Operators
and
Matrices
Volume 13, Number 3 (2019), 855–865
doi:10.7153/oam-2019-13-61

ON MAPS SENDING RANK– κ IDEMPOTENTS TO IDEMPOTENTS

HAYDEN JULIUS

(*Communicated by L. Molnár*)

Abstract. We characterize bijective linear maps on complex-valued $n \times n$ matrices such that rank- κ idempotents are mapped to idempotents, where $2 \leqslant \kappa < n-1$.

1. Introduction

A common type of a *linear preserver problem* (LPP) concerns characterizing linear maps with an invariant subset. To be precise, if $\mathscr{M}$ is a matrix algebra over a field and $\mathscr{S}$ is a proper subset of $\mathscr{M}$, then we say a linear map ϕ *preserves* $\mathscr{S}$ if $\phi(\mathscr{S}) \subseteq \mathscr{S}$. Roughly speaking, LPPs of this type often have the usual conclusion that ϕ is of the form $\phi(A) = MAN$ or $\phi(A) = MA^TN$ for $A \in \mathscr{M}$, where A^T denotes the tranpose of A. Moreover, there is typically some fixed relation on M and N, such as $\det(MN) = 1$ or $N = M^{-1}$. Conclusions in this form have a reasonably high occurence rate in the literature and specific examples can be found in papers by Brešar, Šemrl [4], Li, Tsing [8], Omladič, Šemrl [9], and Šemrl [11]. In particular, for the case of $M_n(\mathbb{C})$, the set of complex-valued $n \times n$ matrices, linear maps that preserve the set of idempotents have this form (see Brešar, Šemrl [2], [3]) and linear maps that preserve the rank of a matrix have this form (see Beasley [1]). Both of these results motivate the upcoming discussion.

Consider the idempotent LPP found in [2]. Let $\mathscr{M}$ be $n \times n$ matrices over a field and let $\mathscr{P}$ be the subset of all idempotents in $\mathscr{M}$. Theorem 2.1 in [2] showed that any linear map θ such that $\theta(\mathscr{P}) \subseteq \mathscr{P}$ is a Jordan homomorphism. A careful inspection of the proof reveals that preserving $\mathscr{P}$ in its entirety is superfluous. Indeed, sending rank-one and rank-two idempotents to idempotents is sufficient to get the theorem (note that no assumptions are made about rank in the image of θ). It is not clear, however, what happens when only idempotents of a given rank are examined.

QUESTION 1. *If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map that sends rank- κ idempotents to idempotents for $1 \leqslant \kappa \leqslant n-1$, is ϕ a Jordan homomorphism?*

Mathematics subject classification (2010): 15A04.
Keywords and phrases: Idempotent, Jordan homomorphism.

Paper OaM-13-61

Note that if $\mathscr{R}_\kappa$ denotes the subset of all rank-κ matrices, then Question 1 can be stated as an LPP of the form $\phi(\mathscr{P}\cap\mathscr{R}_\kappa)\subseteq\mathscr{P}$. This is not a usual invariant subset type of LPP; in fact, this map does not fully preserve the set of idempotents, nor does it preserve rank. Will our "hybrid" problem have the usual conclusion as the idempotent preserver and rank preserver problem? As it turns out the answer is NO.

We will show that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map such that $\phi(\mathscr{P}\cap\mathscr{R}_\kappa)\subseteq\mathscr{P}$ with $2\leqslant\kappa<n-1$, then ϕ is either an automorphism, antiautomorphism, or of the form $\phi(A) = -f(A)+\alpha I_n$, where f is an automorphism or antiautomorphism, $\alpha\in\mathbb{C}$ depends on A and κ, and I_n is the $n\times n$ identity matrix. If the third description occurs, ϕ is not a Jordan homomorphism.

A problem similar to Question 1 appears in the recent paper by Pankov [10] concerning linear operators on the real vector space formed by all self-adjoint operators of finite rank on an infinite-dimensional Hilbert space. Namely, operators sending projections of a fixed rank to projections of other fixed rank are described. The map $\phi(A) = -f(A)+\alpha I_n$ appears as Example 2 in [10].

However, a compromise must be made regarding rank. The method described below uses elementary tools, and as a consequence, the cases where $\kappa = 1$ and $\kappa = n-1$ were not able to be obtained. It would be a relevant and useful extension of this work to obtain those cases using similar tools here. The interested reader should note that proving statements (e), (f), (g) in Lemma 5 in the $\kappa = 1$ and $\kappa = n-1$ cases would suffice. Ideally these statements could be shown with Lemma 3 only, if possible. This would allow a complete bypass of Lemma 4 and all other results would follow.

CONJECTURE 2. *Suppose $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map that sends rank-one idempotents to idempotents. Then ϕ is either of the form*

(I) $\phi(A) = f(A)$, *or*

(II) $\phi(A) = -f(A)+\mathrm{tr}(A)I_n$,

where I_n is the $n\times n$ identity matrix and f is the automorphism $f(A) = UAU^{-1}$ or the antiautomorphism $f(A) = UA^TU^{-1}$ for an invertible matrix $U\in M_n(\mathbb{C})$.

It is natural to expect the same in the $\kappa = n-1$ case.

Notation

We reserve $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ to always denote a linear map such that $\phi(P)^2 = \phi(P)$ whenever P is a rank-κ idempotent with $2\leqslant\kappa<n-1$ and $n\geqslant 4$. Both κ and n are arbitrary but fixed.

The *Jordan product* of x and y is denoted $x\circ y = xy+yx$. If $\circ$ is to be used as function composition, the word "composition" will appear directly before the use of the $\circ$ symbol. Otherwise, $\circ$ will always denote the Jordan product. It is easy to see that $x\circ y = y\circ x$ and $(x+y)^2 = x^2+y^2+x\circ y$. A *Jordan homomorphism* is a linear map that respects the Jordan product; i.e., $\phi(A\circ B) = \phi(A)\circ\phi(B)$ for all $A,B\in M_n(\mathbb{C})$. Since we are working over a commutative ring with $\frac{1}{2}$, an equivalent definition of a Jordan homomorphism is a linear map such that $\phi(A^2) = \phi(A)^2$ for all $A\in M_n(\mathbb{C})$.

The $n\times n$ identity matrix is denoted I_n. The transpose of A is denoted A^T and the trace of A is denoted $\mathrm{tr}(A)$. The symbol e_{ij} denotes a matrix with 1 in the (i,j)-entry and zeros elsewhere. Such matrices are called *matrix units*. The symbol (h_{ij}) represents an $n\times n$ matrix with entries $h_{ij}, 1\leqslant i,j,\leqslant n$. We denote the set of all trace-zero $n\times n$ matrices by sl_n. It is well known that sl_n is generated by the linear span of $\{e_{ij}\mid i\neq j\}\cup\{e_{kk}-e_{11}\mid k\neq 1\}$. In addition, we will also use the fact that sl_n is generated by the linear span of square-zero $n\times n$ matrices.

2. Results

It would be cumbersome to work with rank- κ idempotents if κ is large. Finding an intrinsic property that is shared by all linear maps that send rank- κ idempotents to idempotents will shorten proofs and maintain generality. As the following will show, mapping idempotents of a given rank to idempotents also preserves certain square-zero matrices. For LPPs concerning maps that preserve the set of square-zero matrices, see the papers by Šemrl [11] and Chebotar, Ke, Lee [5].

LEMMA 3. $\phi(N)^2=0$ *whenever* N *is a rank-one nilpotent.*

Proof. Every rank-one matrix has minimal polynomial $x^2-\lambda x$ for some $\lambda\in\mathbb{C}$, so every rank-one nilpotent is square-zero. For every rank-one square-zero matrix N, there exists an invertible $n\times n$ matrix U such that $U^{-1}NU=e_{1n}$. Let $f_N: M_n(\mathbb{C})\to M_n(\mathbb{C})$ be the inner automorphism defined by $f_N(X)=UXU^{-1}$ so that $f_N(e_{1n})=N$. Since f_N preserves rank *and* idempotence, it follows that the composition $\gamma:=\phi\circ f_N$ is a linear map that sends rank- κ idempotents to idempotents with $\gamma(e_{1n})=\phi(N)$. Proving $\gamma(e_{1n})^2=0$ gives the result.

Let E denote the rank- κ idempotent $e_{11}+\cdots+e_{\kappa\kappa}$. It is clear that $E+\lambda e_{1n}$ is also a rank- κ idempotent with $\lambda\in\{1,-1\}$. By hypothesis, $\gamma(E+\lambda e_{1n})^2=\gamma(E+\lambda e_{1n})$. Expanding this equation it is clear that

$$\gamma(E)^2+\gamma(e_{1n})^2+\lambda\gamma(E)\circ\gamma(e_{1n})=\gamma(E+\lambda e_{1n}).$$

Cancelling $\gamma(E)^2=\gamma(E)$ from both sides and adding the two distinct equations obtained by letting $\lambda=1$ and $\lambda=-1$ shows that $\gamma(e_{1n})^2=0$. $\square$

REMARK 1. It is easy to see that Lemma 3 also holds in the $\kappa=1$ and $\kappa=n-1$ case. Considering $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$, the set of square-zero matrices is precisely the set of rank-one nilpotent matrices. Hence in the $n=2$ and $n=3$ case, Question 1 reduces to the square-zero preserver problem described in the paper by Šemrl [11]. This is some justification for our assumption that $n\geqslant 4$, namely, so that nilpotents of higher rank appear.

LEMMA 4. $\phi(M)^2=0$ *whenever* M *is a rank-two square-zero matrix.*

Proof. Analogous to the proof of Lemma 3, there is a map $\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that sends rank-κ idempotents to idempotents with $\gamma(e_{ij}+e_{kl}) = \phi(M)$ for any rank-two square-zero matrix M. The goal is to show that $\phi(M)^2 = \gamma(e_{ij}) \circ \gamma(e_{kl}) = 0$.

Suppose first that γ sends rank-two idempotents to idempotents. Let $\lambda \in \{1, -1\}$ and consider the rank-two idempotent matrix $e_{jj} + e_{kk} + \lambda(e_{ij} + e_{kl})$, which may be visualized as the following matrix with the i, j, k, l rows and columns labeled:

$$\begin{array}{c} \\ i \\ j \\ k \\ l \end{array} \begin{array}{c} \begin{array}{cccc} i & j & k & l \end{array} \\ \begin{pmatrix} 0 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

A matrix represented this way has zeros in all other rows or columns, which are omitted from the display so that there is no loss in generality regarding the dimension of $M_n(\mathbb{C})$. The indices i, j, k, l are arbitrary but distinct.

Let $P_2 = e_{jj} + e_{kk}$. By hypothesis,

$$[\gamma(P_2) + \lambda\gamma(e_{ij} + e_{kl})]^2 = \gamma(P_2) + \lambda\gamma(e_{ij} + e_{kl}). \tag{2.1}$$

Expanding the left-hand side yields

$$\gamma(P_2)^2 + \gamma(e_{ij} + e_{kl})^2 + \lambda\gamma(P_2) \circ \gamma(e_{ij} + e_{kl}),$$

which simplifies to

$$\gamma(P_2) + \gamma(e_{ij}) \circ \gamma(e_{kl}) + \lambda\gamma(P_2) \circ \gamma(e_{ij} + e_{kl})$$

by Lemma 3 and the fact that P_2 is a rank-two idempotent. Returning to equation (2.1),

$$\gamma(e_{ij}) \circ \gamma(e_{kl}) + \lambda\gamma(P_2) \circ \gamma(e_{ij} + e_{kl}) = \lambda\gamma(e_{ij} + e_{kl}).$$

Adding the equations obtained by letting $\lambda = 1$ and $\lambda = -1$ shows that $\gamma(e_{ij}) \circ \gamma(e_{kl}) = 0$.

Suppose now that γ sends rank-κ idempotents to idempotents and assume that $2 < \kappa < n-1$. In particular, since $\kappa > 2$, we may find $\kappa - 2$ more diagonal matrix units e_{qq} with $q \notin \{i, j, k, l\}$. The sum of these $\kappa - 2$ diagonal matrix units forms a diagonal rank-$(\kappa - 2)$ idempotent P' such that $P_\kappa := e_{jj} + e_{kk} + P'$ is a diagonal rank-κ idempotent. Moreover, $P_\kappa + \lambda(e_{ij} + e_{kl})$ is a rank-κ idempotent (again with $\lambda \in \{1, -1\}$), and

$$[\gamma(P_\kappa) + \lambda\gamma(e_{ij} + e_{kl})]^2 = \gamma(P_\kappa) + \lambda\gamma(e_{ij} + e_{kl}). \tag{2.2}$$

Expanding the left-hand side of equation (2.2) and adding the equations obtained by letting $\lambda = 1$ and $\lambda = -1$ (an identical argument to the above) gives $\gamma(e_{ij}) \circ \gamma(e_{kl}) = 0$ in the general case. The proof is complete. $\square$

LEMMA 5. *For distinct indices* $1 \leqslant i,j,k,l \leqslant n$, ϕ *has the following properties:*

(a) $\phi(e_{ij}) \circ \phi(e_{ij}) = \phi(e_{ij}) \circ \phi(e_{ik}) = \phi(e_{ji}) \circ \phi(e_{ki}) = 0$,

(b) $\phi(e_{jj} - e_{ii}) \circ \phi(e_{jj} - e_{ii}) = 2\phi(e_{ij}) \circ \phi(e_{ji})$,

(c) $\phi(e_{jj} - e_{ii}) \circ \phi(e_{ij} - e_{ji}) = 0$,

(d) $\phi(e_{ii} - e_{jj}) \circ \phi(e_{ik}) = \phi(e_{ij}) \circ \phi(e_{jk})$,

(e) $\phi(e_{ij}) \circ \phi(e_{kl}) = 0$,

(f) $\phi(e_{jj} - e_{ii}) \circ \phi(e_{kl}) = \phi(e_{jj} - e_{ii}) \circ \phi(e_{lk}) = 0$, *and*

(g) $\phi(e_{jj} - e_{ii}) \circ \phi(e_{kk} - e_{ll}) = 0$.

Proof. By Lemma 3, $\phi(e_{ij})^2 = 0$ and so $\phi(e_{ij}) \circ \phi(e_{ij}) = 0$. Consider now the rank-one nilpotent matrix $e_{ij} + e_{ik}$. Since $\phi(e_{ij} + e_{ik})^2 = 0$, we have by direct calculation that $\phi(e_{ij}) \circ \phi(e_{ik}) = 0$. A similar appeal to the rank-one nilpotent $e_{ji} + e_{ki}$ also shows that $\phi(e_{ji}) \circ \phi(e_{ki}) = 0$. Hence property (a) is proved.

Consider now the rank-one nilpotent matrix

$$\begin{array}{c} \\ i \\ j \\ k \end{array} \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{pmatrix} -1 & -\lambda & -\mu \\ \lambda & 1 & \lambda\mu \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

where $\lambda \in \{1, -1\}$ and $\mu \in \{-1, 0, 1\}$. This means

$$[\phi(e_{jj} - e_{ii}) + \lambda\phi(e_{ji} - e_{ij}) + \mu\phi(\lambda e_{jk} - e_{ik})]^2 = 0,$$

and after expanding,

$$\begin{aligned} \phi(e_{jj} - e_{ii})^2 - \phi(e_{ij}) \circ \phi(e_{ji}) + \mu^2\phi(\lambda e_{jk} - e_{ik})^2 + \lambda\phi(e_{jj} - e_{ii}) \circ \phi(e_{ji} - e_{ij}) \\ + \mu\phi(e_{jj} - e_{ii}) \circ \phi(\lambda e_{jk} - e_{ik}) + \lambda\mu\phi(e_{ji} - e_{ij}) \circ \phi(\lambda e_{jk} - e_{ik}) = 0. \end{aligned} \tag{2.3}$$

Since $\lambda e_{jk} - e_{ik}$ is a rank-one nilpotent it is clear that the third term in equation (2.3) is zero. Suppose that $\mu = 0$. Equation (2.3) reduces to

$$\phi(e_{jj} - e_{ii})^2 - \phi(e_{ij}) \circ \phi(e_{ji}) + \lambda\phi(e_{jj} - e_{ii}) \circ \phi(e_{ji} - e_{ij}) = 0. \tag{2.4}$$

Adding the equations formed by taking $\lambda = 1$ and $\lambda = -1$ in equation (2.4) yields property (b) since $2\phi(e_{jj} - e_{ii})^2 = \phi(e_{jj} - e_{ii}) \circ \phi(e_{jj} - e_{ii})$. Applying property (b), equation (2.4) also implies property (c) directly. Thus equation (2.3) simplifies greatly to

$$\mu\phi(e_{jj} - e_{ii}) \circ \phi(\lambda e_{jk} - e_{ik}) + \lambda\mu\phi(e_{ji} - e_{ij}) \circ \phi(\lambda e_{jk} - e_{ik}) = 0. \tag{2.5}$$

Since Jordan products distribute over sums, equation (2.5) can be written as

$$\lambda\mu\phi(e_{jj}-e_{ii})\circ\phi(e_{jk})-\mu\phi(e_{jj}-e_{ii})\circ\phi(e_{ik})$$
$$+\mu\phi(e_{ji}-e_{ij})\circ\phi(e_{jk})-\lambda\mu\phi(e_{ji}-e_{ij})\circ\phi(e_{ik})=0.$$

Fixing μ and adding equations with $\lambda=1,-1$ isolates the terms with μ as coefficient, so

$$-\phi(e_{jj}-e_{ii})\circ\phi(e_{ik})+\phi(e_{ji}-e_{ij})\circ\phi(e_{jk})=0$$

and applying property (a), we have

$$\phi(e_{ii}-e_{jj})\circ\phi(e_{ik})-\phi(e_{ij})\circ\phi(e_{jk})=0.$$

This is precisely property (d). Isolating the terms with $\lambda\mu$ as a coefficient instead would give an analogous conclusion with different indices.

Property (e) follows directly from Lemma 4. Consider now the rank-two square-zero matrix

$$\begin{array}{c@{}c} & \begin{array}{cccc} i & j & k & l \end{array} \\ \begin{array}{c} i \\ j \\ k \\ l \end{array} & \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Using properties (b) and (e) implies that $\phi(e_{jj}-e_{ii})\circ\phi(e_{kl})=0$. An analogous argument with the transpose of the above matrix yields $\phi(e_{jj}-e_{ii})\circ\phi(e_{lk})=0$ and so (f) is proved. Lastly, consider the rank-two square-zero matrix

$$\begin{array}{c@{}c} & \begin{array}{cccc} i & j & k & l \end{array} \\ \begin{array}{c} i \\ j \\ k \\ l \end{array} & \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Thus we have the relation

$$[\phi(e_{jj}-e_{ii})+\phi(e_{ji}-e_{ij})+\phi(e_{ll}-e_{kk})+\phi(e_{lk}-e_{kl})]^2=0$$

and applying properties (b), (c), (e), and (f) proves property (g). □

REMARK 2. Now that the computational method has been explored, it is hopefully more evident that the extremal $\kappa=1$ and $\kappa=n-1$ cases of Question 1 might not be obtainable using only the conclusion of Lemma 3. To give a general idea, rank-one idempotents and rank-one nilpotents tend to give the same redundant information during computations, which makes them unable to isolate terms like $\phi(e_{ij})\circ\phi(e_{kl})$ for distinct i,j,k,l. Likewise rank-$(n-1)$ idempotents are restrictive, in the sense that matrix units e_{ij} and e_{kl} cannot be easily inserted into a typical idempotent with such a high rank. However, the rank-one case is likely of considerable interest. Results concerning linear maps that preserve the set of rank-one idempotents or rank-one nilpotents of $n\times n$ matrices can be found in the papers by Jing, Li, Lu [6] and Kuzma [7].

The next lemma extends the structural properties of ϕ proved in Lemma 5 to all trace-zero matrices.

LEMMA 6. *$\phi(A)\circ\phi(B)=\phi(X)\circ\phi(Y)$ whenever $A\circ B=X\circ Y$ for all $A,B,X,Y\in sl_n$.*

Proof. Since ϕ is linear, it suffices to show that ϕ preserves Jordan products on a generating set for sl_n, namely $\{e_{ij}\mid i\neq j\}\cup\{e_{kk}-e_{11}\mid k\neq 1\}$. This amounts to showing the following statements hold:

(1) $\phi(e_{ij})\circ\phi(e_{ij})=\phi(e_{ij})\circ\phi(e_{ik})=\phi(e_{ji})\circ\phi(e_{ki})=\phi(e_{ij})\circ\phi(e_{kl})=0$ for distinct $i,j,k,$ and l,

(2) $\phi(e_{jj}-e_{11})\circ\phi(e_{jj}-e_{11})=2\phi(e_{1j})\circ\phi(e_{j1})$ for $j\neq 1$,

(3) $\phi(e_{jj}-e_{11})\circ\phi(e_{kl})=\phi(e_{jj}-e_{11})\circ\phi(e_{lk})=0$ for $j\neq 1, k\neq l$, and $k,l\notin\{1,j\}$,

(4) $\phi(e_{ii}-e_{11})\circ\phi(e_{jj}-e_{11})=\phi(e_{kk}-e_{11})\circ\phi(e_{ll}-e_{11})$ for $i,j,k,l\neq 1$,

(5) $\phi(e_{jj}-e_{11})\circ\phi(e_{j1})=\phi(e_{jj}-e_{11})\circ\phi(e_{1j})=0$ for $j\neq 1$,

(6) $\phi(e_{ij})\circ\phi(e_{jk})=\phi(e_{il})\circ\phi(e_{lk})$ for distinct indices $i,j,k,$ and l, and

(7) $\phi(e_{ij})\circ\phi(e_{jk})=\phi(e_{ii}-e_{11})\circ\phi(e_{ik})=\phi(e_{kk}-e_{11})\circ\phi(e_{ik})$ for $i\neq 1,\ k\neq 1$, $i\neq k$, and $j\notin\{i,k\}$.

Refer back to the properties of ϕ proved in Lemma 5. Statements (1), (2), and (3) follow directly from properties (a), (b), (e), and (f). Statement (4) follows from property (g) and the calculation

$$\begin{aligned}\phi(e_{ii}-e_{11})\circ\phi(e_{jj}-e_{11})&=\phi(e_{ii}-e_{kk}+e_{kk}-e_{11})\circ\phi(e_{jj}-e_{11})\\&=\phi(e_{kk}-e_{11})\circ\phi(e_{jj}-e_{11})\\&=\phi(e_{kk}-e_{11})\circ\phi(e_{jj}-e_{ll}+e_{ll}-e_{11})\\&=\phi(e_{kk}-e_{11})\circ\phi(e_{ll}-e_{11}).\end{aligned}$$

Stopping the above calculation at the second equality is sufficient in case $n=4$.

Define the diagonal idempotents $P=e_{jj}+\sum_{q\notin\{1,j\}}e_{qq}$ and $P'=e_{11}+\sum_{q\notin\{1,j\}}e_{qq}$ so that both P and P' have rank κ. Observe that $P+e_{1j}$ and $P'+e_{1j}$ are also rank- κ idempotents, therefore

$$\phi(P+e_{1j})^2=\phi(P+e_{1j})\quad\text{and}\quad\phi(P'+e_{1j})^2=\phi(P'+e_{1j})$$

implies that

$$\phi(P)\circ\phi(e_{1j})=\phi(e_{1j})\quad\text{and}\quad\phi(P')\circ\phi(e_{1j})=\phi(e_{1j}).$$

Subtracting the two equations yields $\phi(P-P')\circ\phi(e_{1j}) = \phi(e_{jj}-e_{11})\circ\phi(e_{1j}) = 0$. Replacing e_{1j} by e_{j1} and running the same argument yields the same conclusion, and so statement (5) is proved.

Statement (6) follows at once from properties (d) and (f):

$$\begin{aligned}0 = \phi(e_{jj}-e_{ll})\circ\phi(e_{ik}) &= \phi(e_{jj}-e_{ii})\circ\phi(e_{ik}) + \phi(e_{ii}-e_{ll})\circ\phi(e_{ik}) \\ &= -\phi(e_{ij})\circ\phi(e_{jk}) + \phi(e_{il})\circ\phi(e_{lk}).\end{aligned}$$

To prove statement (7), we first show that $\phi(e_{ii}-e_{11})\circ\phi(e_{ik}) = \phi(e_{kk}-e_{11})\circ\phi(e_{ik})$, provided that $i\neq 1, k\neq 1$, and $i\neq k$. Indeed,

$$\phi(e_{ii}-e_{11})\circ\phi(e_{ik}) = \phi(e_{ii}-e_{kk}+e_{kk}-e_{11})\circ\phi(e_{ik}) = \phi(e_{kk}-e_{11})\circ\phi(e_{ik}).$$

by the preceding statement (5).

Because property (d) implies that $\phi(e_{i1})\circ\phi(e_{1k}) = \phi(e_{ii}-e_{11})\circ\phi(e_{ik})$ and $\phi(e_{i1})\circ\phi(e_{1k}) = \phi(e_{ij})\circ\phi(e_{jk})$ for $j\notin\{i,k\}$ by statement (6), the equality throughout statement (7) has been established. The lemma is proved. □

There are two important corollaries to Lemma 6. The first gives a complete description of ϕ on sl_n and the second restores some information about rank among idempotents in the image of ϕ.

COROLLARY 7. *$\phi(N)^2 = 0$ whenever $N^2 = 0$. Moreover, if ϕ is bijective, then $\phi(sl_n) = sl_n$ and the restriction of ϕ to sl_n is either of the form*

(i) $\phi(A) = cUAU^{-1}$, *or*

(ii) $\phi(A) = cUA^TU^{-1}$,

where $U\in M_n(\mathbb{C})$ is an invertible matrix and $c\in\mathbb{C}$ is a nonzero complex number.

Proof. Let $N\in sl_n$ be a square-zero matrix of any rank. Take $A=B=N$ and $X=Y=0$ as in the conclusion of Lemma 6 to get $\phi(N)^2=0$. Thus ϕ preserves the set of square-zero matrices and since square-zero matrices generate sl_n, it follows that $\phi(sl_n)=sl_n$ when ϕ is bijective. From Corollary 2 in [11] (where bijectivity is indispensable), the restriction of ϕ to sl_n is of the form (i) or (ii). □

COROLLARY 8. *Suppose ϕ is bijective. If P and Q are rank-κ idempotents, then $\phi(P)$ and $\phi(Q)$ have the same rank.*

Proof. The rank of any idempotent matrix is its trace. If P and Q are both rank-κ idempotents, then $\mathrm{tr}(P-Q)=0$ implies $P-Q\in sl_n$. By Corollary 7 we have that $\phi(P-Q)\in sl_n$, thus $\mathrm{tr}(\phi(P)) = \mathrm{tr}(\phi(Q))$. □

The main result is now within reach. So far, the interplay between idempotents and square-zero matrices furnished a complete description of ϕ on sl_n. The constant c appearing in Corollary 7 may be arbitrary if a map is only assumed to preserve the set of square-zero matrices. However, with the additional structure afforded by the image of ϕ on rank-κ idempotents, the constant c holds the key to a complete description of ϕ on $M_n(\mathbb{C})$.

LEMMA 9. *If ϕ is bijective, then $c = \pm 1$. Moreover, ϕ either sends rank- κ idempotents to rank- κ idempotents or sends rank- κ idempotents to rank- $(n-\kappa)$ idempotents.*

Proof. Let ϕ be bijective and fix c and U as described in Corollary 7. Denote by ψ the bijective linear map defined by

$$\psi(A) = U^{-1}\phi(A)U, \quad A \in M_n(\mathbb{C}). \tag{2.6}$$

Since the restriction of ϕ to sl_n is either of the form (i) or (ii) in Corollary 7, $\psi(A) = cA$ whenever $A \in sl_n$ is symmetric. Additionally ψ also sends rank- κ idempotents to idempotents. As in Lemma 3, denote $E = e_{11} + \cdots + e_{\kappa\kappa}$. If $\psi(E) = H$ with $H = (h_{ij})$ an idempotent, then for all $1 \leqslant j \leqslant \kappa$ and $\kappa < m \leqslant n$, the matrix $H + \psi(e_{mm} - e_{jj})$ must also be an idempotent. Since $e_{mm} - e_{jj}$ is symmetric and trace-zero,

$$H + c(e_{mm} - e_{jj}) = H^2 + c^2(e_{mm} + e_{jj}) + cH(e_{mm} - e_{jj}) + c(e_{mm} - e_{jj})H$$

implies that

$$e_{mm} - e_{jj} = c(e_{mm} + e_{jj}) + H(e_{mm} - e_{jj}) + (e_{mm} - e_{jj})H. \tag{2.7}$$

Multiplying equation (2.7) on the left and right by e_{mm} gives $e_{mm} = ce_{mm} + 2h_{mm}e_{mm}$ and multiplying equation (2.7) on the left and right by e_{jj} gives $-e_{jj} = ce_{jj} - 2h_{jj}e_{jj}$. In addition, if $u \neq m$ and $u \neq j$, then multiplying (2.7) on the left by e_{uu} gives the equation $0 = h_{um}e_{um} - h_{uj}e_{uj}$. It follows that $h_{um} = h_{uj} = 0$. If $v \neq m$ and $v \neq j$, then multiplying (2.7) on the right by e_{vv} gives the equation $0 = h_{mv}e_{mv} - h_{jv}e_{jv}$. It follows that $h_{mv} = h_{jv} = 0$ as well. Equation (2.7) does not furnish a direction implication that $h_{jm} = h_{mj} = 0$, but changing indices can show it. Indeed, replace j in equation (2.7) with a j' such that $1 \leqslant j' \leqslant \kappa$ and $j' \neq j$. Multiply the new equation by e_{jj} on the left to get $h_{jm} = 0$ and multiply the new equation by e_{jj} on the right to get $h_{mj} = 0$.

Combining the above, a complete description of the entries of H is obtained:

$$h_{jj} = \frac{1+c}{2} \text{ and } h_{mm} = \frac{1-c}{2} \text{ if } 1 \leqslant j \leqslant \kappa < m \leqslant n, \tag{2.8}$$

and $h_{jm} = 0$ for any distinct j and m. Since H is diagonal and idempotent, the diagonal entries must be 0 or 1. According to equation (2.8), the only two possibilities are $c = 1$ or $c = -1$.

Therefore $H = E$ when $c = 1$ and $H = I_n - E$ when $c = -1$. Hence H is either rank- κ or rank- $(n-\kappa)$, respectively. By Corollary 8, the bijective map ψ (and therefore ϕ) sends rank- κ idempotents to rank- κ idempotents in the former case and sends rank- κ idempotents to rank- $(n-\kappa)$ idempotents in the latter case. □

THEOREM 10. *Suppose $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map that sends rank- κ idempotents to idempotents. Then ϕ is either of the form*

(I) $\phi(A) = f(A)$, *or*

(II) $\phi(A) = -f(A) + \frac{\mathrm{tr}(A)}{\kappa} I_n$,

where I_n *is the* $n \times n$ *identity matrix and* f *is the automorphism* $f(A) = UAU^{-1}$ *or the antiautomorphism* $f(A) = UA^T U^{-1}$ *for an invertible matrix* $U \in M_n(\mathbb{C})$.

Proof. Lemma 9 supplies two distinct situations according to the constant c. If $c = 1$ then $\phi = f$ is immediate. If $c = -1$, then $\phi(A) = -f(A)$ for every $A \in sl_n$, so the description (II) holds for matrices in sl_n. By linearity, ϕ is completely determined on $M_n(\mathbb{C})$ by its image on a single matrix with nonzero trace. Indeed, recall that $E = e_{11} + \cdots + e_{\kappa\kappa}$ and $\phi(E) = U(I_n - E)U^{-1}$ holds by equation (2.6). This may be written alternatively as

$$\phi(E) = -f(E) + I_n,$$

which is of the form (II) since $\mathrm{tr}(E) = \kappa$. It suffices to show that $\phi(e_{11}) = -f(e_{11}) + \frac{\mathrm{tr}(e_{11})}{\kappa} I_n$. Observe that $\kappa e_{11} = E + \sum_{j=2}^{\kappa}(e_{11} - e_{jj})$. Now,

$$\kappa\phi(e_{11}) = \phi(E) + \sum_{j=2}^{\kappa} \phi(e_{11} - e_{jj}) = -f(E) + I_n - \sum_{j=2}^{\kappa} f(e_{11} - e_{jj}) = -\kappa f(e_{11}) + I_n$$

holds true and the description follows upon division by κ. □

REMARK 3. If $2\kappa \neq n$, then preserving the set of rank-κ idempotents is a necessary and sufficient condition for ϕ to be an automorphism or antiautomorphism. However, if $2\kappa = n$, then preserving the set of rank-κ idempotents is necessary but not sufficient for ϕ to be an automorphism or antiautomorphism.

Acknowledgement. The author would like to thank Dr. Mikhail Chebotar for his assistance and encouragement, and also the referee for his/her helpful suggestions.

REFERENCES

[1] L. B. BEASLEY, *Linear transformations on matrices: The invariance of rank k matrices*, Linear Algebra and Appl., **3** (1970), 407–427.

[2] M. BREŠAR AND P. ŠEMRL, *On local automorphisms and mappings that preserve idempotents*, Canad. J. Math., **45(3)** (1993), 483–496.

[3] M. BREŠAR AND P. ŠEMRL, *On local automorphisms and mappings that preserve idempotents*, Studia Math., **113(2)** (1995), 101–108.

[4] M. BREŠAR AND P. ŠMERL, *Linear preservers on* $\mathscr{B}(X)$, Banach Center Publications, **38** (1997), 49–58.

[5] M. A. CHEBOTAR, W.-F. KE, AND P.-H. LEE, *On maps preserving square-zero matrices*, J. Algebra, **289(2)** (2005), 421–445.

[6] W. JING, P. LI, AND S. LU, *Additive mappings that preserve rank one nilpotent operators*, Linear Algebra Appl., **367** (2003), 213–224.

[7] B. KUZMA, *Additive mappings preserving rank-one idempotents*, Acta Math. Sin (Engl. Ser.), **21(6)** (2005), 1399–1406.

[8] C. LI AND N. TSING, *Linear preserver problems: a brief introduction and some special techniques*, Linear Algebra Appl., **289(2)** (1992), 217–235.

[9] M. OMLADIČ AND P. ŠEMRL, *Additive mappings preserving operators of rank one*, Linear Algebra Appl., **182** (1993), 239–256.

[10] M. PANKOV, *Wigner's type theorem in terms of linear operators which send projections of a fixed rank to projections of other fixed rank*, J. Math. Anal. Appl., **474(2)** (2019), 1238–1249.
[11] P. ŠEMRL, *Linear mappings preserving square-zero matrices*, Bull. Austral. Math. Soc., **48(3)** (1993), 365–370.

(Received April 26, 2019)

Hayden Julius
Department of Mathematical Sciences
Kent State University
Kent OH 44242, U.S.A.
e-mail: hjulius@kent.edu