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SOME INEQUALITIES INVOLVING POSITIVE LINEAR MAPS UNDER CERTAIN CONDITIONS

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Abstract. We demonstrate that several well-known classical inequalities also hold for some positive linear maps on matrix algebra. It is shown that for such maps the Jensen inequality hold for all ordinary convex functions.

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices and let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be a positive linear map [3]. A fundamental inequality of Kadison [11] says that if A is any Hermitian element of $\mathbb{M}_n$, then

$$\Phi(A^2) \geqslant \Phi(A)^2. \tag{1}$$

The inequality (1) is a non-commutative analogue of the classical inequality

$$\mathbb{E}(X^2) \geqslant \mathbb{E}(X)^2, \tag{2}$$

where X is a random variable with finite expectation $\mathbb{E}(X)$. The inequality (2) has a simple proof. For any real number $\alpha, (x-\alpha)^2 \geqslant 0,$ therefore $\mathbb{E}(X^2) \geqslant 2\alpha\mathbb{E}(X) - \alpha^2,$ and choice $\alpha = \mathbb{E}(X)$ yields (2). Kadison [11] remarks that the standard proof of (2) does not apply to give simple proof for linear maps. That is, when $\Phi(A)$ and $\Phi(A^2)$ do not commute, one can not conclude the desired inequality (1) by these means as the following example shows. Choose

$$B = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For every real number α, we have

$$B - 2\alpha C + \alpha^2 I = \begin{bmatrix} (\alpha-1)^2+1 & \sqrt{2} \\ \sqrt{2} & (\alpha-1)^2+1 \end{bmatrix} \geqslant O.$$

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But $B \ngeq C^2$. The non-commutativity of the image of the Φ forces us to abandon the usual techniques and the standard proof does not apply to give simple proof for linear maps. One of the proof of the inequality (1) follows on using the tensor product of matrices [3]. The inequality (2) is included in the more general Jensen's inequality [10]. The Jensen inequality [10] says that if the range of the random variable X is contained in (a,b) and f is a convex function on (a,b), then

$$\mathbb{E}(f(X)) \geqslant f(\mathbb{E}(X)). \tag{3}$$

It is then natural to investigate the general version of Kadison's inequality (1): if A is a Hermitian matrix whose spectrum is contained in (a,b) and f is a convex function on (a,b) then do we have

$$\Phi(f(A)) \geqslant f(\Phi(A)). \tag{4}$$

Davis [8] proved that the inequality (4) is true when f is a matrix convex function and Φ is completely positive. The latter restriction was removed by Choi [7] who proved that the inequality (4) remains valid for all unital positive linear maps Φ provided f is matrix convex function. Bhatia and Sharma [4] have shown that if $\Phi : \mathbb{M}_2 \to \mathbb{M}_k$ then (4) is true for all convex functions f on an open interval containing the eigenvalues of an Hermitian element A of $\mathbb{M}_2$. Sharma and Thakur [12] have proved several other inequalities for unital positive linear maps on 2×2 matrices. Bourin and Ricard [5] have obtained the extension of Kadison's inequality (1) for positive definite matrices. They proved that for $0 \leqslant p \leqslant q$,

$$|\Phi(A^p)\Phi(A^q)| \leqslant \Phi(A^{p+q}), \tag{5}$$

where $|\Phi(A^p)\Phi(A^q)| = \left(\Phi(A^q)\Phi(A^p)^2\Phi(A^q)\right)^{\frac{1}{2}}$ is meant in the sense of spectral calculus. Further, Audenaert and Hiai [1] have studied the conditions for which the operator inequality,

$$\Phi(A^p)^{\frac{1}{p}} \leqslant \Phi(A^q)^{\frac{1}{q}}, \tag{6}$$

holds for every positive definite matrix A. Sharma and Thakur [12] have also proved that if $\Phi : \mathbb{M}_2 \to \mathbb{M}_n$ is a unital positive linear map, then (6) holds for every positive semidefinite matrices A and B and every $p,q \in \mathbb{R}$ with $p \leqslant q$.

In Section 2 we consider the unital positive linear maps which satisfy the conditions given in Lemma 1, (see below). It is shown that for such maps the Jensen inequality (7) holds for all ordinary convex functions (Theorem 1). The operator inequality (8) holds for all real numbers p and q such that $p \leqslant q$ (Corollary 1). The Čebyšev's inequality and its special cases are discussed for these linear maps (Theorem 2 and 3, Corollary 2 and 3). Likewise, we obtain some inequalities related to Schwarz inequalities for such unital positive linear maps (Theorem 4, 5 and 6). In Section 3 we mention several examples of maps for which these inequalities hold for all Hermitian or positive definite matrices and examples of linear maps for which inequalities hold for particular matrices.

2. Main results

We begin with the following lemma in which a condition is imposed on map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$. This assumption on map Φ is motivated by the fact that some well known maps possess this property one such map is known as Werner-Holevo Channel [13] in physics literature. See Section 3.

LEMMA 1. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be a unital positive linear map. Let A be a Hermitian element of $\mathbb{M}_n$ with spectral resolution $A = \sum_{i=1}^{n} \lambda_i P_i$. Then, $\Phi(P_i)$ and $\Phi(P_j)$ commute if and only if $\Phi\left(A^l\right)$ and $\Phi(A^m)$ commute, $l, m = 1, 2, \ldots, n-1$.*

Proof. The assertions of the Lemma 1 follow from the fact that the orthogonal projections are polynomial in A, see [9], and therefore for some scalars c_{ik}, we can write

$$\Phi(P_i) = \sum_{k=1}^{n} c_{ik} \Phi\left(A^{k-1}\right). \quad \square$$

We also need the following lemma to prove Jensen's inequality for the maps which satisfy condition of Lemma 1.

LEMMA 2. *Let $A_i \in \mathbb{M}_n$ be commuting Hermitian matrices and let α_i be real numbers, $i = 1, 2, \ldots, m$. Then for any convex function f defined over the interval containing eigenvalues of A_i's we have*

$$f\left(\sum_{i=1}^{m} \alpha_i A_i\right) \leqslant \sum_{i=1}^{m} f(\alpha_i) A_i.$$

Proof. By spectral theorem if $A \in \mathbb{M}_n$ is Hermitian then there exists a unitary $U \in \mathbb{M}_n$ such that $UAU^* = \mathrm{diag}(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$ where $\lambda_j(A)$, $(j = 1, 2, \ldots, n)$, are the eigenvalues of A and $f(A)$ is defined as

$$f(A) = U f(D) U^*,$$

where $f(D) = \mathrm{diag}(f(\lambda_1(A)), f(\lambda_2(A)), \ldots, f(\lambda_n(A)))$, see [2]. Since commuting matrices are simultaneously diagonalizable, there exists a unitary $U \in \mathbb{M}_n$ such that $UA_iU^* = D_i$ where $D_i = \mathrm{diag}(d_{i1}, d_{i2}, \ldots, d_{in})$. Also, f is a convex function, therefore

$$f\left(\sum_{i=1}^{m} \alpha_i d_{ij}\right) \leqslant \sum_{i=1}^{m} f(\alpha_i) d_{ij},$$

for all $j = 1, 2, \ldots, n$. It follows that

$$f\left(\sum_{i=1}^{m} \alpha_i D_i\right) \leqslant \sum_{i=1}^{m} f(\alpha_i) D_i.$$

So

$$f\left(\sum_{i=1}^{m}\alpha_i A_i\right)=Uf\left(\sum_{i=1}^{m}\alpha_i D_i\right)U^*\leqslant U\sum_{i=1}^{m}f(\alpha_i)D_iU^*=\sum_{i=1}^{m}f(\alpha_i)A_i. \quad \square$$

In the following discussion whenever we say that a map Φ satisfies the conditions of Lemma 1, we mean that $\Phi:\mathbb{M}_n\to\mathbb{M}_k$ is a unital positive linear map and for the given Hermitian matrix A, $\Phi\left(A^l\right)$ and $\Phi(A^m)$ commute, $l,m=1,2,\ldots,n-1$.

THEOREM 1. *(Jensen's inequality) Let a map Φ satisfy the conditions of Lemma 1. Let f be a convex function on an open interval containing the eigenvalues of a Hermitian matrix $A\in\mathbb{M}_n$. Then*

$$f(\Phi(A))\leqslant\Phi(f(A)). \tag{7}$$

Proof. By the spectral theorem and linearity of Φ, we have

$$\Phi(A)=\sum_{i=1}^{n}\lambda_i\Phi(P_i).$$

It follows from Lemma 1 that $\Phi(P_i)$ and $\Phi(P_j)$ commute, for $i,j=1,2,\ldots,n$. Therefore, on using Lemma 2, we conclude that

$$f(\Phi(A))=f\left(\sum_{i=1}^{n}\lambda_i\Phi(P_i)\right)\leqslant\sum_{i=1}^{n}f(\lambda_i)\Phi(P_i)=\Phi\left(\sum_{i=1}^{n}f(\lambda_i)P_i\right)=\Phi(f(A)). \quad \square$$

Audenaert and Hiai [1] have shown that the operator inequality (6) holds for every unital positive linear map $\Phi:\mathbb{M}_2\to\mathbb{M}_k$ and every $p,q\in\mathbb{R}$ with $p\leqslant q$, $A>O$. We show that this is also true for maps $\Phi:\mathbb{M}_n\to\mathbb{M}_k$ satisfying the conditions of Lemma 1.

COROLLARY 1. *For a map Φ satisfying the conditions of Lemma 1, the operator inequality,*

$$\Phi(A^p)^{\frac{1}{p}}\leqslant\Phi(A^q)^{\frac{1}{q}}, \tag{8}$$

holds for every $p,q\in\mathbb{R}$ with $p\leqslant q$ and $A>O$.

Proof. The function $f(x)=x^r$ is convex for $r<0$ or $r\geqslant1$, $x\geqslant0$. It follows from Theorem 1 that

$$\Phi(A^r)\geqslant\Phi(A)^r.$$

Therefore, for $q>p\geqslant0$,

$$\Phi(A^q)=\Phi\left((A^p)^{\frac{q}{p}}\right)\geqslant(\Phi(A^p))^{\frac{q}{p}}. \tag{9}$$

Since $\Phi(A^p)$ and $\Phi(A^q)$ commute, the inequality (8) follows from (9). Likewise, the inequality (8) holds for $p<0$ and $q>0$. For $p \leqslant q \leqslant 0$, we have

$$\Phi(A^p) = \Phi\left((A^q)^{\frac{p}{q}}\right) \geqslant (\Phi(A^q))^{\frac{p}{q}}. \tag{10}$$

The inequality (10) implies (8), $p<0$. $\square$

Let $A = \sum_{i=1}^{n} \lambda_i P_i$ and $B = \sum_{i=1}^{n} \mu_i P_i$ be the spectral resolutions of commuting Hermitian elements $A, B \in \mathbb{M}_n$; [2]. We say that the spectra of A and B are similarly ordered if $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ or $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$. Likewise, the spectra of A and B are oppositely ordered if $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$ or $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$.

We now prove an extension of Čebyšev's inequality [6] for linear maps satisfying the conditions of Lemma 1.

THEOREM 2. *(Čebyšev's inequality) Let Φ be a map that satisfies the conditions of Lemma 1. Let A and B be commuting Hermitian matrices in $\mathbb{M}_n$. If the spectra of A and B are similarly ordered, then*

$$\Phi(AB) \geqslant \Phi(A)\Phi(B). \tag{11}$$

If the spectra of A and B are oppositely ordered, then

$$\Phi(AB) \leqslant \Phi(A)\Phi(B). \tag{12}$$

Proof. It follows from Lemma 1 that $\Phi(P_i)$ and $\Phi(P_j)$ commute, for $i, j = 1, 2, ..., n$. Therefore,

$$\Phi(AB) - \Phi(A)\Phi(B) = \sum_{i<j}^{n} (\lambda_i - \lambda_j)(\mu_i - \mu_j)\Phi(P_i)\Phi(P_j). \tag{13}$$

From (13), we conclude that the inequality (11) holds when $(\lambda_i - \lambda_j)(\mu_i - \mu_j) \geqslant 0$, that is, if the spectra of A and B are similarly ordered. Likewise, (13) implies (12) when $(\lambda_i - \lambda_j)(\mu_i - \mu_j) \leqslant 0$, that is, if the spectra of A and B are oppositely ordered. $\square$
Let $X, Y \in \mathbb{M}_n$. Then $X \geqslant Y$ means that X and Y are Hermitian and $X - Y$ is positive semidefinite. Also, the product of two Hermitian matrices X and Y is Hermitian if and only if X and Y commute. Thus the condition in the above theorem that A and B are commuting Hermitian matrices is necessary.

Note that under the above conditions on Φ we have $|\Phi(A^p)\Phi(A^q)| = \Phi(A^p)\Phi(A^q)$ and (5) becomes $\Phi(A^{p+q}) \geqslant \Phi(A^p)\Phi(A^q)$. Such inequalities now follow easily from Theorem 2.

COROLLARY 2. *Let $A \in \mathbb{M}_n$ be a positive definite matrix and let Φ satisfies the conditions of Lemma 1. If p and q are real numbers such that $pq > 0$, then*

$$\Phi(A^{p+q}) \geqslant \Phi(A^p)\Phi(A^q). \tag{14}$$

If $pq < 0$, the reverse inequality holds.
Also, we have

$$\Phi(A \log A) \geqslant \Phi(A)\Phi(\log A) \tag{15}$$

and

$$\Phi\left(A^{-1}\log A\right) \leqslant \Phi\left(A^{-1}\right)\Phi(\log A). \tag{16}$$

Proof. It is clear that the spectra of A^p and A^q are similarly ordered when $pq > 0$ and oppositely ordered when $pq < 0$. Therefore, the assertions of the corollary about the inequality (14) follows from Theorem 2. Spectra of A and $\log A$ are similarly ordered, (15) therefore follows from (11). Likewise, (12) implies (16). $\square$

COROLLARY 3. *Let the spectra of commuting positive definite matrices $A_1, A_2, \ldots, A_n$ be similarly ordered. Let Φ satisfies the conditions of Lemma 1. Then*

$$\Phi\left(\prod_{i=1}^{n} A_i\right) \geqslant \prod_{i=1}^{n} \Phi(A_i). \tag{17}$$

Let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be a unital positive linear map and $A > O$. By Kadison's inequality (1), we have

$$\Phi\left(A^{p+q}\right) \geqslant \Phi\left(A^{\frac{p+q}{2}}\right)^2. \tag{18}$$

We prove a refinement of the inequality (18) in the following theorem.

THEOREM 3. *Let Φ be a map that satisfies the conditions of Lemma 1. Then, for $A > O$,*

$$\Phi\left(A^{\frac{p+q}{2}}\right)^2 \leqslant \Phi(A^p)\Phi(A^q) \leqslant \Phi\left(A^{p+q}\right), \tag{19}$$

where p and q are real numbers such that $pq > 0$.

Proof. Let λ_i be the eigenvalues of the matrix $A > O$ and let X be a Hermitian matrix that commute with each $\Phi(P_i)$, $i = 1, 2, \ldots, n$. We then have

$$\left(\lambda_i^{\frac{p}{2}} - X\lambda_i^{\frac{q}{2}}\right)^2 \Phi(P_i) \geqslant O, \tag{20}$$

for all $i = 1, 2, \ldots, n$. Add n inequalities (20), we get

$$\Phi(A^p) \geqslant 2X\Phi\left(A^{\frac{p+q}{2}}\right) - X^2\Phi(A^q). \tag{21}$$

On the other hand for any Hermitian matrix X which commute with $\Phi(A^q)$ and $\Phi\left(A^{\frac{p+q}{2}}\right)$ we have

$$\left(\Phi\left(A^{\frac{p+q}{2}}\right) - X\Phi(A^q)\right)^2 \geqslant O. \tag{22}$$

Further if C and D are commuting positive definite matrices then $C^{-1}D$ is also a positive definite matrix, and therefore it follows from (22) that

$$\Phi\left(A^{q}\right)^{-1}\left(\Phi\left(A^{\frac{p+q}{2}}\right)-X\Phi\left(A^{q}\right)\right)^{2}\geqslant O.$$

This gives

$$2X\Phi\left(A^{\frac{p+q}{2}}\right)-X^{2}\Phi\left(A^{q}\right)\leqslant\Phi\left(A^{\frac{p+q}{2}}\right)^{2}\Phi\left(A^{q}\right)^{-1}. \tag{23}$$

The equality is attained in (23) when

$$X=\Phi\left(A^{\frac{p+q}{2}}\right)\Phi\left(A^{q}\right)^{-1}. \tag{24}$$

The inequality (21) holds for all Hermitian matrices X which commute with $\Phi\left(A^{q}\right)$ and $\Phi\left(A^{\frac{p+q}{2}}\right)$. Therefore the inequality (21) must also hold for the Hermitian matrix X in (24). Inserting (24) in (21) we get

$$\Phi\left(A^{p}\right)-\Phi\left(A^{\frac{p+q}{2}}\right)^{2}\Phi\left(A^{q}\right)^{-1}\geqslant O. \tag{25}$$

The matrix $\Phi\left(A^{q}\right)$ is positive definite and commute with the left hand side matrix in (25), therefore

$$\Phi\left(A^{q}\right)\left(\Phi\left(A^{p}\right)-\Phi\left(A^{\frac{p+q}{2}}\right)^{2}\Phi\left(A^{q}\right)^{-1}\right)\geqslant O.$$

This immediately gives the first inequality (19). The second inequality (19) follows from Corollary 2. □

Arguments similar to those in the proof of above theorem give a version of Schwarz's inequality for such maps proved in the following theorem.

THEOREM 4. *(Schwarz's inequality) Let A and B be commuting Hermitian matrices with eigenvalues λ_i and μ_i respectively, $i=1,2,\dots,n$. For a map Φ satisfying the conditions of Lemma 1, we have*

$$\Phi\left(A^{2}\right)\Phi\left(B^{2}\right)\geqslant\Phi\left(AB\right)^{2}. \tag{26}$$

Proof. For a Hermitian matrix X that commute with each $\Phi\left(P_i\right)$, the inequality

$$\left(\lambda_i-X\mu_i\right)^{2}\Phi\left(P_i\right)\geqslant O, \tag{27}$$

holds for all $i=1,2,\dots,n$. Add n inequalities (27), we get

$$\Phi\left(A^{2}\right)\geqslant 2X\Phi\left(AB\right)-X^{2}\Phi\left(B^{2}\right). \tag{28}$$

By Lemma 1, $\Phi\left(P_i\right)$ and $\Phi\left(P_j\right)$ commute, $i,j=1,2,\dots,n$. Therefore, from the inequality

$$\left(\Phi\left(AB\right)-X\Phi\left(B^{2}\right)\right)^{2}\geqslant O,$$

we find that

$$2X\Phi(AB) - X^2\Phi\left(B^2\right) \leqslant \Phi(AB)^2\Phi\left(B^2\right)^{-1}. \tag{29}$$

Equality is attained in (29) when

$$X = \Phi(AB)\Phi\left(B^2\right)^{-1}. \tag{30}$$

Insert (30) in (28); we immediately get (26). $\square$

A special case of the above theorem says that if $A > O$,

$$\Phi\left(A^3\right) \geqslant \Phi\left(A^2\right)^2\Phi(A)^{-1} \geqslant \Phi\left(A^2\right)\Phi(A) \geqslant \Phi(A)^3.$$

A related classical inequality can be generalized immediately for positive unital linear functionals,

$$\varphi\left(A^4\right) \geqslant \frac{\left(\varphi\left(A^3\right)-\varphi(A)\varphi\left(A^2\right)\right)^2}{\varphi\left(A^2\right)-\varphi(A)^2} + \varphi\left(A^2\right)^2. \tag{31}$$

We show that inequalities of the type (31) also hold for maps Φ satisfying the conditions of Lemma 1. We prove an analogous inequality for $\Phi\left(A^{-1}\right)$ in the following theorem.

THEOREM 5. *For a positive definite matrix A, the inequality*

$$\Phi\left(A^{-1}\right) \geqslant \left(\Phi\left(A^3\right) - 2\Phi(A)\Phi\left(A^2\right) + \Phi(A)^3\right)\left(\Phi(A)\Phi\left(A^3\right) - \Phi\left(A^2\right)^2\right)^{-1}, \tag{32}$$

holds for maps Φ satisfying the conditions of Lemma 1.

Proof. Let X and Y be Hermitian matrices which commute with $\Phi(P_i)$, $i = 1,2,...,n$. Let λ_i be the eigenvalues of A, then the inequality

$$\frac{1}{\lambda_i}\left(\lambda_i^2 - \lambda_i X + Y\right)^2\Phi(P_i) \geqslant O, \tag{33}$$

holds for all $i = 1,2,...,n$. Add n inequalities (33), we get

$$\Phi\left(A^3\right) \geqslant 2X\Phi\left(A^2\right) - \left(X^2 + 2Y\right)\Phi(A) + 2XY - Y^2\Phi\left(A^{-1}\right). \tag{34}$$

The inequality (34) is valid for all Hermitian matrices X and Y which commute with $\Phi(P_i)$, $i = 1,2,...,n$. We see that the Hermitian matrices

$$X = \left(\Phi\left(A^2\right) - \Phi(A)\Phi\left(A^{-1}\right)^{-1}\right)\left(\Phi(A) - \Phi\left(A^{-1}\right)^{-1}\right)^{-1} \tag{35}$$

and

$$Y = \left(\Phi\left(A^2\right) - \Phi(A)^2\right)\left(\Phi(A) - \Phi\left(A^{-1}\right)^{-1}\right)^{-1}\Phi\left(A^{-1}\right)^{-1}, \tag{36}$$

commute with all $\Phi(P_i)$. Insert (35) and (36) into (34), we get

$$\Phi\left(A^3\right) \geqslant \left(\Phi\left(A^2\right) - \Phi(A)\Phi\left(A^{-1}\right)^{-1}\right)^2 \left(\Phi(A) - \Phi\left(A^{-1}\right)^{-1}\right)^{-1} + \Phi(A)^2\Phi\left(A^{-1}\right)^{-1}. \tag{37}$$

A little computation shows that (37) implies (32). □

It may be noted here that the inequality (32) can be written in the following equivalent form:

$$\begin{vmatrix} \Phi\left(A^{-1}\right) & I & \Phi(A) \\ I & \Phi(A) & \Phi\left(A^2\right) \\ \Phi(A) & \Phi\left(A^2\right) & \Phi\left(A^3\right) \end{vmatrix} \geqslant 0.$$

Also note that the inequality (32) gives a refinement of the Choi inequality [7],

$$\Phi\left(A^{-1}\right) \geqslant \Phi(A)^{-1}. \tag{38}$$

THEOREM 6. *Under the conditions of Lemma 1, the inequality*

$$\Phi\left(A^4\right) \geqslant \left(\Phi\left(A^3\right) - \Phi(A)\Phi\left(A^2\right)\right)^2 \left(\Phi\left(A^2\right) - \Phi(A)^2\right)^{-1} + \Phi\left(A^2\right)^2, \tag{39}$$

holds for all Hermitian matrices, $A \in \mathbb{M}_n$.

Proof. Let X and Y be Hermitian matrices which commute with $\Phi(P_i)$, $i = 1,2,\ldots,n$. Let λ_i be an eigenvalues of A, then

$$\left(\lambda_i^2 - \lambda_i X + Y\right)^2 \Phi(P_i) \geqslant O, \tag{40}$$

for all $i = 1,2,\ldots,n$. Add n inequalities (40), we get

$$\Phi\left(A^4\right) \geqslant 2X\Phi\left(A^3\right) - \left(X^2 + 2Y\right)\Phi\left(A^2\right) + 2XY\Phi(A) - Y^2. \tag{41}$$

The inequality (41) is valid for all Hermitian matrices X and Y which commute with $\Phi(P_i)$, $i = 1,2,\ldots,n$. We see that the Hermitian matrices

$$X = \left(\Phi\left(A^3\right) - \Phi(A)\Phi\left(A^2\right)\right)\left(\Phi\left(A^2\right) - \Phi(A)^2\right)^{-1} \tag{42}$$

and

$$Y = \left(\Phi\left(A^3\right)\Phi(A) - \Phi\left(A^2\right)\right)\left(\Phi\left(A^2\right) - \Phi(A)^2\right)^{-1}, \tag{43}$$

commute with all $\Phi(P_i)$. Insert (42) and (43) into (41), we get (39). □

It may be noted here that the inequality (39) can be written in the following equivalent form:

$$\begin{vmatrix} I & \Phi(A) & \Phi\left(A^2\right) \\ \Phi(A) & \Phi\left(A^2\right) & \Phi\left(A^3\right) \\ \Phi\left(A^2\right) & \Phi\left(A^3\right) & \Phi\left(A^4\right) \end{vmatrix} \geqslant 0.$$

3. Numerical examples

We give some examples of the unital positive linear maps which satisfy the conditions of Lemma 1 and consequently all the above inequalities hold for these maps. Immediate examples are $\Phi(A) = A^T$, $\Phi(A) = \frac{\mathrm{tr}A}{n}I$ and $\Phi(A) = U^*AU$, where U is a unitary matrix. All the maps $\varphi : \mathbb{M}_n \to \mathbb{C}$ and $\Phi : \mathbb{M}_2 \to \mathbb{M}_k$ also satisfy the conditions of Lemma 1 and inequalities in the subsequent theorems. Note that for $n = 2$, $\Phi(P_1) + \Phi(P_2) = I$ and so $\Phi(P_1)$ and $\Phi(P_2) = I - \Phi(P_1)$ always commute with each other. The map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ that replaces all off-diagonal entries by 0 is a unital positive linear map. In this case, $\Phi(A)$ and $\Phi(B)$ commute for all $A,\ B \in \mathbb{M}_n$. We have $\Phi(A) = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

EXAMPLE 1. The map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ defined by

$$\Phi(A) = \frac{1}{n-1}(\mathrm{tr}AI - A),$$

is a unital positive linear map (Positivity follows from the fact that $A \leqslant \mathrm{tr}A$). This map satisfies the conditions of Lemma 1. Therefore, all the above inequalities hold for this map.

EXAMPLE 2. The map $\Phi : \mathbb{M}_n \to \mathbb{M}_2$ defined by

$$\Phi(A) = \begin{bmatrix} a_{11} & a_{1n} \\ \overline{a_{1n}} & a_{11} \end{bmatrix};\ n > 1,$$

is a unital positive linear map. This map does not satisfy the conditions of Lemma 1 for all Hermitian matrices. But for the Toeplitz matrix

$$A = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & a_1 \\ a_{n-1} & a_{n-2} & \ldots & a_0 \end{bmatrix},$$

with real entries, $\Phi\left(A^l\right)$ and $\Phi(A^m)$ commute, $l, m = 1, 2, \ldots, n-1$, and hence the above inequalities hold for this map on Toeplitz matrix.

EXAMPLE 3. The map $\Phi : \mathbb{M}_n \to \mathbb{M}_2$ defined by

$$\Phi(A) = \begin{bmatrix} \frac{a_{ii}+a_{jj}}{2} & \frac{a_{ij}+a_{ji}}{2} \\ \frac{a_{ji}+a_{ij}}{2} & \frac{a_{ii}+a_{jj}}{2} \end{bmatrix};\ i \neq j, \tag{44}$$

satisfies the conditions of Lemma 1. Therefore, all the above inequalities hold for map (44). Let

$$A = \begin{bmatrix} 1 & 2 & 2+\imath \\ 2 & 5 & 1-\imath \\ 2-\imath & 1+\imath & 25 \end{bmatrix}.$$

Using (44), for $i=1$ and $j=2$, we have

$$\Phi(A)=\begin{bmatrix}3&2\\2&3\end{bmatrix}.$$

From (32) and (38), we respectively have

$$\Phi(A^{-1})\geqslant\frac{1}{15481}\begin{bmatrix}19070&-15508\\-15508&19070\end{bmatrix}\quad\text{and}\quad\Phi(A^{-1})\geqslant\frac{1}{5}\begin{bmatrix}3&-2\\-2&3\end{bmatrix}.$$

We see that our inequality (32) is better than (38).

EXAMPLE 4. Let

$$A=\begin{bmatrix}1&2&3\\2&1&4\\3&4&1\end{bmatrix}$$

and let $\Phi(A)$ be the compression map that takes an $n\times n$ matrix to a 2×2 block in its top left corner, then

$$\Phi(A)\Phi\left(A^2\right)=\begin{bmatrix}46&58\\44&53\end{bmatrix},\ \Phi\left(A^2\right)\Phi(A)=\begin{bmatrix}46&44\\58&53\end{bmatrix}$$

and

$$\Phi\left(A^4\right)-\Phi(A)^4=\begin{bmatrix}607&716\\716&852\end{bmatrix}\geqslant O.$$

So $\Phi(A)$ and $\Phi\left(A^2\right)$ do not commute, but $\Phi\left(A^4\right)\geqslant\Phi(A)^4$. Hence the conditions in Theorem 1 are sufficient rather than necessary for (7) to be true. Let

$$A=\begin{bmatrix}4&2&1\\2&2&0\\1&0&2\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}2&3&4\\3&2&4\\4&4&7\end{bmatrix}.$$

Here, $A>O$, $\Phi\left(A^3\right)\ngeqslant\Phi(A)^3$ and $\Phi\left(A^4\right)\ngeqslant\Phi(A)^4$. The matrix B is not positive definite, $\Phi(B)$ and $\Phi\left(B^2\right)$ commute, and $\Phi\left(B^3\right)\geqslant\Phi(B)^3$.

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