}{s}) = 2h(a),$$

for $r,s \in \mathbb{R} \setminus \{0\}$ on Banach Lie triple systems.

For convenience, we use the following abbreviation for given mappings $f,g,h : \mathscr{A} \to \mathscr{A}$,

$$D_\mu(f,g,h)(a,b) := rf(\frac{\mu a + \mu b}{r}) + sg(\frac{\mu a - \mu b}{s}) - 2\mu h(a),$$

for all $a,b \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$.

We recall the alternative of the fixed point theorem by Margolis and Diaz.

THEOREM 2. [16] *Let (X,d) be a complete generalized metric space and let $J : \mathscr{X} \longrightarrow \mathscr{X}$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in \mathscr{X}$, either $d(J^n x, J^{n+1} x) = \infty$ for all $n \geqslant 0$ or there exists a natural number n_0 such that,*

1. *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geqslant n_0$,*
2. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J,*
3. *y^* is the unique fixed point of J in the set $\mathscr{Y} = \{y \in \mathscr{X} : d(J^{n_0} x, y) < \infty\}$,*
4. *$d(y,y^*) \leqslant \frac{1}{1-L} d(y,Jy)$ for all $y \in \mathscr{Y}$.*

LEMMA 1. [19] *Let $\mathscr{X}$ and $\mathscr{Y}$ be linear spaces and let $f : \mathscr{X} \longrightarrow \mathscr{Y}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in \mathscr{X}$ and $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

We aim to investigate the generalized Hyers-Ulam-Rassias stability of left δ-centralizers.

THEOREM 3. *Let $f,g,h,k:\mathscr{A}\longrightarrow\mathscr{A}$ be mappings with $f(0)=g(0)=h(0)=k(0)=0$ for which there exist functions $\varphi:\mathscr{A}^2\longrightarrow[0,\infty)$ and $\psi:\mathscr{A}^5\longrightarrow[0,\infty)$ such that*

$$\lim_{n\to\infty}2^n\varphi(\frac{a}{2^n},\frac{b}{2^n})=\lim_{n\to\infty}\frac{1}{2^n}\psi(2^na,2^nb,2^nc,2^nu,2^nv)=0, \tag{9}$$

$$||D_\mu(f,g,h)(a,b)||\leqslant\varphi(a,b), \tag{10}$$

$$||f([a,b,c]-[f(a),k(b),k(c)])-k(\mu u+v)-\mu k(u)-k(v)||\leqslant\psi(u,v,a,b,c), \tag{11}$$

for all $u,v,a,b,c\in\mathscr{A}$ and $\mu\in\mathbb{T}^1$. If there exist constants $0<L_1,L_2<1$ such that the functions

$$\psi_1(a):=\varphi(a,0)+\varphi(\frac{a}{2},\frac{a}{2})+\varphi(\frac{a}{2},-\frac{a}{2}),$$
$$\psi_2(a):=\psi(\frac{a}{2},\frac{a}{2},0,0,0),$$

have the property $\psi_1(a)\leqslant\frac{L_1}{2}\psi_1(2a)$ and $\psi_2(a)\leqslant 2L_2\psi_2(\frac{a}{2})$ for all $a\in\mathscr{A}$, then there exists a unique left δ- centralizer $T:\mathscr{A}\to\mathscr{A}$ such that

$$T(a)=\lim_{n\to\infty}2^nf(\frac{a}{2^n})=\lim_{n\to\infty}2^ng(\frac{a}{2^n})=\lim_{n\to\infty}2^nh(\frac{a}{2^n}),$$
$$\delta(a)=\lim_{n\to\infty}2^nk(\frac{a}{2^n}),$$
$$||h(a)-T(a)||\leqslant\frac{1}{2-2L_1}\psi_1(a),$$
$$||f(a)-T(a)||\leqslant\frac{1}{r}\varphi(\frac{ra}{2},\frac{ra}{2})+\frac{1}{r-rL_1}\psi_1(\frac{ra}{2}),$$
$$||g(a)-T(a)||\leqslant\frac{1}{s}\varphi(\frac{sa}{2},-\frac{sa}{2})+\frac{1}{s-sL_1}\psi_1(\frac{sa}{2}),$$
$$||k(a)-\delta(a)||\leqslant\frac{L_2}{2-2L_2}\psi_2(a),$$

for all $a\in\mathscr{A}$.

Proof. Setting $\mu=1$ and $b=0$ in (10), we get

$$||rf(\frac{a}{r})+sg(\frac{a}{s})-2h(a)||\leqslant\varphi(a,0). \tag{12}$$

Setting $\mu=1$ and $b=a,-a$ and then replacing a by $\frac{a}{2}$ in (10), we get the following inequalities,

$$||rf(\frac{a}{r})-2h(\frac{a}{2})||\leqslant\varphi(\frac{a}{2},\frac{a}{2}), \tag{13}$$

$$||sg(\frac{a}{s})-2h(\frac{a}{2})||\leqslant\varphi(\frac{a}{2},-\frac{a}{2}), \tag{14}$$

for all $a \in \mathscr{A}$. Thus, it follows from (12), (13) and (14) that

$$||h(a) - 2h(\frac{a}{2})|| \leqslant \frac{1}{2}\psi_1(a), \tag{15}$$

for all $a \in \mathscr{A}$. Let $\mathscr{S} := \{g : \mathscr{A} \longrightarrow \mathscr{A} : g(0) = 0\}$. We introduce a generalized metric on $\mathscr{S}$ as follows

$$d_1(g,h) := \inf\{t \in (0,\infty) : ||g(a) - h(a)|| \leqslant t\psi_1(a), \forall a \in \mathscr{A}\}.$$

It is easy to show that $(\mathscr{S}, d_1)$ is a generalized complete metric space and the mapping $J_1 : \mathscr{S} \longrightarrow \mathscr{S}$ given by $(J_1 g)(a) := 2g(\frac{a}{2})$ is a strictly contractive mapping with the Lipschitz constant L_1. It follows from (15) that $d_1(J_1 h, h) \leqslant \frac{1}{2}$. By Theorem 2 the sequence $\{J_1^n h\}$ converges to a fixed point T of J_1, i.e.,

$$T : \mathscr{A} \longrightarrow \mathscr{A},\ T(a) = \lim_{n\to\infty}(J_1^n h)(a) = \lim_{n\to\infty} 2^n h(\frac{a}{2^n}),$$

and $T(a) = 2T(\frac{a}{2})$ for all $a \in \mathscr{A}$. Also, T is the unique fixed point of J_1 in the set $U = \{g \in \mathscr{S} : d_1(g,h) < \infty\}$ and

$$d_1(h,T) \leqslant \frac{1}{1-L_1} d_1(h, J_1 h) \leqslant \frac{1}{2-2L_1},$$

i.e., the inequality

$$||h(a) - T(a)|| \leqslant \frac{1}{2-2L_1}\psi_1(a), \tag{16}$$

holds for all $a \in \mathscr{A}$. It follows from the definition of T, (13), (14) and (15) that

$$\lim_{n\to\infty} 2^n r f(\frac{a}{2^n r}) = \lim_{n\to\infty} 2^n s g(\frac{a}{2^n s}) = T(a), \tag{17}$$

for all $a \in \mathscr{A}$. Hence, we get from (9) and (10) that

$$T(\mu a + \mu b) + T(\mu a - \mu b) = 2\mu T(a), \tag{18}$$

for all $a, b \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1$. Letting $\mu = 1$, $a + b = x$ and $a - b = y$, we get

$$T(x) + T(y) = 2T(\frac{x+y}{2}) = T(x+y),$$

i.e., T is additive. We have, $T(\mu a) = \mu T(a)$ by setting $b = 0$ in (18). By Lemma 1, we conclude that T is $\mathbb{C}$-linear. Since T is $\mathbb{C}$-linear, it follows from (17) that

$$\lim_{n\to\infty} 2^n f(\frac{a}{2^n}) = \lim_{n\to\infty} 2^n g(\frac{a}{2^n}) = T(a),$$

for all $a \in \mathscr{A}$. Thus, (13) and (16) imply

$$\begin{aligned} ||f(a) - T(a)|| &\leqslant ||f(a) - \frac{2}{r}h(\frac{ra}{2})|| + \frac{2}{r}||h(\frac{ra}{2}) - T(\frac{ra}{2})|| \\ &\leqslant \frac{1}{r}\varphi(\frac{ra}{2}, \frac{ra}{2}) + \frac{1}{r - rL_1}\psi_1(\frac{ra}{2}), \end{aligned}$$

for all $a \in \mathscr{A}$. In a similar way, we obtain the following inequality

$$||g(a) - T(a)|| \leqslant \frac{1}{s}\varphi(\frac{sa}{2}, -\frac{sa}{2}) + \frac{1}{s - sL_1}\psi_1(\frac{sa}{2}),$$

for all $a \in \mathscr{A}$. Setting $\mu = 1$, $u = v$ and $a = b = c = 0$ in (11), we obtain

$$||k(2u) - 2k(u)|| \leqslant \psi(u, u, 0, 0, 0). \tag{19}$$

It follows from (19) that

$$||\frac{1}{2}k(2u) - k(u)|| \leqslant \frac{1}{2}\psi_2(2u) \leqslant L_2\psi_2(u). \tag{20}$$

for all $u \in \mathscr{A}$. We introduce another generalized metric on $\mathscr{S}$ as follows

$$d_2(g,h) := \inf\{t \in (0,\infty) : ||g(a) - h(a)|| \leqslant t\psi_2(a), \forall a \in \mathscr{A}\}.$$

It is easy to show that $(\mathscr{S}, d_2)$ is a generalized complete metric space and the mapping $J_2 : \mathscr{S} \longrightarrow \mathscr{S}$ given by $(J_2g)(a) := \frac{1}{2}g(2a)$ is a strictly contractive mapping with the Lipschitz constant L_2. It follows from (20) that $d_2(J_2k, k) \leqslant L_2$. From Theorem 2 it follows that the sequence $\{J_2^n k\}$ converges to a fixed point δ of J_2, i.e.,

$$\delta : \mathscr{A} \longrightarrow \mathscr{A},\ \delta(a) = \lim_{n\to\infty}(J_2^n k)(a) = \lim_{n\to\infty}\frac{1}{2^n}k(2^n a),$$

and $\delta(2a) = 2\delta(a)$ for all $a \in \mathscr{A}$. Also, δ is the unique fixed point of J_2 in the set $V = \{g \in \mathscr{S} : d_2(g,k) < \infty\}$ and

$$d_2(k,d) \leqslant \frac{1}{1-L_2}d_2(k, J_2k) \leqslant \frac{L_2}{2-2L_2}$$

and so,

$$||k(a) - \delta(a)|| \leqslant \frac{L_2}{2-2L_2}\psi_2(a), \tag{21}$$

holds for all $a \in \mathscr{A}$.

We show that $\delta : \mathscr{A} \longrightarrow \mathscr{A}$ is $\mathbb{C}$-linear. It follows from (11) by setting $a = b = c = 0$ that

$$||k(\mu u + v) - \mu k(u) - k(v)|| \leqslant \phi_2(u, v, 0, 0, 0). \tag{22}$$

Replace u, v in (22) by $2^n u, 2^n v$, respectively, and divide both sides by 2^n. Passing the limit as $n \longrightarrow \infty$ and applying (9) we obtain

$$\delta(\mu u + v) = \mu\delta(u) + \delta(v), \tag{23}$$

for all $\mu \in \mathbb{T}^1$ and all $u, v \in \mathscr{A}$. Letting $\mu = 1$ in (23) we conclude that δ is additive and setting $v = 0$ we have, $\delta(\mu u) = \mu\delta(u)$. Thus, Lemma1 implies δ is $\mathbb{C}$-linear. Setting $u = v = 0$, replacing a, b, c by $2^n a, 2^n b, 2^n c$ in (11), dividing both sides by 2^{3n}, taking the limit as $n \longrightarrow \infty$ and applying (9), we obtain

$$T[a,b,c] = [T(a), \delta(b), \delta(c)],$$

for all $a, b, c \in \mathscr{A}$. Hence, T is a left δ-centralizer on $\mathscr{A}$. $\square$

COROLLARY 2. *Let $p > 1$, $0 < q < 1$, $\beta,\gamma > 0$, and $f,g,h,k : \mathscr{A} \longrightarrow \mathscr{A}$ with $f(0) = g(0) = h(0) = k(0) = 0$ be mappings such that*

$$||D_\mu(f,g,h)(a,b)|| \leqslant \beta(||a||^p + ||b||^p) + \gamma||a||^{\frac{p}{2}}||b||^{\frac{p}{2}}, \tag{24}$$

$$\begin{aligned}&||f([a,b,c] - [f(a),k(b),k(c)]) - k(\mu u + v) - \mu k(u) - k(v)||\\ \leqslant&\beta(||a||^q + ||b||^q + ||c||^q + ||u||^q + ||v||^q) + \gamma||a||^{\frac{q}{2}}||b||^{\frac{q}{2}}||c||^{\frac{q}{2}}||u||^{\frac{q}{2}}||v||^{\frac{q}{2}},\end{aligned} \tag{25}$$

for all $u,v,a,b,c \in \mathscr{A}$ and $\mu \in \mathbb{T}^1$. Then there exists a unique left δ- centralizer $T : \mathscr{A} \longrightarrow \mathscr{A}$ such that

$$\begin{aligned}||h(a) - T(a)|| &\leqslant \frac{(2 + 2^{p-1})\beta + \gamma}{2^p - 2}||a||^p,\\ ||f(a) - T(a)|| &\leqslant \frac{3\beta + \gamma}{2^p - 2}r^{p-1}||a||^p,\\ ||g(a) - T(a)|| &\leqslant \frac{3\beta + \gamma}{2^p - 2}s^{p-1}||a||^p,\\ ||k(a) - \delta(a)|| &\leqslant \frac{2\beta}{2 - 2^q}\|a\|^q,\end{aligned}$$

for all $a \in \mathscr{A}$.

Proof. The proof follows from Theorem 3 by taking

$$\begin{aligned}\varphi(a,b) &:= \beta(||a||^p + ||b||^p) + \gamma||a||^{\frac{p}{2}}||b||^{\frac{p}{2}},\\ \psi(u,v,a,b,c) &:= \beta(||u||^q + ||v||^q + ||a||^q + ||b||^q + ||c||^q) + \gamma||u||^{\frac{q}{5}}||v||^{\frac{q}{5}}||a||^{\frac{q}{5}}||b||^{\frac{q}{5}}||c||^{\frac{q}{5}}.\end{aligned}$$

We have, $\psi_1(a) = \frac{(2+2^{p-1})\beta+\gamma}{2^{p-1}}||a||^p$ and $\psi_2(a) = \frac{\beta}{2^{q-1}}||a||^q$. We can obtain the desired results by choosing $L_1 = 2^{1-p}$ and $L_2 = 2^{q-1}$. $\square$

The next result is a dual to the Theorem 3 in some sense.

THEOREM 4. *Let $f,g,h,k : \mathscr{A} \longrightarrow \mathscr{A}$ be mappings with $f(0) = g(0) = h(0) = k(0) = 0$ for which there exist functions $\varphi : \mathscr{A}^2 \longrightarrow [0,\infty)$ and $\psi : \mathscr{A}^5 \longrightarrow [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n a, 2^n b) = \lim_{n\to\infty} 2^n \psi(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}, \frac{u}{2^n}, \frac{v}{2^n}) = 0, \tag{26}$$

$$||D_\mu(f,g,h)(a,b)|| \leqslant \varphi(a,b), \tag{27}$$

$$||f([a,b,c] - [f(a),k(b),k(c)]) - k(\mu u + v) - \mu k(u) - k(v)|| \leqslant \psi(u,v,a,b,c), \tag{28}$$

for all $u,v,a,b,c \in \mathscr{A}$ and $\mu \in \mathbb{T}^1$. If there exist constants $0 < L_1, L_2 < 1$ such that the functions

$$\begin{aligned}\psi_1(a) &:= \varphi(a,0) + \varphi(\frac{a}{2},\frac{a}{2}) + \varphi(\frac{a}{2},-\frac{a}{2}),\\ \psi_2(a) &:= \psi(\frac{a}{2},\frac{a}{2},0,0,0),\end{aligned}$$

have the property $\psi_1(a) \leqslant 2L_1\psi_1(\frac{a}{2})$ *and* $\psi_2(a) \leqslant \frac{L_2}{2}\psi_2(2a)$ *for all* $a \in \mathscr{A}$, *then there exists a unique left* δ*- centralizer* $T : \mathscr{A} \to \mathscr{A}$ *such that*

$$||h(a)-T(a)|| \leqslant \frac{L_1}{2-2L_1}\psi_1(a),$$
$$||f(a)-T(a)|| \leqslant \frac{1}{r}\varphi(\frac{ra}{2},\frac{ra}{2})+\frac{L_1}{r-rL_1}\psi_1(\frac{ra}{2}),$$
$$||g(a)-T(a)|| \leqslant \frac{1}{s}\varphi(\frac{sa}{2},-\frac{sa}{2})+\frac{L_1}{s-sL_1}\psi_1(\frac{sa}{2}),$$
$$||k(a)-\delta(a)|| \leqslant \frac{1}{1-L_2}\psi_2(a),$$

for all $a \in \mathscr{A}$.

Proof. Using the same method as in the proof of Theorem 3 and replacing a by $2a$ in (15) and dividing by 2 we have, $||\frac{1}{2}h(2a)-h(a)|| \leqslant \frac{1}{4}\psi_1(2a)$ for all $a \in \mathscr{A}$. Thus,

$$||\frac{1}{2}h(2a)-h(a)|| \leqslant \frac{L_1}{2}\psi_1(a),$$

for all $a \in \mathscr{A}$. Also, replacing u by $\frac{u}{2}$ in (19), we get

$$||k(u)-2k(\frac{u}{2})|| \leqslant \psi_2(u),$$

for all $u \in \mathscr{A}$. We introduce the same definitions for $\mathscr{S}$, d_1, and d_2 as in the proof of Theorem 3 such that $(\mathscr{S},d_1)$ and $(\mathscr{S},d_2)$ become a generalized complete metric space. Define $J_1 : \mathscr{S} \longrightarrow \mathscr{S}$ by $(J_1g)(a) = \frac{1}{2}g(2a)$ and $J_2 : \mathscr{S} \longrightarrow \mathscr{S}$ by $(J_2g)(a) = 2g(\frac{a}{2})$. Hence, $d_1(h,J_1h) < \frac{L_1}{2}$ and $d_2(k,J_1k) < 1$. Due to Theorem 2 the sequences $\{J_1^nh\}$ and $\{J_2^nk\}$ converge to a fixed points T and δ, i.e.,

$$T : \mathscr{A} \longrightarrow \mathscr{A},\ T(a) = \lim_{n\to\infty}(J_1^nh)(a) = \lim_{n\to\infty}\frac{1}{2^n}h(2^na),$$
$$\delta : \mathscr{A} \longrightarrow \mathscr{A},\ \delta(a) = \lim_{n\to\infty}(J_2^nk)(a) = \lim_{n\to\infty}2^nk(\frac{a}{2^n}).$$

So $T(2a) = 2T(a)$ and $\delta(a) = 2\delta(\frac{a}{2})$ for all $a \in \mathscr{A}$. Again by applying Theorem 2, we obtain

$$d_1(h,T) \leqslant \frac{1}{1-L_1}d_1(h,J_1h) \leqslant \frac{L_1}{2-2L_1},$$
$$d_2(k,\delta) \leqslant \frac{1}{1-L_2}d_2(k,J_1k) \leqslant \frac{1}{1-L_2},$$

and therefore

$$||h(a)-T(a)|| \leqslant \frac{L_1}{2-2L_1}\psi_1(a),$$
$$||k(a)-\delta(a)|| \leqslant \frac{1}{1-L_2}\psi_2(a),$$

for all $a \in \mathscr{A}$. The reminder is similar to the Theorem 3. $\square$

COROLLARY 3. *Let $0 < p < 1$, $q > 1$, $\beta,\gamma > 0$, and $f,g,h,k : \mathscr{A} \longrightarrow \mathscr{A}$ with $f(0) = g(0) = h(0) = k(0) = 0$ be mappings such that*

$$||D_\mu(f,g,h)(a,b)|| \leqslant \beta(||a||^p + ||b||^p) + \gamma||a||^{\frac{p}{2}}||b||^{\frac{p}{2}}, \tag{29}$$

$$\begin{aligned}&||f([a,b,c] - [f(a),k(b),k(c)]) - k(\mu u + v) - \mu k(u) - k(v)||\\ \leqslant&\beta(||a||^q + ||b||^q + ||c||^q + ||u||^q + ||v||^q) + \gamma||a||^{\frac{q}{2}}||b||^{\frac{q}{2}}||c||^{\frac{q}{2}}||u||^{\frac{q}{2}}||v||^{\frac{q}{2}},\end{aligned} \tag{30}$$

for all $u,v,a,b,c \in \mathscr{A}$ and $\mu \in \mathbb{T}^1$. Then there exists a unique left δ- centralizer $T : \mathscr{A} \longrightarrow \mathscr{A}$ such that

$$\begin{aligned}||h(a) - T(a)|| &\leqslant \frac{(2+2^{p-1})\beta + \gamma}{2-2^p}||x||^p,\\ ||f(a) - T(a)|| &\leqslant \frac{(8-2^p)\beta + (4-2^p)\gamma}{2^p(2-2^p)}r^{p-1}||x||^p,\\ ||g(a) - T(a)|| &\leqslant \frac{(8-2^p)\beta + (4-2^p)\gamma}{2^p(2-2^p)}s^{p-1}||x||^p,\\ ||k(a) - \delta(a)|| &\leqslant \frac{2\beta}{2^q-2}||x||^q,\end{aligned}$$

for all $a \in \mathscr{A}$.

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