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CHARACTERIZATION OF SOME CLASSES OF COMPACT AND MATRIX OPERATORS ON THE SEQUENCE SPACES OF CESÀRO MEANS

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Abstract. In this study, we give characterization of the matrix classes $(|C_{-1}|_k, X)$, where the spaces $|C_{-1}|_k, k \geqslant 1$ have been defined and studied by Hazar and Sarıgöl in [15] and $X = \{c_0, c, \ell_\infty\}$. Also, we determine the Hausdorff measures of noncompactness of certain matrix operators on the spaces $|C_{-1}|_k$ and apply our results to characterize some classes of compact operators on those spaces. So, we extend some well known results.

1. Background, notations and preliminaries

Let ω be the set of all complex sequences. Any vector subspace of ω is called a sequence space. We shall write $\ell_\infty, c,\ c_0$, and ϕ for the spaces of all bounded, convergent, null and finite sequences, respectively. Also by c_s and ℓ_k $(k \geqslant 1,\ \ell_1 = \ell)$, we denote the spaces of all convergent and k-absolutely convergent series, respectively. Let $A = (a_{nj})$ be an arbitrary infinite matrix of complex numbers and A_n be the sequence in the n-th row of A, that is, $A_n = (a_{nv})_{v=0}^{\infty}$ for every $n \in \mathbb{N}$, and also let X and Y be subspaces of w.

By $Ax = (A_n(x))$, we denote the A-transform of the sequence $x = (x_j)$, i.e.,

$$A_n(x) = \sum_{j=0}^{\infty} a_{nj} x_j,$$

provided that the series is convergent for each $n \in \mathbb{N}$. Further, we say that A defines a matrix transformation from X into Y, and it is denoted by $A \in (X,Y)$ or $A : X \to Y$ if sequence Ax exists and $Ax = (A_n(x)) \in Y$ for every sequence $x \in X$ and also the sets $X^\beta = \{\varepsilon = (\varepsilon_v) \in w : \varepsilon x = (\varepsilon_v x_v) \in c_s \text{ for all } x = (x_v) \in X\}$ and

$$X_A = \{x \in w : Ax \in X\} \tag{1.1}$$

are said to be the β- dual of X and the domain of the matrix A in X, respectively. Thus $A \in (X,Y)$ if and only if $A_n = (a_{nv})_{v=0}^{\infty} \in X^\beta$ for each n and $Ax \in Y$ for all $x \in X$.

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An infinite matrix $A=(a_{nv})$ is called a triangle if $a_{nn}\neq 0$, and $a_{nv}=0$ for $v>n$, which has a unique inverse [39]. Throughout paper k^* denotes the conjugate of $k>1$, i.e., $1/k+1/k^*=1$, and $1/k^*=0$ for $k=1$.

The theory of BK- spaces has a special importance in the characterization of matrix transformation between sequence spaces due to the various properties which they have.

A BK- space is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ defined by $P_n(x)=x_n$ for $n\geqslant 0$, where $\mathbb{C}$ denotes the complex field. Also, a BK- space $X\supset\phi$ has AK if every sequence $x=(x_v)\in X$ has a unique representation $x=\sum_{v=0}^{\infty} x_v e^{(v)}$, where $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n\in\mathbb{N}$ [1, p. 225]. For example, the sequence spaces ℓ_∞, c and c_0 are BK-spaces with the same sup-norm given by $\|x\|_{\ell_\infty}=\sup_v |x_v|$, where the supremum is taken over all $v\in\mathbb{N}$. Further, the space ℓ_k is a BK-space with respect to the natural norm

$$\|x\|_{\ell_k}=\left(\sum_{v=0}^{\infty}|x_v|^k\right)^{1/k} \quad (1\leqslant k<\infty).$$

Moreover, ℓ_k $(1\leqslant k<\infty)$ and c_0 have AK [25, Examples 1.13;1.20].

Let X and Y be normed spaces. Then, a linear operator $A : X\to Y$ is called bounded if there exists a constant $M\geqslant 0$ such that

$$\|Ax\|_Y \leqslant M\|x\|_X \text{ for all } x\in X.$$

The collection of all bounded linear operators from X into Y is denoted by $\mathscr{B}(X,Y)$. If $A\in\mathscr{B}(X,Y)$, then the norm of A is defined by

$$\|A\|_{(X,Y)}=\sup\{\|Ax\| : \|x\|\leqslant 1\}.$$

If $X\supset\phi$ is a BK- space and $a=(a_v)\in w$, then we write

$$\|a\|_X^* = \sup_{x\in S_X}\left|\sum_{v=0}^{\infty} a_v x_v\right| \tag{1.2}$$

provided the expression on the right is defined and finite which is the case whenever $a\in X^\beta$, where S_X denotes the unit sphere in X, that is $S_X=\{x\in X : \|x\|=1\}$ [24, p. 35].

We need following important results for our investigation.

LEMMA 1.1. *([25, Theorem 1.29]) Let $1<k<\infty$. Then, we have $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$, $\ell_1^\beta=\ell_\infty$ and $\ell_k^\beta=\ell_{k^*}$. Furthermore, let X denote any of the spaces $\ell_\infty, c, c_0, \ell_1$ and ℓ_k. Then, we have $\|a\|_X^*=\|a\|_{X^\beta}$ for all $a\in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the dual space X^β.*

LEMMA 1.2. *([25, Theorem 1.23(a)]) Let X and Y be BK spaces. Then, we have $(X,Y)\subset\mathscr{B}(X,Y)$, that is, every matrix $A\in(X,Y)$ defines an operator $L_A\in\mathscr{B}(X,Y)$ by $L_A(x)=Ax$ for all $x\in X$.*

LEMMA 1.3. *([25, Lemma 2.2]) Let $X \supset \phi$ be a BK space and Y be any of the spaces ℓ_∞, c, c_0. If $A \in (X,Y)$, then $\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty$.*

LEMMA 1.4. *([33]) Let $1 < k < \infty$. Then, $A \in (\ell_k, \ell)$ if and only if*

$$\|A\|'_{(\ell_k,\ell)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty,$$

and there exists $1 \leqslant \xi \leqslant 4$ such that $\|A\|'_{(\ell_k,\ell)} = \xi \|A\|_{(\ell_k,\ell)}$.

LEMMA 1.5. *([22]) Let $1 \leqslant k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if*

$$\|A\|_{(\ell,\ell_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{1/k} < \infty.$$

If S and H are subsets of a metric space (X,d) and $\varepsilon > 0$ then S is called an ε-net of H, if, for every $h \in H$, there exists $s \in S$ such that $d(h,s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-*net* of H. By $\mathscr{M}_X$, we denote the collection of all bounded subsets of X. If $Q \in \mathscr{M}_X$, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathscr{M}_X \to [0,\infty)$ is called the Hausdorff measure of noncompactness [31].

Note that if X and Y are Banach spaces then a linear operator $L : X \to Y$ is said to be compact if its domain is all of X and for every bounded sequence $x = (x_n) \in X$, the sequence $(L(x_n))$ has convergent subsequence in Y. We denote the class of such operators by $\mathscr{C}(X,Y)$.

The most effective way to characterize compact operators between Banach spaces is to apply the Hausdorff measure of noncompactness, which is achieved as follows.

LEMMA 1.6. *([25]) Let X and Y be Banach spaces, $L \in \mathscr{B}(X,Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by*

$$\|L\|_\chi = \chi(L(S_X)),$$

and L is compact iff $\|L\|_\chi = 0$.

2. The absolute Cesàro spaces and matrix operators

Throughout this paper, let Σx_n be an infinite series with partial sums s_n and σ_n^α be the nth Cesàro mean (C,α) of order $\alpha > -1$ of the sequence (s_n), i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v,$$

where

$$A_0^\alpha = 1,\ A_n^\alpha = \binom{\alpha+n}{n},\ A_{-n}^\alpha = 0,\ n \geqslant 1.$$

Then, the series Σx_n is said to be summable $|C,\alpha|_k$ with index $k \geqslant 1$ if (see [11])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k < \infty, \sigma_{-1}^\alpha = 0.$$

More recently, Sarıgöl has defined the series space $|C_\alpha|_k$ for $\alpha > -1$, as the set of all series summable by the method $|C,\alpha|_k$, and studied its some properties and related matrix mappings in [32]. Also some compact operators on it are investigated in [16].

Note that this method does not work for $\alpha = -1$ while Cesàro summability (C,α) is studied usually for $\alpha \geqslant -1$ (see [12]), and so, Thorpe separately [38] defined that if the series to sequence transformation

$$T_n = \sum_{v=0}^{n-1} x_v + (n+1)x_n \tag{2.1}$$

tends to a finite number s as n tends to infinity, then the series Σx_n is summable by Cesàro summability $(C,-1)$ to the number s [38].

Also, Hazar and Sarıgöl [15] have introduced the space $|C_{-1}|_k$ as the set of all series summable of the method $|C,-1|_k$, as follows.

$$|C_{-1}|_k = \left\{ x = (x_v) : \sum_{n=1}^{\infty} n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty \right\},$$

where (T_n) is defined by (2.1), or

$$|C_{-1}|_k = \left\{ x = (x_v) : \sum_{n=1}^{\infty} n^{k-1} \left|(n+1)x_n - (n-1)x_{n-1}\right|^k < \infty \right\}.$$

The β - duals of the space $|C_{-1}|_k$ have been determined and some related matrix classes have been characterized. We refer the reader to [13,15] for the most recent work on this topic. Also, independently of these studies, some sequence spaces have been generated and examined by several authors (see [1,2,3,4,5,6,7,8,9,10], [14], [17,18,19,20,21], [23], [26,27,28,29], [34,35,36]).

In this study, we give the characterization of the classes of infinite matrices $(|C_{-1}|_k, X)$ for $X = \{c_0, c, \ell_\infty\}$, establish estimates for the norms and the Hausdorff measures of noncompactness of bounded linear operators defined by those matrix mappings, and characterize the corresponding subclasses of compact operators.

Now, we may remind some properties of the space $|C_{-1}|_k$, they can be found in detail [15].

We may restate $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$ in view of the identity (1.1), where the matrix $T^{(k)} = (t_{nv}^{(k)})$ is defined by $t_{00}^{(k)} = 1$ and

$$t_{nv}^{(k)} = \begin{cases} -n^{1/k^*}(n-1), & v = n-1, \\ n^{1/k^*}(n+1), & v = n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Since the matrix $T^{(k)} = (t_{nv}^{(k)})$ is triangle, there exists the inverse matrix $S^{(k)} = (s_{nv}^{(k)})$ defined by $s_{00}^{(k)} = 1$ and

$$s_{nv}^{(k)} = \begin{cases} \dfrac{v^{1/k}}{n(n+1)}, & 1 \leqslant v \leqslant n, \\ 0, & v > n. \end{cases} \tag{2.3}$$

For any sequence $x = (x_v) \in |C_{-1}|_k$, if we define the associated sequence $y = T^{(k)}(x)$ as

$$T_0^{(k)}(x) = x_0, T_n^{(k)}(x) = n^{1/k^*}\left[(n+1)x_n - (n-1)x_{n-1}\right], n \geqslant 1, x_{-1} = 0 \tag{2.4}$$

then, it is seen that $x \in |C_{-1}|_k$ if and only if $y \in \ell_k$, furthermore, if $x \in |C_{-1}|_k$, then $\|x\|_{|C_{-1}|_k} = \|y\|_{\ell_k}$. In fact, the linear operator $T^{(k)}(x) : |C_{-1}|_k \to \ell_k$, which maps every sequence $x \in |C_{-1}|_k$ to its associated sequence $y \in \ell_k$, is bijective and norm preserving.

Thus, we can note that $|C_{-1}|_k$ is a BK-space with respect to the norm [15]

$$\|x\|_{|C_{-1}|_k} = \left\| T^{(k)}(x) \right\|_{\ell_k}. \tag{2.5}$$

Finally, we define the following notations:

$$D_1 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sum_{v=r}^{\infty} \frac{\varepsilon_v}{v(v+1)} \text{ converges}, r \geqslant 1 \right\},$$

$$D_2 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_{m,r} \left| r \sum_{v=r}^{m} \frac{\varepsilon_v}{v(v+1)} \right| < \infty, r \geqslant 1 \right\},$$

$$D_3 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_{m} \sum_{r=1}^{m} \left| r^{1/k} \sum_{v=r}^{m} \frac{\varepsilon_v}{v(v+1)} \right|^{k^*} < \infty \right\}.$$

The following results play an important role in our study.

LEMMA 2.1.

a) $A \in (\ell, c) \Leftrightarrow (i)$ $\lim_n a_{nv}$ *exists,* $v \geqslant 0$, (ii) $\sup_{n,v} |a_{nv}| < \infty$.

b) $A \in (\ell, \ell_\infty) \Leftrightarrow (ii)$ *holds.*

c) If $1 < k < \infty$, *then,* $A \in (\ell_k, c) \Leftrightarrow (i)$ *holds,* (iii) $\sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty$.

d) If $1 < k < \infty$, *then,* $A \in (\ell_k, \ell_\infty) \Leftrightarrow (iii)$ *holds.*

e) If $1 < k < \infty$, *then,* $A \in (\ell_k, c_0) \Leftrightarrow (iii)$ *holds,* (iv) $\lim_n a_{nv} = 0$, $v \geqslant 0$.

f) $A \in (\ell, c_0) \Leftrightarrow (ii)$ *and* (iv) *holds [37].*

LEMMA 2.2. *Let* $1 \leqslant k < \infty$. *If* $a = (a_v) \in (|C_{-1}|_k)^{\beta}$, *then* $\tilde{a}^{(k)} = \left(\tilde{a}_v^{(k)}\right) \in \ell_{k^*}$ $\left(\tilde{a}^{(1)} \in \ell_\infty, \text{ for } k = 1\right)$ *and we have*

$$\sum_{v=1}^{\infty} a_v x_v = \sum_{v=1}^{\infty} \tilde{a}_v^{(k)} y_v \tag{2.6}$$

for every $x = (x_k) \in |C_{-1}|_k$ *with* $y = T^{(k)}(x)$, *where*

$$\tilde{a}_v^{(k)} = v^{1/k} \sum_{r=v}^{\infty} \frac{a_r}{r(r+1)} = \sum_{r=v}^{\infty} a_r s_{rv}^{(k)}. \tag{2.7}$$

By Lemma 2.2, we deduce following lemma.

LEMMA 2.3. $|C_{-1}|_k^{\beta} = D_1 \cap D_3$ *for* $1 < k < \infty$ *and* $(|C_{-1}|)^{\beta} = D_1 \cap D_2$ *for* $k = 1$ *[15].*

LEMMA 2.4. *If* $1 < k < \infty$, *then, we have* $\|a\|_{|C_{-1}|_k}^{*} = \left\|\tilde{a}^{(k)}\right\|_{\ell_{k^*}}$ *and if* $k = 1$, *then, we have* $\|a\|_{|C_{-1}|}^{*} = \left\|\tilde{a}^{(1)}\right\|_{\ell_\infty}$, *for all* $a \in (|C_{-1}|_k)^{\beta}$, *where* $\tilde{a}^{(k)} = \left(\tilde{a}_v^{(k)}\right)$ *is defined by* (2.7).

Proof. Let $1 < k < \infty$ and $a \in (|C_{-1}|_k)^{\beta}$ be given. Then it follows from Lemma 2.2 that $\tilde{a}^{(k)} = \left(\tilde{a}_v^{(k)}\right) \in \ell_{k^*}$ and the equality (2.6) holds for all sequences $x \in |C_{-1}|_k$ and $y \in \ell_k$ which are connected by the relation $y = T^{(k)}(x)$. Also, by (2.5), $x \in S_{|C_{-1}|_k}$ if and only if $y \in S_{\ell_k}$. So, we get from (1.2) and (2.6) that

$$\|a\|_{|C_{-1}|_k}^{*} = \sup_{x \in S_{|C_{-1}|_k}} \left| \sum_{v=1}^{\infty} a_v x_v \right| = \sup_{y \in S_{\ell_k}} \left| \sum_{v=1}^{\infty} \tilde{a}_v^{(k)} y_v \right| = \left\|\tilde{a}^{(k)}\right\|_{\ell_k}^{*}$$

which completes the proof. □

For $k = 1$, since it can be similarly proved, we omit detail.

Throughout this paper we use following notation.

For an infinite matrix $A = (a_{nv})$, we define the associated matrix $\tilde{A}^{(k)} = \left(\tilde{a}_{nv}^{(k)}\right)$ by

$$\tilde{a}_{nv}^{(k)} = v^{1/k} \sum_{r=v}^{\infty} \frac{a_{nr}}{r(r+1)} = \sum_{r=v}^{\infty} a_{nr} s_{rv}^{(k)} \tag{2.8}$$

provided the series on the right converges for all $n, v \geqslant 1$.

LEMMA 2.5. *Let* Z *be a sequence space,* $A = (a_{nv})$ *an infinite matrix and* $1 \leqslant k < \infty$. *If* $A \in (|C_{-1}|_k, Z)$, *then* $\tilde{A}^{(k)} \in (\ell_k, Z)$ *such that* $Ax = \tilde{A}^{(k)} y$ *for all* $x \in |C_{-1}|_k$ *and* $y \in \ell_k$ *which are connected by* (2.4), *where* $\tilde{A}^{(k)}$ *associated matrix is defined by* (2.8).

Proof. This is immediate by Lemma 2.2. □

Finally, we complete this section with following Lemmas on operator norms.

LEMMA 2.6. *Let $A = (a_{nv})$ be an infinite matrix and $\tilde{A}^{(k)}$ associated matrix defined by* (2.8). *If A is in any of the classes* $(|C_{-1}|_k, c_0), (|C_{-1}|_k, c)$ *and* $(|C_{-1}|_k, \ell_\infty)$ *then, we have for* $1 < k < \infty$,

$$\|L_A\| = \|A\|_{(|C_{-1}|_k, \ell_\infty)} = \sup_n \left\| \tilde{A}_n^{(k)} \right\|_{\ell_{k^*}}$$

and for $k = 1$,

$$\|L_A\| = \|A\|_{(|C_{-1}|, \ell_\infty)} = \sup_n \left\| \tilde{A}_n^{(1)} \right\|_{\ell_\infty}.$$

Proof. It is deduced by combining Lemmas 1.2, 1.3 and 2.4. □

LEMMA 2.7. *Let $A = (a_{nv})$ be an infinite matrix and $\tilde{A}^{(k)} = \left(\tilde{a}_{nv}^{(k)}\right)$ associated matrix defined by* (2.8). *Then, we have*

a) If $A \in (|C_{-1}|, \ell_k)$, then for $k \geqslant 1$,

$$\|L_A\| = \|A\|_{(|C_{-1}|, \ell_k)} = \left\| \tilde{A}^{(1)} \right\|_{(\ell, \ell_k)}.$$

b) If $A \in (|C_{-1}|_k, \ell)$, then for $1 < k < \infty$, there exists $1 \leqslant \xi \leqslant 4$ such that

$$\|L_A\| = \|A\|_{(|C_{-1}|_k, \ell)} = \left\| \tilde{A}^{(k)} \right\|_{(\ell_k, \ell)} = \frac{1}{\xi} \left\| \tilde{A}^{(k)} \right\|'_{(\ell_k, \ell)}.$$

Proof. It is immediate by combining Lemmas 1.2, 1.4, 1.5 and 2.5. □

3. Some compact and matrix operators on $|C_{-1}|_k$

In this section, we give characterization of the matrix classes $(|C_{-1}|_k, X)$, where $k \geqslant 1, X = \{c_0, c, \ell_\infty\}$. Also, we determine the Hausdorff measures of noncompactness of certain matrix operators on the spaces $|C_{-1}|_k$ and apply our results to characterize some classes of compact operators on those spaces. So, we extend some well known results.

Now, we prove our first main result related to characterization of matrix classes.

THEOREM 3.1. *Let $A = (a_{nr})$ be an infinite matrix of complex numbers for all $n, r \geqslant 1$, the associated matrix $\tilde{A}^{(1)} = \left(\tilde{a}_{nr}^{(1)}\right)$ and the matrix $S^{(1)} = (s_{nv}^{(1)})$ be defined by* (2.8) *and* (2.3), *for $k = 1$, respectively.*

a) Then, $A \in (|C_{-1}|, \ell_\infty)$ if and only if

$$\sum_{v=r}^{\infty} s_{vr}^{(1)} a_{nv} \text{ converges, for all } n, r \geqslant 1, \tag{3.1}$$

$$\sup_{m,r} \left| \sum_{v=r}^{m} s_{vr}^{(1)} a_{nv} \right| < \infty, \text{ for all } n, r \geqslant 1, \tag{3.2}$$

$$\sup_{n,r} \left| \tilde{a}_{nr}^{(1)} \right| < \infty. \tag{3.3}$$

b) *Then,* $A \in (|C_{-1}|, c)$ *if and only if* $(3.1), (3.2), (3.3)$ *hold and*

$$\lim_n \tilde{a}_{nr}^{(1)} \text{ exists for each } r. \tag{3.4}$$

c) *Then,* $A \in (|C_{-1}|, c_0)$ *if and only if* $(3.1), (3.2), (3.3)$ *hold and*

$$\lim_n \tilde{a}_{nr}^{(1)} = 0, \text{ for each } r. \tag{3.5}$$

Proof.

a) $A \in (|C_{-1}|, \ell_\infty)$ iff $(a_{nv})_{v=1}^{\infty} \in (|C_{-1}|)^\beta$ and $Ax \in \ell_\infty$ for every $x \in |C_{-1}|$, and also by Lemma 2.3, $(a_{nv})_{v=1}^{\infty} \in (|C_{-1}|)^\beta$ iff (3.1) and (3.2) hold. Moreover, the series $\Sigma a_{nv} x_v$ converges uniformly in n and so

$$\lim_n A_n(x) = \sum_{v=1}^{\infty} \lim_n a_{nv} x_v. \tag{3.6}$$

Now to prove necessity and sufficiency of condition (3.3), consider the operator $T^{(1)} : |C_{-1}| \to \ell$ defined by (2.2) with $k = 1$. It is clear that this operator is bijection and the matrix corresponding to this operator is triangle. Further, let $x \in |C_{-1}|$ be given. Then $T^{(1)}(x) = y \in \ell$ iff $x = S^{(1)}(y)$, where $S^{(1)}$ is the inverse of $T^{(1)}$ and it is defined by (2.3) with $k = 1$. We can write that

$$\sum_{v=1}^{m} a_{nv} x_v = \sum_{j=1}^{m} \left(\sum_{v=j}^{m} a_{nv} s_{vj}^{(1)} \right) y_j = \sum_{j=1}^{m} r_{mj}^{(n)} y_j$$

where the matrix $R^{(n)} = \left(r_{mj}^{(n)} \right)$, for $j, m = 1, 2, ...,$ is defined by

$$r_{mj}^{(n)} = \begin{cases} \sum_{v=j}^{m} a_{nv} s_{vj}^{(1)}, & 1 \leqslant j \leqslant m \\ 0, & j > m. \end{cases}$$

Also, by (3.1) and (3.2), by applying the matrix $R^{(n)} = \left(r_{mj}^{(n)} \right)$ to (3.6), we obtain that

$$A_n(x) = \lim_m \sum_{j=1}^{m} r_{mj}^{(n)} y_j = \sum_{j=1}^{\infty} \left(\sum_{v=j}^{\infty} a_{nv} s_{vj}^{(1)} \right) y_j = \sum_{j=1}^{\infty} \tilde{a}_{nj}^{(1)} y_j = \tilde{A}_n^{(1)}(y)$$

converges for all $n \geqslant 1$, and $\tilde{A}^{(1)} = \left(\tilde{a}_{nr}^{(1)} \right)$ is defined by (2.8) for $k = 1$.This gives us that the mapping sequence $Ax = (A_n(x))$ exists. So, it can be written that $A : |C_{-1}| \to \ell_\infty$ iff $\tilde{A}^{(1)} : \ell \to \ell_\infty$, and also it is easily seen that $\tilde{A}^{(1)} = AoS^{(1)}$. Thus, it follows by applying Lemma 2.1 with the matrix $\tilde{A}^{(1)}$ that $\tilde{A}^{(1)} : \ell \to \ell_\infty$ iff (3.3) holds, which completes the proof of the part of a).

Since b) and c) are proved easily as in a), so we omit the detail. □

THEOREM 3.2. *Let* $A = (a_{nr})$ *be an infinite matrix of complex numbers for all* $n, r \geqslant 1$ *and the associated matrix* $\tilde{A}^{(k)} = \left(\tilde{a}_{nr}^{(k)}\right)$ *and the matrix* $S^{(k)} = (s_{nv}^{(k)})$ *be defined by* (2.8) *and* (2.3)*, respectively, and* $1 < k < \infty$*. Then,*

a) $A \in (|C_{-1}|_k, \ell_\infty)$ *if and only if* (3.1) *holds, and*

$$\sup_m \sum_{r=1}^{m} \left| \sum_{v=r}^{m} s_{vr}^{(k)} a_{nv} \right|^{k^*} < \infty, \ n \geqslant 1, \tag{3.7}$$

$$\sup_n \sum_{r=1}^{\infty} \left| \tilde{a}_{nr}^{(k)} \right|^{k^*} < \infty. \tag{3.8}$$

b) $A \in (|C_{-1}|_k, c)$ *if and only if* $(3.1), (3.7), (3.8)$ *hold, and*

$$\lim_n \tilde{a}_{nr}^{(k)} \text{ exists for each } r. \tag{3.9}$$

c) $A \in (|C_{-1}|_k, c_0)$ *if and only if* $(3.1), (3.7), (3.8)$ *hold, and*

$$\lim_n \tilde{a}_{nr}^{(k)} = 0 \text{ for each } r. \tag{3.10}$$

Proof.

a) $A \in (|C_{-1}|_k, \ell_\infty)$ iff $(a_{nv})_{v=1}^{\infty} \in |C_{-1}|_k^{\beta}$ and $Ax \in \ell_\infty$ for every $x \in |C_{-1}|_k$. Also, by Lemma 2.3, $(a_{nv})_{v=1}^{\infty} \in |C_{-1}|_k^{\beta}$ iff (3.1) and (3.7) hold. Moreover, the series $\Sigma a_{nv} x_v$ converges uniformly in n and so (3.6) holds.

To obtain (3.8), as in the proof of Theorem 3.1 consider the operator $T^{(k)}$: $|C_{-1}|_k \to \ell_k$ given by (2.2). Then, the inverse matrix $S^{(k)}$ of $T^{(k)}$ is given by (2.3). Also, by (3.1) and (3.7), by applying the matrix $B^{(n)} = \left(b_{mj}^{(n)}\right)$ to (3.6), we obtain that

$$A_n(x) = \lim_m \sum_{j=1}^{m} b_{mj}^{(n)} y_j = \sum_{j=1}^{\infty} \left(\sum_{v=j}^{\infty} a_{nv} s_{vj}^{(k)} \right) y_j = \sum_{j=1}^{\infty} \tilde{a}_{nj}^{(k)} y_j = \tilde{A}_n^{(k)}(y)$$

converges for all $n \geqslant 1$, where, for $j, m = 1, 2, \ldots$,

$$b_{mj}^{(n)} = \begin{cases} \sum_{v=j}^{m} a_{nv} s_{vj}^{(k)}, & 1 \leqslant j \leqslant m, \\ 0, & j > m, \end{cases}$$

and $\tilde{A}^{(k)} = \left(\tilde{a}_{nr}^{(k)}\right)$ is defined by (2.8), also $A : |C_{-1}|_k \to \ell_\infty$ if and only if $\tilde{A}^{(k)}$: $\ell_k \to \ell_\infty$. Further, it can be easily calculated that $\tilde{A}^{(k)} = AoS^{(k)}$. Thus, by Lemma 2.1, we get that $\tilde{A}^{(k)} : \ell_k \to \ell_\infty$, *i.e.*, equivalently, (3.8) holds, and this proves the part of a). □

We may state the following lemma to characterize some classes of compact operators on the spaces $|C_{-1}|_k$.

LEMMA 3.3. *([30, Theorem 3.7]) Let $X \supset \phi$ be a BK space. Then, we have:*

a) If $A \in (X, \ell_\infty)$, then

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|A_n\|_X^* .$$

b) If $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_X^* .$$

c) If X has AK or $X = \ell_\infty$ and $A \in (X, c)$, then

$$\frac{1}{2}\limsup_{n\to\infty} \|A_n - \alpha\|_X^* \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|A_n - \alpha\|_X^* ,$$

where $\alpha = (\alpha_\nu)$ with $\alpha_\nu = \lim_{n\to\infty} a_{n\nu}$ for all $\nu \in \mathbb{N}$.

Now, if we take $X = \ell_k$, which has AK, in Lemma 3.3, we deduce the following result by combining Lemmas 2.4, 2.5 and 3.3.

THEOREM 3.4. *Let $A = (a_{n\nu})$ be an infinite matrix and $\tilde{A}^{(k)} = \left(\tilde{a}_{n\nu}^{(k)}\right)$ the associated matrix defined by (2.8). Then, we have for $k \geqslant 1$*

a) If $A \in (|C_{-1}|_k, \ell_\infty)$, then

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \left\|\tilde{A}_n^{(k)}\right\|_{\ell_k}^* \tag{3.11}$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty} \left\|\tilde{A}_n^{(k)}\right\|_{\ell_k}^* = 0. \tag{3.12}$$

b) If $A \in (|C_{-1}|_k, c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \left\|\tilde{A}_n^{(k)}\right\|_{\ell_k}^* , \tag{3.13}$$

$$L_A \text{ is compact if and only if } \lim_{n\to\infty} \left\|\tilde{A}_n^{(k)}\right\|_{\ell_k}^* = 0. \tag{3.14}$$

c) If $A \in (|C_{-1}|_k, c)$, then

$$\frac{1}{2}\limsup_{n\to\infty} \left\|\tilde{A}_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \left\|\tilde{A}_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* , \tag{3.15}$$

$$L_A \text{ is compact if and only if } \lim_{n\to\infty} \left\|\tilde{A}_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* = 0, \tag{3.16}$$

where $\tilde{\alpha} = (\tilde{\alpha}_\nu)$ with $\tilde{\alpha}_\nu = \lim_{n\to\infty} \tilde{a}_{n\nu}^{(k)}$ for all $\nu \geqslant 1$.

Proof. From Lemma 1.6, it is obvious that (3.12), (3.14) and (3.16) are the results of $(3.11), (3.13)$ and (3.15), respectively. Then, we may show that the conditions $(3.11), (3.13)$ and (3.15) hold.

(3.11) and (3.13) are obtained by combining parts a) and b) of Lemma 3.3, respectively and Lemma 2.4, since $|C_{-1}|_k, k \geqslant 1$ is a BK-space.

Now we may prove that the condition (3.16) is satisfied. Let $A \in (|C_{-1}|_k, c)$ be given and $\tilde{A}^{(k)} = \left(\tilde{a}_{nv}^{(k)}\right)$ be the associated matrix defined by (2.8), then it follows from Lemma 2.5 that $\tilde{A}^{(k)} \in (\ell_k, c)$. Considering part c) of Lemma 3.3 we get

$$\frac{1}{2} \limsup_{n\to\infty} \left\| \tilde{A}_n^{(k)} - \tilde{\alpha} \right\|_{\ell_k}^{*} \leqslant \left\| L_{\tilde{A}^{(k)}} \right\|_{\chi} \leqslant \limsup_{n\to\infty} \left\| \tilde{A}_n^{(k)} - \tilde{\alpha} \right\|_{\ell_k}^{*}, \tag{3.17}$$

where $\tilde{\alpha} = (\tilde{\alpha}_v)$ with $\tilde{\alpha}_v = \lim_{n\to\infty} \tilde{a}_{nv}^{(k)}$ for all $v \geqslant 1$.

On the otherhand, since $Ax = \tilde{A}^{(k)} y$ by Lemma 2.5, it can be written that $x \in S_{|C_{-1}|_k}$ if and only if $y \in S_{\ell_k}$. So, this gives us

$$\|L_A\|_{\chi} = \chi\left(AS_{|C_{-1}|_k}\right) = \chi\left(\tilde{A}^{(k)} S_{\ell_k}\right) = \left\|L_{\tilde{A}^{(k)}}\right\|_{\chi}, \tag{3.18}$$

by Lemma 1.2, Lemma 1.6 and Lemma 2.5, which concludes the proof. □

The following result is most powerful tool to compute the Hausdorff measure of noncompactness of a bounded subset of the BK space ℓ_k $(1 \leqslant k < \infty)$.

LEMMA 3.5. *([31]) Let Q be a bounded subset of the normed space X, where $X = \ell_k$ for $1 \leqslant k < \infty$. If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, \ldots)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r\to\infty} \sup_{x\in Q} \|(I - P_r)(x)\|_{\ell_k},$$

where I is the identity operator on X.

Using Lemma 3.5, we can state following result.

THEOREM 3.6. *Let $A = (a_{nv})$ be an infinite matrix and $\tilde{A}^{(k)} = \left(\tilde{a}_{nv}^{(k)}\right)$ the associated matrix defined by* (2.8).

a) If $A \in (|C_{-1}|, \ell_k)$, then for $1 \leqslant k < \infty$,

$$\|L_A\|_{\chi} = \lim_{r\to\infty} \sup_{v} \left(\sum_{n=r+1}^{\infty} \left| \tilde{a}_{nv}^{(1)} \right|^k \right)^{1/k} \tag{3.19}$$

and

$$L_A \text{ is compact iff } \lim_{r\to\infty} \sup_{v} \sum_{n=r+1}^{\infty} \left| \tilde{a}_{nv}^{(1)} \right|^k = 0. \tag{3.20}$$

b) If $A \in (|C_{-1}|_k, \ell)$, then for $k > 1$, there exists $1 \leqslant \xi \leqslant 4$ such that

$$\|L_A\|_{\chi} = \frac{1}{\xi} \lim_{r\to\infty} \left(\sum_{v=1}^{\infty} \left(\sum_{n=r+1}^{\infty} \left| \tilde{a}_{nv}^{(k)} \right| \right)^{k^*} \right)^{1/k^*}$$

and

$$L_A \text{ is compact iff } \lim_{r\to\infty}\sum_{v=1}^{\infty}\left(\sum_{n=r+1}^{\infty}\left|\tilde{a}_{nv}^{(k)}\right|\right)^{k^*}=0.$$

Proof.

a) Let $S=\{x\in|C_{-1}|:\|x\|=1\}$. Then, using Lemma 1.5, 1.6, 2.5 and 3.5, we obtain

$$\begin{aligned}\|L_A\|_\chi &= \chi(AS)=\chi\left(\tilde{A}^{(1)}T^{(1)}S\right)\\ &= \lim_{r\to\infty}\sup_{y\in T^{(1)}S}\left\|(I-P_r)\tilde{A}^{(1)}(y)\right\|_{\ell_k}=\lim_{r\to\infty}\sup_{v}\left(\sum_{n=r+1}^{\infty}\left|\tilde{a}_{nv}^{(1)}\right|^k\right)^{1/k}\end{aligned}$$

where $P_r:\ell_k\to\ell_k$ is defined by $P_r(y)=(y_0,y_1,...,y_r,0,...)$, which completes the asserted.

Also, from Lemma 1.6, it is obvious that the condition (3.20) is the consequence of (3.19).

b) Since (b) is proved easily as in (a) using Lemma 1.4 instead of 1.5, so we omit the detail. □

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