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SELFADJOINT OPERATORS, NORMAL OPERATORS, AND CHARACTERIZATIONS

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Abstract. Let $\mathfrak{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space H. We shall show that:

1. The class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by scalars is characterized by

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geqslant 2\left\|SXS\right\|,\quad (S \in \mathfrak{B}(H)).$$

2. The class of all normal operators in $\mathfrak{B}(H)$ is characterized by each of the three following properties (where $D_S = S^*S - SS^*$, for $S \in \mathfrak{B}(H)$),

   (i) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geqslant 2\left\|SXS\right\|, (S \in \mathfrak{B}(H))$,

   (ii) $S^*D_SS = 0 = SD_SS^*, (S \in \mathfrak{B}(H))$,

   (iii) $S^*D_SS \geqslant 0 \geqslant SD_SS^*, (S \in \mathfrak{B}(H))$.

1. Introduction and preliminaries

Let $\mathfrak{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space H.

We denote by:

- $D_S = S^*S - SS^*$, the selfcommutator of $S \in \mathfrak{B}(H)$,
- $(M)_1 = M \cap \{x \in X : \|x\| = 1\}$, where X is any normed space and M is a subset of X,
- $x \otimes y$ (where $x, y \in H$), the operator (of rank less or equal to one) on H defined by $(x \otimes y)z = \langle z, y\rangle x$, for every $z \in H$.

It is known that:

(i) for $S \in \mathfrak{B}(H)$, S is with closed range if and only there exits a unique operator S^+ (the Moore-Penrose inverse of S) in $\mathfrak{B}(H)$ such that $SS^+S = S$, $S^+SS^+ = S^+$, $(SS^+)^* = SS^+$, and $(S^+S)^* = S^+S$.

(ii) if $S \in \mathfrak{B}(H)$ and is invertible, then $S^{-1} = S^+$.

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1.1. Selfadjoint operators and arithmetic-geometric-mean inequality

In [4], Heinz proved that for every two positive operators P and Q in $\mathfrak{B}(H)$, and for every $\alpha \in [0,1]$, the following operator inequality holds

$$\forall X \in \mathfrak{B}(H),\ \|PX + XQ\| \geqslant \left\|P^{\alpha}XQ^{1-\alpha} + P^{1-\alpha}XQ^{\alpha}\right\|. \tag{HI}$$

Making $\alpha = \frac{1}{2}$ in the above inequality, we obtain the well known arithmetic-geometric mean inequality given by

$$\forall A,B,X \in \mathfrak{B}(H),\ \|A^*AX + XBB^*\| \geqslant 2\,\|AXB\|. \tag{$AGMI1$}$$

Note that the proof of (HI) given by Heinz is somewhat complicated. For this reason, McIntosh [5] with an elegant proof, proved that the operator inequality $(AGMI1)$ holds, and deduced from it the Heinz inequality by iteration method.

Independently of the work of Heinz and McIntosh, Corach et al. proved in [1], that for every invertible selfadjoint operator S in $\mathfrak{B}(H)$, the following inequality holds

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\,\|X\|, \tag{$CPRI1$}$$

In [2], it was proved that the three above inequalities are equivalent, and we found an easy proof of $(CPR1)$. This gives us another proof of Heinz inequality.

Recently [11], it was given three other inequalities equivalent to the Heinz inequality listed as follows:

- For every selfadjoint operator with closed range S in $\mathfrak{B}(H)$, the following inequality holds

$$\forall X \in \mathfrak{B}(H), \left\|SXS^{+} + S^{+}XS\right\| \geqslant 2\left\|SS^{+}XS^{+}S\right\|. \tag{$S1$}$$

- For every selfadjoint operator with closed range S in $\mathfrak{B}(H)$, the following inequality holds

$$\forall X \in \mathfrak{B}(H), \left\|S^2X + XS^2\right\| \geqslant 2\,\|SXS\|. \tag{$S2$}$$

- For every selfadjoint operator S in $\mathfrak{B}(H)$, the following inequality holds

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geqslant 2\,\|SXS\|. \tag{$S3$}$$

Then, we have a family of four operator inequalities generated by a selfadjoint operator (invertible, with closed range, and any) that are equivalent to the Heinz inequality (or to $(AGMI1)$), and each of them holds.

It is natural to know about the maximal form of each inequality of this family. This kind of problem was introduced in [6], where it was proved that an invertible operator $S \in \mathfrak{B}(H)$ satisfies the inequality $(CPRI1)$ if and only if S is an invertible selfadjoint operator multiplied by a nonzero scalar. In [9, 10], it was shown that an operator with closed range $S \in \mathfrak{B}(H)$ satisfies the inequality $(S1)$ (or $(S2)$) if and only if S is a selfadjoint operator with closed range multiplied by a scalar. Note that this last characterization concerning the operator with closed range case follows from the first characterization with the invertible operator case.

In this note, we shall show in the general situation, that an operator $S \in \mathfrak{B}(H)$ satisfies the inequality $(S3)$ if and only if S is a selfadjoint operator multiplied by a scalar.

1.2. Normal operators and arithmetic-geometric mean inequality

Recently [11], a second family of operator inequalities was given, a family of four inequalities that are equivalent to a second arithmetic-geometric mean inequality given by

$$\forall A,B,X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XBB^*\| \geqslant 2\|AXB\|. \tag{AGMI2}$$

This family is given by:

- For every invertible normal operator S in $\mathfrak{B}(H)$, the following inequality holds
$$\forall X \in \mathfrak{B}(H), \left\| SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\|. \tag{CPRI2}$$
- For every normal operator with closed range S in $\mathfrak{B}(H)$, the following inequality holds
$$\forall X \in \mathfrak{B}(H), \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \geqslant 2\left\|SS^{+}XS^{+}S\right\|. \tag{N1}$$
- For every normal operator with closed range S in $\mathfrak{B}(H)$, the following inequality holds
$$\forall X \in \mathfrak{B}(H), \left\|S^2X\right\| + \left\|XS^2\right\| \geqslant 2\|SXS\|. \tag{N2}$$
- For every normal operator S in $\mathfrak{B}(H)$, the following inequality holds
$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geqslant 2\|SXS\|. \tag{N3}$$

Note that the inequality $(AGMI2)$ follows immediately from $(AGMI1)$. In [11], a direct proof of $(AGMI2)$ was given independently of $(AGMI1)$.

As with the first above family, it is interesting to know about the maximal form of each inequality of this second family. In [7], it was proved that an invertible operator $S \in \mathfrak{B}(H)$ satisfies the inequality $(CPRI2)$ if and only if S is an invertible normal operator. In [9, 10], it was shown that an operator with closed range $S \in \mathfrak{B}(H)$ satisfies the inequality $(N1)$ (or $(N2)$) if and only if S is a normal operator with closed range. Note that this last characterization concerning the operator with closed range case follows from the first characterization with the invertible operator case.

In this note, we shall show in the general situation, that an operator $S \in \mathfrak{B}(H)$ satisfies the inequality $(N3)$ if and only if S normal.

By using this last characterization in term of operator inequality of the class of normal operators, we shall show two general characterizations of this class given by

- $S^*D_SS = 0 = SD_SS^*,\ \ (S \in \mathfrak{B}(H))$.
- $S^*D_SS \geqslant 0 \geqslant SD_SS^*,\ \ (S \in \mathfrak{B}(H))$.

2. Characterizations of some distinguished classes of operators in terms of operator inequalities

In this section, we shall give characteristic properties in terms of operator inequalities for the two distinguished classes of operators, namely the class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by scalars, and the class of all normal operators in $\mathfrak{B}(H)$. Also, we shall present two general characterizations of this last class.

Along this section, let $S \in \mathfrak{B}(H)$, and since the class of all operators with closed ranges in $\mathfrak{B}(H)$ is dense in $\mathfrak{B}(H)$ (see [3, Ex. 140, p. 75]), we denote by $(S_n)_{n\geqslant 1}$ a sequence of operators with closed ranges in $\mathfrak{B}(H)$ that converges uniformly to S.

PROPOSITION 1. *The two following properties are equivalent:*
(i) *S is a selfadjoint operator multiplied by a scalar,*
(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geqslant 2\left\|SXS\right\|$.

Proof. We may assume without loss of generality that $\|S\| = 1$.
$(i) \Rightarrow (ii)$. This implication follows immediately from $(AGMI1)$.
$(ii) \Rightarrow (i)$. Assume (ii) holds.
Applying triangular inequality in (ii), we deduce that $\left\|S^2\right\| = \|S\|^2 = 1$.
Define the real function F on the complete metric space $(\mathfrak{B}(H))_1$ by $F(X) = \left\|S^2X + XS^2\right\| - 2\left\|SXS\right\|$, for $X \in (\mathfrak{B}(H))_1$; and for $n \geqslant 1$, define the real function F_n on $(\mathfrak{B}(H))_1$ by $F_n(X) = \left\|S_n^2X + XS_n^2\right\| - 2\left\|S_nXS_n\right\|$, for $X \in (\mathfrak{B}(H))_1$.
Put $D = \{X \in (\mathfrak{B}(H))_1 : F(X) > 0\}$. Then there are two cases, $D = \varnothing$, $D \neq \varnothing$.
Case 1. $D = \varnothing$. So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| = 2\left\|SXS\right\|. \qquad (*)$$

From this equality, we have

$$\forall x, y \in H,\ \left\|S^2x \otimes y + x \otimes S^{*2}y\right\| = 2\left\|Sx\right\|\left\|S^*y\right\|.$$

Using this last equality and since $S^2 \neq 0$, we deduce that $\ker S^* = \{0\}$. Hence, S is with dense range. Using again this last equality, we obtain the following inequality,

$$\forall x, y \in (H)_1,\ \left\|S^2x\right\| + 2\left\|Sx\right\|\left\|S^*y\right\| \geqslant \left\|S^{*2}y\right\|.$$

By taking the supremum over $y \in (H)_1$, we obtain that $\|Sx\| \geqslant \frac{1}{3}\|x\|$, for every $x \in H$. Thus, S is bounded below with dense range. Hence, S is invertible. So, from $(*)$, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\left\|X\right\|.$$

Then from [8], S is a unitary reflection operator in $\mathfrak{B}(H)$ multiplied by a nonzero scalar. This gives us that (i) holds.

Case 2. $D \neq \varnothing$. From the fact that F is a positive continuous map on $(\mathfrak{B}(H))_1$, it follows that

$$\overline{D} = \overline{F^{-1}((0,\infty))} = F^{-1}([0,\infty)) = \{X \in (\mathfrak{B}(H))_1 : F(X) \geqslant 0\} = (\mathfrak{B}(H))_1.$$

Let $X \in D$, and $\varepsilon > 0$. Since $S_n \to S$ uniformly, then there exists an integer $N \geqslant 1$ (depends only in ε) such that

$$\forall n \geqslant N,\ \forall Y \in (\mathfrak{B}(H))_1,\ |F(Y) - F_n(Y)| \leqslant \varepsilon.$$

If there exists $n \geqslant N$ such that $F_n(X) < 0$, then using this last inequality, we have $0 \leqslant F(X) < \varepsilon$, for every $\varepsilon > 0$; thus $F(X) = 0$, leading a contradiction with $X \in D$.

From this fact, it follows that

$$\forall X \in D,\ \forall n \geqslant N,\ \ F_n(X) \geqslant 0.$$

Since each F_n is a continuous map on $(\mathfrak{B}(H))_1$ and D is dense in $(\mathfrak{B}(H))_1$, then

$$\forall X \in (\mathfrak{B}(H))_1,\ \forall n \geqslant N,\ F_n(X) \geqslant 0.$$

So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \forall n \geqslant N,\ \left\|S_n^2 X + X S_n^2\right\| \geqslant 2\left\|S_n X S_n\right\|.$$

Since for each $n \geqslant 1$, S_n is with closed range, then from the corresponding characterization in the case of operators with closed ranges [9, 10], we obtain that S_n is a selfadjoint operator with closed range multiplied by a scalar, for every $n \geqslant N$. Since $S_n \to S$ uniformly, and the class of all selfadjoint operators in $\mathfrak{B}(H)$ is closed, then S is a selfadjoint operator multiplied by a scalar. □

REMARK 1.

1. The class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by scalars is characterized by

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2 X + X S^2\right\| \geqslant 2\left\|S X S\right\|, \quad (S \in \mathfrak{B}(H)).$$

2. From a part of the above proof, the class of all unitary reflection operators in $\mathfrak{B}(H)$ multiplied by scalars is characterized by

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2 X + X S^2\right\| = 2\left\|S X S\right\|, \quad (S \in \mathfrak{B}(H)).$$

3. Proposition 1 gives us a positive answer of Problem 1 in [11].

PROPOSITION 2. *The two following properties are equivalent:*

(i) S *is normal,*

(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2 X\right\| + \left\|X S^2\right\| \geqslant 2\left\|S X S\right\|$.

Proof. We may assume without loss of generality that $\|S\| = 1$.

$(i) \Rightarrow (ii)$. Assume (i) holds and let $X \in \mathfrak{B}(H)$. From the fact that S is normal and applying $(AGMI2)$, we deduce

$$\left\|S^2 X\right\| + \left\|X S^2\right\| = \left\|S^* S X\right\| + \left\|X S S^*\right\| \geqslant 2\left\|S X S\right\|.$$

$(ii) \Rightarrow (i)$. Assume (ii) holds.

Then $\|S^2\| = \|S\|^2 = 1$.

Define the real function G on the complete metric space $(\mathfrak{B}(H))_1$ by $G(X) = \|S^2X\| + \|XS^2\| - 2\|SXS\|$, for $X \in (\mathfrak{B}(H))_1$; and for $n \geqslant 1$, define the real function G_n on $(\mathfrak{B}(H))_1$ by $G_n(X) = \|S_n^2X\| + \|XS_n^2\| - 2\|S_nXS_n\|$, for $X \in (\mathfrak{B}(H))_1$.

Put $L = \{X \in (\mathfrak{B}(H))_1 : G(X) > 0\}$. Then there are two cases, $L = \varnothing$, $L \neq \varnothing$.

Case 1. $L = \varnothing$. So, it follows that

$$\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| = 2\|SXS\|. \qquad (*)$$

From this equality, we have

$$\forall x, y \in H,\ \|S^2x\|\|y\| + \|x\|\|S^{*2}y\| = 2\|Sx\|\|S^*y\|.$$

From this, and using the same argument as used in the same case of Proposition 1, we obtain that S is invertible.

So, from $(*)$, it follows that

$$\forall X \in \mathfrak{B}(H),\ \|SXS^{-1}\| + \|S^{-1}XS\| = 2\|X\|.$$

Then from [8], S is a unitary operator in $\mathfrak{B}(H)$ multiplied by a nonzero scalar. This gives us that (i) holds.

Case 2. $L \neq \varnothing$. Using the same argument as used in the proof of Proposition 1 with the function F, we find $\overline{L} = (\mathfrak{B}(H))_1$, and there exists an integer $N \geqslant 1$ such that

$$\forall X \in L,\ \forall n \geqslant N,\ G_n(X) \geqslant 0.$$

Since each G_n is a continuous map on $(\mathfrak{B}(H))_1$ and L is dense in $(\mathfrak{B}(H))_1$, then

$$\forall X \in (\mathfrak{B}(H))_1,\ \forall n \geqslant N,\ G_n(X) \geqslant 0.$$

So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \forall n \geqslant N,\ \|S_n^2X\| + \|XS_n^2\| \geqslant 2\|S_nXS_n\|.$$

Since for each $n \geqslant 1$, S_n is with closed range, then from the corresponding characterization in the case of operators with closed ranges in [9, 10], we obtain that S_n is a normal operator, for every $n \geqslant N$. Since $S_n \to S$ uniformly and the class of all normal operators in $\mathfrak{B}(H)$ is closed, then S is a normal operator. □

REMARK 2. 1. The class of all normal operators in $\mathfrak{B}(H)$ is characterized by

$$\forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| \geqslant 2\|SXS\|, \quad (S \in \mathfrak{B}(H)).$$

2. From a part of the above proof, the class of all unitary operators in $\mathfrak{B}(H)$ multiplied by scalars is characterized by

$$\forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| = 2\|SXS\|, \quad (S \in \mathfrak{B}(H)).$$

3. Proposition 2 gives us a positive answer of Problem 2 in [11].

3. Application: new general characterizations of the class of all normal operators

For $S \in \mathfrak{B}(H)$, from the definition, S is normal if and only if $D_S = 0$; and it is well known that S is normal if and only if $SD_S = D_S S$. In this section, we shall present two new general forms of characterizations of the class of all normal operators.

PROPOSITION 3. *Let $S \in \mathfrak{B}(H)$. The following properties are equivalent:*

(i) S *is normal,*

(ii) $S^* D_S S = 0 = SD_S S^*$,

(iii) $S^* D_S S \geqslant 0 \geqslant SD_S S^*$.

Proof. The two implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.
$(iii) \Rightarrow (i)$. Assume (iii) holds. So, we have

$$\begin{cases} \forall x \in H, \ \left\|S^2 x\right\| \geqslant \|S^* S x\|, \\ \forall x \in H, \ \left\|S^{*2} x\right\| \geqslant \|SS^* x\|. \end{cases}$$

Hence,

$$\begin{cases} \forall X \in \mathfrak{B}(H), \ \left\|S^2 X\right\| \geqslant \|S^* S X\|, \\ \forall X \in \mathfrak{B}(H), \ \left\|X S^2\right\| \geqslant \|X S S^*\|. \end{cases}$$

So we obtain,

$$\forall X \in \mathfrak{B}(H), \ \left\|S^2 X\right\| + \left\|X S^2\right\| \geqslant \|S^* S X\| + \|X S S^*\|.$$

Using $(AGMI2)$, we deduce that

$$\forall X \in \mathfrak{B}(H), \ \left\|S^2 X\right\| + \left\|X S^2\right\| \geqslant 2\|S X S\|.$$

Applying Theorem 2, (i) holds. □

REMARK 3. The class of all normal operators in $\mathfrak{B}(H)$ is characterized by each of the two following properties:

- $S^* D_S S = 0 = SD_S S^*, \ (S \in \mathfrak{B}(H))$.
- $S^* D_S S \geqslant 0 \geqslant SD_S S^*, \ (S \in \mathfrak{B}(H))$.

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