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MAPS PRESERVING EQUIVALENCE BY PRODUCTS OF INVOLUTIONS

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Abstract. Let $\mathscr{B}(\mathscr{X})$ be the algebra of bounded linear operators on a complex Banach space $\mathscr{X}$. Two operators A and $B \in \mathscr{B}(\mathscr{X})$ are said to be equivalent by products of involutions, if $A = TBS$ for T and S being a products of finitely many involutions. We will give description of linear bijective maps ϕ on $\mathscr{B}(\mathscr{X})$ satisfying that $\phi(A)$ and $\phi(B)$ are equivalent (i.e. $A = TBS$ for some invertible $T, S \in \mathscr{B}(\mathscr{X})$) whenever A and B are equivalent by products of involutions.

1. Introduction and the main result

Let $\mathscr{X}$ be, if not stated otherwise, a complex Banach space of dimension at least two, $\mathscr{X}'$ its topological dual, $\ker f$ the kernel of $f \in \mathscr{X}'$, $\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$ and $\mathscr{F}(\mathscr{X})$ the ideal of all finite rank operators.

Over the past decades, there has been a considerable interest in the study of linear or merely additive maps on operator algebras that leave certain relations invariant. A lot of interest, among others, has been devoted to the similarity relation (operators A and B are similar, if $B = SAS^{-1}$ for some invertible operator S) and to the classification of similarity-preserving linear or additive maps ϕ (i.e. if operators A and B are similar, then $\phi(A)$ and $\phi(B)$ are similar as well), for instance [2, 3, 4, 6, 7, 10, 11, 13, 16]. Although a lot of results regarding similarity relation exist, let us expose the result due to Lu and Peng, [11]. They proved that if $\mathscr{X}$ is an infinite-dimensional complex Banach space and $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ is a surjective similarity-preserving linear map, then there exist either a non-zero $c \in \mathbb{C}$, an invertible $T \in \mathscr{B}(\mathscr{X})$ and a similarity-invariant linear functional h on $\mathscr{B}(\mathscr{X})$ with $h(I) \neq -c$ such that

$$\phi(X) = cTXT^{-1} + h(X)I, \qquad \text{for every } X \in \mathscr{B}(\mathscr{X}), \tag{1}$$

or there exist a non-zero $c \in \mathbb{C}$, invertible bounded linear operator $T : \mathscr{X}' \to \mathscr{X}$ and a similarity-invariant linear functional h on $\mathscr{B}(\mathscr{X})$ with $h(I) \neq -c$ such that

$$\phi(X) = cTX'T^{-1} + h(X)I, \qquad \text{for every } X \in \mathscr{B}(\mathscr{X}), \tag{2}$$

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where X' stands for the adjoint of the operator X, and a similarity-invariant functional h means that $h(A) = h(B)$ whenever A is similar to B. Qin and Lu, [19], modified the problem and presented it in another way.

An operator $J \in \mathscr{B}(\mathscr{X})$ is called an involution if $J^2 = I$, the identity operator on $\mathscr{X}$. By P-Inv$(\mathscr{X})$ we denote the set of all finite products of involutions. Obviously, P-Inv$(\mathscr{X})$ is a subset of $\mathscr{G}(\mathscr{X})$, the multiplicative group of all invertible operators in $\mathscr{B}(\mathscr{X})$. Moreover, due to Radjavi [15] it is known that P-Inv$(\mathscr{X}) = \{A \in \mathscr{B}(\mathscr{X}) | \det A = \pm 1\}$ in the case of finite dimensional space $\mathscr{X}$, and P-Inv$(\mathscr{X}) = \mathscr{G}(\mathscr{X})$ if $\mathscr{X}$ is an infinite-dimensional complex Hilbert space. In a general infinite-dimensional complex Banach space $\mathscr{X}$ the problem whether P-Inv$(\mathscr{X})$ coincides with $\mathscr{G}(\mathscr{X})$ is connected with the existence of a non-trivial multiplicative functional $f \in \mathscr{X}'$. As stated in [1, 12, 17, 20] there exists a Banach space $\mathscr{X}$ having a non-trivial multiplicative $f \in \mathscr{X}'$, so P-Inv$(\mathscr{X})$ can be a proper subset of $\mathscr{G}(\mathscr{X})$.

Two operators A and B are called p-similar, if $B = SAS^{-1}$ for some $S \in$ P-Inv$(\mathscr{X})$, and a linear map $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ is said to be p-similarity preserving if $\phi(A)$ and $\phi(B)$ are similar whenever A is p-similar to B. Note that similarity preserving is stronger assumption than p-similarity preserving with which Qin and Lu were occupied. They proved that a linear bijection $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ being only a p-similarity preserving is (as in the similarity-preserving case) either of the form (1) or of the form (2).

We now define another equivalence relations on $\mathscr{B}(\mathscr{X})$. Two operators A and $B \in \mathscr{B}(\mathscr{X})$ are said to be *equivalent*, denoted by $A \sim B$, if $A = TBS$ for some $T, S \in \mathscr{G}(\mathscr{X})$, and are *equivalent by products of involutions*, denoted by $A \sim_p B$, if $A = TBS$ for some $T, S \in$ P-Inv$(\mathscr{X})$.

The aim of this note is to refine the result stated in [14], where linear bijection $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ with $A \sim B \Rightarrow \phi(A) \sim \phi(B)$ were determined. It was proved that in the case of $\mathscr{X}$ being an infinite-dimensional reflexive complex Banach space either there exist $T, S \in \mathscr{G}(\mathscr{X})$ such that $\phi(X) = TXS$ for every $X \in \mathscr{B}(\mathscr{X})$, or there exist bounded bijective linear operators $T : \mathscr{X}' \to \mathscr{X}$ and $S : \mathscr{X} \to \mathscr{X}'$ such that $\phi(X) = TX'S$ for every $X \in \mathscr{B}(\mathscr{X})$.

Our main result reads as follows.

THEOREM 1. *Let $\mathscr{X}$ be a complex Banach space of dimension at least two and $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ a surjective linear map such that*

$$A \sim_p B \qquad \Rightarrow \qquad \phi(A) \sim \phi(B),$$

for every $A, B \in \mathscr{B}(\mathscr{X})$. Then one and only one of the following statements holds.

(i) $\phi(F) = 0$, *for every $F \in \mathscr{F}(\mathscr{X})$.*

(ii) There exist invertible $T, S \in \mathscr{B}(\mathscr{X})$ such that

$$\phi(X) = TXS, \qquad \text{for every } X \in \mathscr{B}(\mathscr{X}).$$

(iii) There exist invertible bounded linear operators $T : \mathscr{X}' \to \mathscr{X}$ and $S : \mathscr{X} \to \mathscr{X}'$ such that

$$\phi(X) = TX'S, \qquad \text{for every } X \in \mathscr{B}(\mathscr{X}),$$

where X' stands for the adjoint of the operator X.

Case (iii) can only occur if $\mathscr{X}$ is reflexive.

Let us remark that the problem stated in Theorem 1 is not of any general type of LPPs. We actually determine those surjective linear maps where from equivalence by products of involutions of A and B follows that $\phi(A)$ is equivalent to $\phi(B)$ and not equivalent by products of involutions as we would expect.

2. Preliminaries

Every rank-one operator can be written as $x \otimes f$ for some non-zero vector $x \in \mathscr{X}$ and some non-zero functional $f \in \mathscr{X}'$, and is defined by $(x \otimes f)z = f(z)x$ for every $z \in \mathscr{X}$, $A(x \otimes f) = Ax \otimes f$ and $(x \otimes f)A = x \otimes A'f$ for every $A \in \mathscr{B}(\mathscr{X})$, where A' stands for the adjoint operator of A; operator $x \otimes f$ is idempotent if $f(x) = 1$ and it is nilpotent if $f(x) = 0$.

It is obvious that all rank-one operators are mutually equivalent. But, when we are speaking about equivalence orbit of a rank-one operator under equivalence by products of involutions, the problem is a little bit more complicated. With the following Proposition and some subsequent Lemmas we will be able to determine all operators that are equivalent by products of involutions to a fixed rank-one operator in $\mathscr{B}(\mathscr{X})$.

PROPOSITION 1. [19, Proposition 2.1] *Let $N \in \mathscr{B}(\mathscr{X})$ be a non-zero finite-rank operator with $N^2 = 0$. Then $I + N$ is a product of two involutions.*

LEMMA 1. *Let $0 \neq x \in \mathscr{X}$ and $0 \neq f \in \mathscr{X}'$. Then $x \otimes f \sim_p y \otimes f$ for every non-zero $y \in \mathscr{X}$.*

Proof. Take any non-zero $y \in \mathscr{X}$. If y is linearly independent of x, then there exist $g_1, g_2 \in \mathscr{X}'$ such that $g_1(x) = 1 = g_2(y)$ and $g_1(y) = 0 = g_2(x)$. Let it be $N = (x - y) \otimes (g_1 + g_2)$. As $N \neq 0$ and $N^2 = 0$, the operator $I + N$ is a product of two involutions by Proposition 1. Thus

$$y \otimes f \sim_p (I+N)(y \otimes f) = (I + (x-y) \otimes (g_1 + g_2))y \otimes f = x \otimes f, \tag{3}$$

as desired. Next, let x and y be linearly dependent. As $\dim \mathscr{X} \geqslant 2$, there exists a non-zero $z \in \mathscr{X}$ such that x, z and y, z are linearly independent, respectively. Apply (3) to get $x \otimes f \sim_p z \otimes f$ and $y \otimes f \sim_p z \otimes f$. By the transitivity we have $x \otimes f \sim_p y \otimes f$. $\square$

LEMMA 2. *Let $0 \neq x \in \mathscr{X}$ and $0 \neq f \in \mathscr{X}'$. Then $x \otimes f \sim_p x \otimes g$ for every non-zero $g \in \mathscr{X}'$.*

Proof. Take any non-zero $g \in \mathscr{X}'$. If $\ker g = \ker f$, then g is linearly dependent on f: $g = \alpha f$ for some $\alpha \neq 0$. In turn we have

$$x \otimes g = x \otimes \alpha f = \alpha x \otimes f \sim_p x \otimes f,$$

by Lemma 1. Otherwise, when $\ker g \neq \ker f$, there exist linearly independent $y_1, y_2 \in \mathscr{X}$ such that $f(y_1) = 1 = g(y_2)$ and $f(y_2) = 0 = g(y_1)$. By setting $N = (y_1 + y_2) \otimes (f - g)$ we can see that $N \neq 0$ and $N^2 = 0$. Therefore, by Proposition 1, we obtain $x \otimes g \sim_p (x \otimes g)(I + N) = x \otimes f$. $\square$

PROPOSITION 2. *All rank-one operators in $\mathscr{B}(\mathscr{X})$ are mutually equivalent by products of involutions.*

Proof. Take any non-zero $x, y \in \mathscr{X}$ and any non-zero $f, g \in \mathscr{X}'$. The straightforward consequence of Lemmas 1 and 2 is that $x \otimes f \sim_p y \otimes f \sim_p y \otimes g$. By the transitivity we complete the proof. $\square$

Our first step will be reducing the problem to the case of rank-one preserving map, i.e. if $A \in \mathscr{B}(\mathscr{X})$ is of rank one, then $\phi(A)$ is of rank one too. We will use a result due to Kuzma regarding rank-one-non-increasing additive mappings.

THEOREM 2. [8, Theorem 2.3] *Let $\phi : \mathscr{F}(\mathscr{X}) \to \mathscr{F}(\mathscr{X})$ be an additive map, which maps rank-one operators to operators of rank at most one. Then one and only one of the following statements holds.*

(i) There exist an $f_0 \in \mathscr{X}'$ and an additive map $\tau : \mathscr{F}(\mathscr{X}) \to \mathscr{X}$, such that

$$\phi(X) = \tau(X) \otimes f_0, \qquad \text{for every } X \in \mathscr{F}(\mathscr{X}).$$

(ii) There exist an $x_0 \in \mathscr{X}$ and an additive map $\varphi : \mathscr{F}(\mathscr{X}) \to \mathscr{X}'$, such that

$$\phi(X) = x_0 \otimes \varphi(X), \qquad \text{for every } X \in \mathscr{F}(\mathscr{X}).$$

(iii) There exist additive maps $T : \mathscr{X} \to \mathscr{X}$ and $S : \mathscr{X}' \to \mathscr{X}'$ such that

$$\phi(x \otimes f) = Tx \otimes Sf, \qquad \text{for every } x \in \mathscr{X} \text{ and every } f \in \mathscr{X}'.$$

(iv) There exist additive maps $T : \mathscr{X}' \to \mathscr{X}$ and $S : \mathscr{X} \to \mathscr{X}'$ such that

$$\phi(x \otimes f) = Tf \otimes Sx, \qquad \text{for every } x \in \mathscr{X} \text{ and every } f \in \mathscr{X}'.$$

REMARK 1. If ϕ is in addition linear, it is easy to verify that τ and φ from (i) and (ii) as well as T and S from (iii) and (iv) are linear maps.

We will close the section with two simple Lemmas applying invertible operators.

LEMMA 3. [18, Lemma 3.3] *Let $x \in \mathscr{X}$ and $f \in \mathscr{X}'$. Then $I - x \otimes f$ is invertible in $\mathscr{B}(\mathscr{X})$ if and only if $f(x) \neq 1$.*

LEMMA 4. [11, Lemma 2.5] *Let $x, y \in \mathscr{X}$ and $f, g \in \mathscr{X}'$. Then $I - (x \otimes f + y \otimes g)$ is invertible if and only if $(f(x) - 1)(g(y) - 1) \neq f(y)g(x)$.*

3. Proof of the main result

Let $\mathscr{X}$ be a complex Banach space with $\dim\mathscr{X} \geqslant 2$ and $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ a surjective linear map such that $A \sim_p B$ implies $\phi(A) \sim \phi(B)$ for every $A, B \in \mathscr{B}(\mathscr{X})$.

If $\mathscr{X}$ is finite-dimensional, then P-Inv$(\mathscr{X})$ is equal to $\{A \in \mathscr{B}(\mathscr{X}) \,|\, \det A = \pm 1\}$ and by [5, Theorem 4.1] the proof is completed. In the case of $\mathscr{X}$ being an infinite-dimensional, we set up the proof through several steps.

STEP 1. *ϕ is rank-one-non-increasing linear map, i.e. rank $\phi(A) \leqslant 1$ for every rank-one $A \in \mathscr{B}(\mathscr{X})$.*

Take any $P \in \mathscr{B}(X)$ of rank one. By the surjectivity of ϕ there exists an $A \in \mathscr{B}(\mathscr{X})$ such that

$$\phi(A) = P.$$

If A is of rank one, then we have, by Proposition 2, $A \sim_p E$ for every $E \in \mathscr{B}(\mathscr{X})$ of rank one. Acting by ϕ on this relation implies $P = \phi(A) \sim \phi(E)$. Thus $\phi(E)$ is of rank one for every $E \in \mathscr{B}(\mathscr{X})$ of rank one. In other words, ϕ is rank-one preserving.

In the other case, if A is not of rank one, there exist linearly independent $x_1, x_2 \in \mathscr{X}$ such that Ax_1 and Ax_2 are linearly independent too. Choose linearly independent $f_1, f_2 \in \mathscr{X}'$ such that $f_1(x_1) = 1 = f_2(x_2)$ and $f_1(x_2) = 0 = f_2(x_1)$. Set

$$N = (x_1 - x_2) \otimes (f_1 + f_2) \neq 0.$$

As $N^2 = 0$ and $(-N)^2 = 0$, the operators $I+N$ and $I-N \in$ P-Inv$(\mathscr{X})$ by Proposition 1. From the relation $A \sim_p A(I \pm N) = A \pm AN$ we get

$$P = \phi(A) \sim \phi(A \pm AN) = P \pm \phi(AN).$$

It follows that both $P + \phi(AN)$ as well as $P - \phi(AN)$ are of rank one. Since Ax_1, Ax_2 and f_1, f_2 are linearly independent, respectively, the operator $AN = (Ax_1 - Ax_2) \otimes (f_1 + f_2)$ is of rank one and either

$$\phi(AN) = 0 \qquad \text{or} \qquad \phi(AN) \neq 0.$$

Firstly assume that $\phi(AN) = 0$. Then, by Proposition 2, we have $\phi(E_1) = 0$ for every $E_1 \in \mathscr{B}(\mathscr{X})$ of rank one. Using the fact that every finite-rank operator $F \in \mathscr{F}(\mathscr{X})$ can be written as a sum of rank-one operators, it is obvious that $\phi(\mathscr{F}(\mathscr{X})) = 0$. But, if there exists at least one finite-rank operator in $\mathscr{B}(\mathscr{X})$ which is not mapped to zero operator, then $\phi(AN) \neq 0$. Thus, by [14, Lemma 2.2], the operator $\phi(AN)$ is of rank one. As we have found one operator of rank one which is mapped to an operator of rank one, $\phi(E_2)$ is of rank one for every rank-one $E_2 \in \mathscr{B}(\mathscr{X})$.

Taking both possibilities into consideration, we conclude that ϕ is rank-one-non-increasing map.

By the proof of STEP 1 we have seen that either $\phi(\mathscr{F}(\mathscr{X}))=0$ or ϕ is rank-one preserving. Hence, from now on we can and we will assume that ϕ is rank-one preserving.

STEP 2. *ϕ is injective.*

By the surjectivity of ϕ take an $A\in\mathscr{B}(\mathscr{X})$ such that $\phi(A)=0$. If $A\neq 0$, then there exists an $x\in\mathscr{X}$ with $Ax\neq 0$. Choose any non-zero $f\in\mathscr{X}'$ with $f(x)=0$ and, by Lemma 1, the operator $I+x\otimes f$ is a product of two involutions. Acting by ϕ on the relation $A\sim_p A(I+x\otimes f)=A+Ax\otimes f$ we get $0=\phi(A)\sim\phi(A+Ax\otimes f)=\phi(Ax\otimes f)$ which further implies $\phi(Ax\otimes f)=0$, a contradiction with the rank-one preserving property. So, $A=0$ which proves the claim.

STEP 3. *Either there exist linear maps $T:\mathscr{X}\to\mathscr{X}$ and $S:\mathscr{X}'\to\mathscr{X}'$ such that $\phi(x\otimes f)=Tx\otimes Sf$, for every $x\in\mathscr{X}$ and every $f\in\mathscr{X}'$, or there exist linear maps $T:\mathscr{X}'\to\mathscr{X}$ and $S:\mathscr{X}\to\mathscr{X}'$ such that $\phi(x\otimes f)=Tf\otimes Sx$, for every $x\in\mathscr{X}$ and every $f\in\mathscr{X}'$.*

Since ϕ is rank-one preserving, we can apply Theorem 2. Assume firstly that $\phi(X)=\tau(X)\otimes g_0$ for some non-zero $g_0\in\mathscr{X}'$ and some linear map $\tau:\mathscr{F}(\mathscr{X})\to\mathscr{X}$. Choose any non-zero $y\in\mathscr{X}'$ and any $g_1\in\mathscr{X}'$ linearly independent of g_0. By the surjectivity of ϕ there exists a non-zero $A\in\mathscr{B}(\mathscr{X})$ such that

$$\phi(A)=y\otimes g_1.$$

It is obvious that A is not of rank one, thus there exist linearly independent $x_1,x_2\in\mathscr{X}$ such that Ax_1 and Ax_2 are linearly independent too. For each $i=1,2$ choose $f_i\in\mathscr{X}'$ with $f_i(x_i)=1$. Then it is easy to verify that the operator $I-2x_i\otimes f_i$ is involutive, so acting by ϕ on the relation $A\sim_p A(I-2x_i\otimes f_i)=A-2Ax_i\otimes f_i$ implies $y\otimes g_1\sim y\otimes g_1-\tau(2Ax_i\otimes f_i)\otimes g_0$, for $i=1,2$, and consequently $y\otimes g_1-2\tau(Ax_i\otimes f_i)\otimes g_0$ is of rank one for $i=1,2$. Hence, both $\tau(Ax_1\otimes f_1)$ as well as $\tau(Ax_2\otimes f_2)$ are scalars multiplied of y. It follows that there exists $\alpha\in\mathbb{C}$ such that $\tau(Ax_1\otimes f_1)=\alpha\tau(Ax_2\otimes f_2)$ and in turn $\phi(Ax_1\otimes f_1)=\phi(\alpha Ax_2\otimes f_2)$. By the injectivity of ϕ, $Ax_1\otimes f_1=\alpha Ax_2\otimes f_2$, a contradiction with linear independency of Ax_1, Ax_2 and f_1, f_2, respectively.

Therefore (i), and similarly (ii), from Theorem 2 cannot occur.

We will assume that there exist linear maps $T:\mathscr{X}\to\mathscr{X}$ and $S:\mathscr{X}'\to\mathscr{X}'$ such that

$$\phi(x\otimes f)=Tx\otimes Sf,\qquad \text{for every } x\in\mathscr{X} \text{ and every } f\in\mathscr{X}'.$$

STEP 4. *T and S are bijective.*

The injectivity of T and S follows immediately from the bijectivity of the map ϕ. The surjectivity of T will be proved by a contradiction, so let us assume that T is not surjective. Then there exists a non-zero $y\in\mathscr{X}$ such that y is not contained in the range of T. Choose any non-zero $g \in \mathscr{X}'$. Since ϕ is surjective, there exists an $A \in \mathscr{B}(\mathscr{X})$ such that

$$\phi(A) = y \otimes g.$$

Obviously, $A \neq 0$. Hence, there exists an $x \in \mathscr{X}$ such that $Ax \neq 0$. Take linearly independent $f_1, f_2 \in \mathscr{X}'$ with $f_1(x) = 0 = f_2(x)$. According to Proposition 1, the operator $I + x \otimes f_i \in \text{P-Inv}(\mathscr{X})$, for $i = 1, 2$. Acting by ϕ on the relation $A \sim_p A(I + x \otimes f_i) = A + Ax \otimes f_i$ we obtain

$$y \otimes g \sim y \otimes g + TAx \otimes Sf_i, \qquad \text{for } i = 1, 2,$$

which further implies that $y \otimes g + TAx \otimes Sf_i$ is of rank one. Observe that $TAx \otimes Sf_i \neq 0$. Since y and TAx are linearly independent, the linear functionals Sf_1 and Sf_2 are scalars multiplied of g. Therefore, Sf_1 and Sf_2 are linearly dependent and, by the injectivity of S, f_1 and f_2 are linearly dependent, a contradiction.

By the same method we can see that S is surjective as well.

STEP 5. *Let $\phi(A) = I$ for some non-zero $A \in \mathscr{B}(\mathscr{X})$. Then there exist non-zero $\mu, \upsilon \in \mathbb{C}$ such that*

$$(Sf)(TAx) = \mu f(x) \qquad \textit{and} \qquad \left(SA'f\right)(Tx) = \upsilon f(x), \tag{4}$$

for every $x \in \mathscr{X}$ and every $f \in \mathscr{X}'$. Consequently, A and A' are injective.

Choose any non-zero $x_0 \in \mathscr{X}$ and any non-zero $f_0 \in \mathscr{X}'$ such that $f_0(x_0) = 0$. By Proposition 1, the operator $I + \lambda_0 x_0 \otimes f_0 \in \text{P-Inv}(\mathscr{X})$ for every $\lambda_0 \in \mathbb{C}$. From the relation $A \sim_p A(I + \lambda_0 x_0 \otimes f_0) = A + \lambda_0 Ax_0 \otimes f_0$ it follows

$$I \sim I + \lambda_0 TAx_0 \otimes Sf_0, \qquad \text{for every } \lambda_0 \in \mathbb{C}.$$

Thus, $I + \lambda_0 TAx_0 \otimes Sf_0$ is invertible, so $\lambda_0(Sf_0)(TAx_0) \neq -1$ for every $\lambda_0 \in \mathbb{C}$ by Lemma 3. Therefore,

$$(Sf_0)(TAx_0) = 0, \qquad \text{for every nilpotent } x_0 \otimes f_0 \in \mathscr{B}(\mathscr{X}).$$

Following the steps similar to those used in [16, Remark after Proposition 3.1] we prove that there exists a $\mu \in \mathbb{C}$ such that

$$(Sf)(TAx) = \mu f(x), \qquad \text{for every } x \in \mathscr{X} \text{ and every } f \in \mathscr{X}'.$$

Next we want to see that $\mu \neq 0$. To do this, let us assume the contrary, $\mu = 0$. By the surjectivity of S we have $g(TAx) = 0$ for every $g \in \mathscr{X}'$, which implies $TAx = 0$ for every $x \in \mathscr{X}$. The injectivity of T forces that $Ax = 0$ for every $x \in \mathscr{X}$, a contradiction with $A \neq 0$.

If we started the proof of this Step by $A \sim_p (I + \lambda_0 x_0 \otimes f_0)A$ instead of $A \sim_p A(I + \lambda_0 x_0 \otimes f_0)$ and then continuing the proof in the same way, we would get the second equality of (4). To show that A and A' are injective is then an elementary exercise.

STEP 6. *T and S are continuous.*

We are essentially following the lines of the proof of Step 4 of Theorem 3.3 in [14]. For the sake of completeness, the proof is included.

Firstly we will prove the continuity of the operator TA. Let $(x_n)_{n\in\mathbb{N}} \to 0$ and $(TAx_n)_{n\in\mathbb{N}} \to y \in \mathscr{X}$. Applying (4) gives $(Sf)(y) = 0$ for every $f \in \mathscr{X}'$. As S is surjective, we obtain $y = 0$. By the Closed graph theorem, the operator TA is continuous.

By the bijectivity of S and according to (4) once again we have $\left(S^{-1}f\right)(x) = \mu^{-1}f(TAx)$ for every $x \in \mathscr{X}$ and every $f \in \mathscr{X}'$. Then

$$\left|\left(S^{-1}f\right)(x)\right| = \left|\mu^{-1}f(TAx)\right| \leqslant \left|\mu^{-1}\right| \cdot \|f\| \cdot \|TA\| \cdot \|x\|, \tag{5}$$

for every $x \in \mathscr{X}$. Hence $\|S^{-1}f\| \leqslant \left|\mu^{-1}\right| \cdot \|TA\| \cdot \|f\|$ for every $f \in \mathscr{X}'$. It turns out that $\|S^{-1}\| \leqslant \left|\mu^{-1}\right| \cdot \|TA\|$, so S^{-1} as well as S is continuous.

In the same way, from $(SA'f)(x) = \upsilon f\left(T^{-1}x\right)$, for every $x \otimes f \in \mathscr{B}(\mathscr{X})$, yields the continuity of the operator SA'. As a consequence, T^{-1} and T are continuous too.

Observe that the injectivity of A' immediately implies that A has dense range. After that choose any non-zero $x \in \mathscr{X}$. Because S is bijective, there exists an $f_x \in \mathscr{X}'$ such that $\|S^{-1}f_x\| = 1$ and $\left(S^{-1}f_x\right)(x) = \|x\|$. From the first property it follows $\|f_x\| = \|SS^{-1}f_x\| \leqslant \|S\|$. By the same approach as in (5), the second property of f_x provides

$$\begin{aligned}\|x\| &= \left|\left(S^{-1}f_x\right)(x)\right| = |\mu|^{-1} \cdot |f_x(TAx)| \leqslant |\mu|^{-1} \cdot \|f_x\| \cdot \|TAx\| \\ &\leqslant |\mu|^{-1} \cdot \|S\| \cdot \|T\| \cdot \|Ax\|.\end{aligned}$$

As x was arbitrary, the operator A having dense range is bounded below. Thus, it is invertible. Therefore, TA is invertible and in turn, $(TA)'$ as well.

By (4) it is obvious that $\mu f(x) = \left((TA)'Sf\right)(x)$, for every $x \in \mathscr{X}$ and every $f \in \mathscr{X}'$. Hence $\mu I = (TA)'S$ and consequently, $S = \mu\left((TA)'\right)^{-1}$. Now we can replace ϕ by the map $X \mapsto \mu^{-1}T^{-1}\phi(X)TA$, which is clearly bijective and satisfies: $\phi(B_1) \sim \phi(B_2)$ whenever $B_1 \sim_p B_2$, for $B_1, B_2 \in \mathscr{B}(\mathscr{X})$. Moreover,

$$\phi(x \otimes f) = x \otimes f, \qquad \text{for every } x \in \mathscr{X} \text{ and every } f \in \mathscr{X}'.$$

Let us remark that supposing the alternate form of ϕ (i.e. $\phi(x \otimes f) = Tf \otimes Sx$ for every $x \in \mathscr{X}$ and every $f \in \mathscr{X}'$) the proof of invertibility of linear maps $T : \mathscr{X}' \to \mathscr{X}$ and $S : \mathscr{X} \to \mathscr{X}'$ goes through similarly. Then it is obvious that T' is invertible too. By denoting $\phi^{-1}(I) = A$ we can see that there exists a $0 \neq \mu \in \mathbb{C}$ such that $\mu f(x) = (SAx)(Tf) = (T'SAx)(f)$, for every $x \in \mathscr{X}$ and every $f \in \mathscr{X}'$. As for every non-zero $x \in \mathscr{X}$ exists an $f_x \in \mathscr{X}'$ with $\|f_x\| = 1$ and $f_x(x) = \|x\|$, it follows that $\|x\| \leqslant |\mu|^{-1}\|T'\| \cdot \|S\| \cdot \|Ax\|$ for every $x \in \mathscr{X}$. Then it is easy to verify that A is invertible. Therefore $i = \mu^{-1}T'SA$ is bijective, where $i : \mathscr{X} \to \mathscr{X}''$ is canonical isometric embedding of $\mathscr{X}$. In other words, $\mathscr{X}$ is reflexive. Now we can replace ϕ by the map $X \mapsto \mu^{-1}S^{-1}\phi(X)'SA$. Note that $\mathscr{X}'$ is reflexive too and so $j : \mathscr{X}' \to \mathscr{X}'''$

is bijective canonical isometric embedding of $\mathscr{X}'$. In this special case we can obtain $i' = j^{-1}$. For this reason we have

$$\begin{aligned}\phi(x\otimes f) &= \mu^{-1}S^{-1}(Tf\otimes Sx)'SA = \mu^{-1}S^{-1}Sx\otimes(SA)'(Tf)'' \\ &= \mu^{-1}x\otimes\left(\mu T'^{-1}i\right)'T''f'' = x\otimes i'(f'') = x\otimes f,\end{aligned}$$

for every $x\in\mathscr{X}$ and every $f\in\mathscr{X}'$, and then we continue in the same way.

STEP 7. $\phi(A) = A$ *for every* $A\in\mathbb{C}I+\mathscr{F}(\mathscr{X})$.

By the linearity of ϕ, it is sufficient to prove that $\phi(I) = I$. Denote $\phi^{-1}(I) = J$. Now, we may and we do assume that T and S are identities on $\mathscr{X}$ and $\mathscr{X}'$, respectively. So, apply (4) to get existence of such $0\neq\alpha\in\mathbb{C}$ that $\alpha f(x) = (Sf)(TJx) = f(Jx)$ for every $x\in\mathscr{X}$ and every $f\in\mathscr{X}'$. Consequently, $J=\alpha I$ and thus $\phi(\alpha I) = I$.

In order to see that $\alpha = 1$, choose linearly independent $x_1,x_2\in\mathscr{X}$ and linearly independent $f_1,f_2\in\mathscr{X}'$ such that $f_1(x_1) = 1 = f_2(x_2)$ and $f_1(x_2) = 0 = f_2(x_1)$. By Proposition 1 it is easy to see that $I+\lambda(x_1+x_2)\otimes(f_1-f_2)\in \text{P-Inv}(\mathscr{X})$ for every $\lambda\in\mathbb{C}$. Moreover, $I-2x_1\otimes f_1$ is an involution. Hence

$$\alpha I\sim_p \alpha I\left(I+\lambda(x_1+x_2)\otimes(f_1-f_2)\right)(I-2x_1\otimes f_1)$$

and thus

$$\alpha I\sim_p \alpha I-\alpha\lambda(x_1+x_2)\otimes(f_1+f_2)-2\alpha x_1\otimes f_1,$$

for every $\lambda\in\mathbb{C}$. Acting by ϕ on this relation implies

$$I\sim I-(\alpha\lambda(x_1+x_2)\otimes(f_1+f_2)+2\alpha x_1\otimes f_1).$$

Therefore, the operator $I-(\alpha\lambda(x_1+x_2)\otimes(f_1+f_2)+2\alpha x_1\otimes f_1)$ is invertible for every $\lambda\in\mathbb{C}$. From Lemma 4 it follows

$$(\alpha\lambda(f_1+f_2)(x_1+x_2)-1)\cdot(2\alpha f_1(x_1)-1)\neq 2\alpha(f_1+f_2)(x_1)\cdot\alpha\lambda f_1(x_1+x_2),$$

which yields $\left(2\alpha^2-2\alpha\right)\lambda+(1-2\alpha)\neq 0$ for every $\lambda\in\mathbb{C}$. Consequently, $2\alpha^2-2\alpha = 0$. As $\alpha\neq 0$, we get $\alpha = 1$, as desired.

STEP 8. *If* $A\notin\mathbb{C}I+\mathscr{F}(\mathscr{X})$, *then there exists an* $\alpha_A\in\mathbb{C}$ *depending on* A *such that* $\phi(A) = A+\alpha_A I$.

Let us suppose $A\in\mathscr{B}(\mathscr{X})$ is not a member of $\mathbb{C}I+\mathscr{F}(\mathscr{X})$ and denote

$$\phi(A) = B.$$

Without loss of generality we can assume that B is invertible. If it is non-invertible, then there exists a non-zero $\gamma\in\mathbb{C}$ such that $B+\gamma I$ becomes invertible. In this case, replace A by $A+\gamma I$.

Choose any non-zero $x \in \mathscr{X}$ and any non-zero $f \in \mathscr{X}'$ such that $f(x) = 0$ and $f\left(B^{-1}x\right) = 0$. By Lemma 1 it is obvious that $I + \lambda x \otimes f \in$ P-Inv$(\mathscr{X})$ for every $\lambda \in \mathbb{C}$. Then from $A \sim_p (I + \lambda x \otimes f)A = A + \lambda x \otimes A'f$ it follows

$$B \sim B + \lambda x \otimes A'f = B\left(I + \lambda B^{-1}x \otimes A'f\right).$$

As B is invertible, $B + \lambda x \otimes A'f$ and in turn $I + \lambda B^{-1}x \otimes A'f$ are invertible too. By Lemma 3 we have $-1 \neq \lambda\,(A'f)\left(B^{-1}x\right) = \lambda f\left(AB^{-1}x\right)$ for every $\lambda \in \mathbb{C}$. So, $f\left(AB^{-1}x\right) = 0$. As x and f were arbitrary with $f(x) = f\left(B^{-1}x\right) = 0$, we can obtain that AB^{-1}, I and B^{-1} are linearly dependent by [9, Lemma 2.4]. Hence, there exists $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, not all zero, such that $\alpha_1 AB^{-1} + \alpha_2 I + \alpha_3 B^{-1} = 0$, which is equivalent to $\alpha_1 A + \alpha_2 B + \alpha_3 I = 0$. Since $A \notin \mathbb{C}I$ it is obvious that $\alpha_2 \neq 0$. Thus

$$\phi(A) = B = \alpha_A A + \beta_A I, \tag{6}$$

for some scalars α_A and β_A. In order to see that $\alpha_A = 1$, take any $C \in \mathscr{F}(X)$ such that A, C and I are linearly independent. Obviously, $A + C \notin \mathbb{C}I + \mathscr{F}(\mathscr{X})$ and by applying (6) we get $\phi(A+C) = \alpha_{A+C}(A+C) + \beta_{A+C}I$. On the other hand, $\phi(A+C) = \phi(A) + \phi(C) = \alpha_A A + \beta_A I + C$. Therefore

$$(\alpha_{A+C} - \alpha_A)A + (\alpha_{A+C} - 1)C + (\beta_{A+C} - \beta_A)I = 0.$$

As A, C and I are linearly independent, $\alpha_A = \alpha_{A+C} = 1$.

STEP 9. *$\phi(A) = A$ for every $A \in \mathscr{B}(\mathscr{X})$.*

We will prove this by a contradiction, so let us assume, by STEP 9, that there exists an $A \notin \mathbb{C}I + \mathscr{F}(\mathscr{X})$ with

$$\phi(A) = A + \alpha_A I,$$

for some non-zero $\alpha_A \in \mathbb{C}$. Choose any $x \in \mathscr{X}$ and any $f \in \mathscr{X}'$ such that $f(x) = 0$. According to Lemma 1, the operator $I + \lambda x \otimes f \in$ P-Inv$(\mathscr{X})$ for every $\lambda \in \mathbb{C}$. From

$$A - \alpha_A I \sim_p (A - \alpha_A I)(I + \lambda x \otimes f) = (A - \alpha_A I) + \lambda\,(A - \alpha_A I)x \otimes f,$$

being valid for every $\lambda \in \mathbb{C}$ and by the action of ϕ it follows

$$A \sim A + \lambda\,(A - \alpha_A I)x \otimes f,$$

for every $\lambda \in \mathbb{C}$. If A is invertible, then

$$A + \lambda\,(A - \alpha_A I)x \otimes f = A\left(I + \lambda\left(I - \alpha_A A^{-1}\right)x \otimes f\right)$$

is invertible too. Thus $I + \lambda\left(I - \alpha_A A^{-1}\right)x \otimes f$ is invertible for every $\lambda \in \mathbb{C}$ and by Lemma 3 we have $\lambda f\left(x - \alpha_A A^{-1}x\right) \neq -1$ for every $\lambda \in \mathbb{C}$. Consequently, $0 = f\left(x - \alpha_A A^{-1}x\right) = -\alpha_A f\left(A^{-1}x\right)$. As $\alpha_A \neq 0$ it follows $f\left(A^{-1}x\right) = 0$ for every $f \in \mathscr{X}'$ with $f(x) = 0$. Hence, $A^{-1}x$ and x are linearly dependent. Since $x \in \mathscr{X}$ was arbitrary, there exists a non-zero $\mu_1 \in \mathbb{C}$ such that $A^{-1} = \mu_1 I$, a contradiction.

Therefore, A is non-invertible. But then there exists a non-zero $\beta \in \mathbb{C}$ such that $A+\beta I$ is invertible. By the method used above, we get $(A+\beta I)^{-1} = \mu_2 I$ for some $\mu_2 \in \mathbb{C}$, a contradiction. $\square$

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