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ALGEBRAIC REFLEXIVITY OF SETS OF BOUNDED LINEAR OPERATORS ON ABSOLUTELY CONTINUOUS FUNCTION SPACES

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Abstract. In this paper we deal with the algebraic reflexivity of sets of bounded linear operators on absolutely continuous vector-valued function spaces. As a consequence, it is shown that the set of all surjective linear isometries, the set of all isometric reflections, and the set of all generalized bi-circular projections on $AC[0,1]$ are algebraically reflexive.

1. Introduction

For arbitrary normed spaces A and B, $\mathscr{L}(A,B)$ stands for the space of all bounded linear operators from A to B. Let $\xi \subseteq \mathscr{L}(A,B)$. We say a map $T \in \mathscr{L}(A,B)$ *belongs locally* to ξ if T pointwise equals an element of ξ, in other words, if for each $a \in A$ there is a $T_a \in \xi$, possibly depending on a, such that $Ta = T_a a$. The collection ξ is said to be *algebraically reflexive* if ξ contains every T belonging locally to ξ. For more information on this concept we refer to the book by Molnár [18].

One important collection ξ is the set $\mathrm{Iso}(A,B)$ of all surjective linear isometries from A onto B. When $A = B$, we write $\mathrm{Iso}(A) = \mathrm{Iso}(A,A)$ which, namely, is the isometry group of all surjective linear isometries of A. Investigations on algebraic reflexivity of the group of isometries of spaces of continuous functions were begun by Molnár and Zalar [19]. They proved that the isometry group of the function space $C(X)$ of all continuous complex-valued functions on a compact Hausdorff space X, endowed with the supremum norm $\|\cdot\|_\infty$, is algebraically reflexive when X is first countable. Motivated by [19], considerable study has been done in this direction. Jarosz and Rao [15] investigated a vector-valued version of this result proving that if X is a first countable compact Hausdorff space and E is a uniformly convex complex Banach space such that $\mathrm{Iso}(E)$ is algebraically reflexive, then $\mathrm{Iso}(C(X,E))$ is algebraically reflexive. Moreover, with respect to subsets of the isometry group, Dutta and Rao in [10] studied the algebraic reflexivity of the set of isometric reflections $\mathrm{Iso}^2(C(X))$ of $C(X)$, and they proved that for any compact Hausdorff space X, if $\mathrm{Iso}(C(X))$ is algebraically reflexive, then so is $\mathrm{Iso}^2(C(X))$. Recently, Jiménez-Vargas et al. in [16] showed that if (X,d) is a compact metric space, then the isometry group and the set of isometric reflections of $\mathrm{Lip(X,d)}$ and $\mathrm{lip(X,d^\alpha)}$ $(0<\alpha<1)$, equipped with the sum norm $\|\cdot\|_\infty + L(\cdot)$, are

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algebraically reflexive, where $L(\cdot)$ denotes the Lipschitz constant of a function. Then, a vector-valued version of the results of [16] was established by Botelho and Jamison [3] for the group of surjective linear isometries preserving a constant function on the space $\mathrm{Lip(X,E)}$ consisting of all E-valued Lipschitz functions on a compact metric space X with respect to the norm $\max\{\|\cdot\|_\infty, L(\cdot)\}$, when is E is a strictly convex complex Banach space. Meantime, more recently, Oi [20] investigated the algebraic reflexivity of groups of surjective linear isometries on $\mathrm{Lip(X,E)}$ with the sum norm for the case where E is $C(Y)$, or $M_n(\mathbb{C})$ (the algebra of all $n\times n$ complex matrices). One can also see related results on the algebraic reflexivity of sets of isometries for various classical spaces of continuous scalar and vector-valued functions in [2, 6, 7, 10, 21].

Motivated by the above results, in this paper we investigate the algebraic reflexivity of the sets of isometries of spaces of absolutely continuous vector-valued functions, endowed with the max norm $\max\{\|\cdot\|_\infty, \mathscr{V}(\cdot)\}$, where $\mathscr{V}(\cdot)$ is the total variation of a function.

Another important class of operators has recently received considerable attention, known as generalized bi-circular projections. The notion of generalized bi-circular projection (abbreviated gbp) was introduced in [11]. In the context of function spaces, Botelho and Jamison [4] characterized gbp's on the spaces of vector-valued continuous functions $C(X,E)$ for certain compact Hausdorff spaces X and Banach spaces E. Then in [10], the authors applied this characterization to show that if $\mathrm{Iso}(C(X))$ is algebraically reflexive, then the set of gbp's of $C(X)$ is algebraically reflexive. Further results on the representation and the algebraic reflexivity problem of gbp's on various function spaces can be found in [1, 3, 5, 12] and the references therein. In this paper, Section 3 concerns the description of gbp's on $AC(X,E)$-spaces which is also essential in our study of the algebraic reflexivity problem for sets of such operators.

As a consequence, in the scalar-valued case, we prove that the isometry group, the set of all isometric reflections, and the set of all generalized bi-circular projections on $AC[0,1]$ with respect to either of the natural norms $\max\{\|\cdot\|_\infty, \mathscr{V}(\cdot)\}$ and $\|\cdot\|_\infty + \mathscr{V}(\cdot)$, are algebraically reflexive.

In the rest of this section, we fix notation and recall some definitions that we shall use in our paper.

Let X be a compact subset of the real line $\mathbb{R}$ with at least two points and E be a nonzero normed space over the filed $\mathbb{R}$ or $\mathbb{C}$. A function $f: X \longrightarrow E$ is called *absolutely continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \varepsilon,$$

for every finite family of non-overlapping open intervals $\{(a_i,b_i) : i = 1,...,n\}$ whose extreme points belong to X with $\sum_{i=1}^{n}(b_i - a_i) < \delta$. We denote by $AC(X,E)$ the space of all absolutely continuous E-valued functions on X. If not explicitly stated otherwise, $AC(X,E)$ is equipped with the norm

$$\|f\| = \max\{\|f\|_\infty, \mathscr{V}(f)\} \quad (f \in AC(X,E)),$$

where $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$ and $\mathscr{V}(f)$ is the total variation of f defined as

$$\mathscr{V}(f) := \sup\Big\{ \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\| : n \in \mathbb{N}, x_0, x_1, \ldots, x_n \in X, x_0 < x_1 < \ldots < x_n \Big\}.$$

For the case where E is the scalar field $\mathbb{R}$ or $\mathbb{C}$, we shall write $AC(X)$ instead of $AC(X,E)$. Furthermore, given $e \in E$, $\hat{e}$ stands for the function in $AC(X,E)$ which is constantly e on X.

For each non-empty set, I denotes the identity map on it.

We finally recall that a nonzero normed space E is said to be *strictly convex* if each point in the unit sphere $\mathscr{S}_E = \{e \in E : \|e\| = 1\}$ is an extreme point of the closed unit ball of E. Meantime, one can observe that $\|e_1\|, \|e_2\| < \max\{\|e_1+e_2\|, \|e_1-e_2\|\}$ for every $e_1, e_2 \in E \setminus \{0\}$.

2. Algebraic reflexivity of sets of isometries on $AC(X,E)$-spaces

Throughout the rest of this paper, we assume that X and Y are compact subsets of the real line with at least two points, and E and F are both strictly convex real normed spaces or both strictly convex complex normed spaces. We denote by $\mathscr{G}(AC(X,E), AC(Y,F))$ the set of all linear isometries from $AC(X,E)$ onto $AC(Y,F)$ such that $T\hat{e}(a) \neq 0$ and $T\hat{e'}(b) \neq 0$ for some $e, e' \in E$, where $a = \min Y$ and $b = \max Y$. According to [14, Theorem 4.1], for a given T in $\mathscr{G}(AC(X,E), AC(Y,F))$ there exist a monotonic absolutely continuous homeomorphism $\varphi : Y \longrightarrow X$, and a surjective linear isometry J in $\mathrm{Iso}(E,F)$ such that

$$Tf(y) = J(f(\varphi(y))) \quad (f \in AC(X,E), y \in Y).$$

Indeed, $\mathscr{G}(AC(X,E), AC(Y,F))$ is the set of all surjective linear isometries of the form of the generalized weighted composition operators (see [14, Remark 4.2]). If $X = Y$ and $E = F$, we denote this set by $\mathscr{G}(AC(X,E))$. We would like to remark that, by [14, Corollary 4.4], $\mathscr{G}(AC[0,1])$ is the isometry group Iso(AC[0,1]) of all surjective linear isometries acting on $AC[0,1]$.

Meantime, here every linear map $T : AC(X,E) \longrightarrow AC(Y,F)$ which belongs locally to the set $\mathscr{G}(AC(X,E), AC(Y,F))$ is naturally termed as a *local isometry*.

The following theorem provides a precise description of local isometries of spaces of absolutely continuous functions.

THEOREM 2.1. *Let $T : AC(X,E) \longrightarrow AC(Y,F)$ be a local isometry. Then there exist a monotonic absolutely continuous homeomorphism $\varphi : Y \longrightarrow X$, and a linear isometry $J : E \longrightarrow F$ belonging locally to* Iso(E,F) *such that*

$$Tf(y) = J(f(\varphi(y))) \quad (f \in AC(X,E), y \in Y).$$

Furthermore, T is surjective if and only if J is surjective.

Proof. First notice that T preserves the supremum norm. To see this, let $f \in AC(X,E)$. Since T is a local isometry, there exists a $T_f \in \mathscr{G}(AC(X,E),AC(Y,F))$ such that $Tf = T_f f$. Hence, from the representation of T_f, it is inferred that $\|T_f f\|_\infty = \|f\|_\infty$, which immediately yields $\|Tf\|_\infty = \|f\|_\infty$. Now, since $AC(X,E)$ (resp. $AC(Y,F)$) is uniformly dense in $C(X,E)$ (resp. $C(Y,F)$) (see [13, Lemma 1] and [17, Corollary 1.2]), we can extend T to a linear isometry from $C(X,E)$ to $C(Y,F)$, which we keep denoting by T. Hence by [8], there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow X$, a function ω from Y_0 to $\mathscr{L}(E,F)$ such that

$$Tf(y) = \omega(y)(f(\varphi(y))) \quad (f \in AC(X,E), y \in Y_0).$$

Next, we claim that ω is a constant function. Let $e \in E$. Since T is a local isometry, there is a surjective linear isometry $T_{\hat{e}}$ in the set $\mathscr{G}(AC(X,E),AC(Y,F))$ such that $T\hat{e} = T_{\hat{e}}\hat{e}$. According to [14, Theorem 4.1], there exist a monotonic absolutely continuous homeomorphism $\varphi_{\hat{e}} : Y \longrightarrow X$, and a surjective linear isometry $J_{\hat{e}} : E \longrightarrow F$ such that

$$T_{\hat{e}} f(y) = J_{\hat{e}}(f(\varphi_{\hat{e}}(y))) \quad (f \in AC(X,E), y \in Y).$$

Hence letting $f = \hat{e}$, from the above relations, it follows that $\omega(y)(e) = J_{\hat{e}}(e)$ for all $y \in Y_0$. This discussion implies that $\omega(y) = \omega(y')$ for all $y, y' \in Y_0$. Therefore, ω is a constant function, as claimed.

Now, we set $J := \omega(y)$ for some $y \in Y_0$. Note that the above argument shows that for each $e \in E$ we have $J(e) = \omega(y)(e) = J_{\hat{e}}(e)$, which says that J belongs locally to $\mathrm{Iso}(E,F)$.

We now assert that $Y_0 = Y$. For this purpose, take $e \in \mathscr{S}_E$ and define $f(x) = (x - \mathtt{a} + 1)e$ for all $x \in X$, where $\mathtt{a} = \min X$. Since T is a local isometry, there exist a monotonic absolutely continuous homeomorphism $\varphi_f : Y \longrightarrow X$, and $J_f \in \mathrm{Iso}(E,F)$ such that

$$J(f(\varphi(y))) = Tf(y) = J_f(f(\varphi_f(y))) \quad (y \in Y_0).$$

Hence we get

$$J((\varphi(y) - \mathtt{a} + 1)e) = J_f((\varphi_f(y) - \mathtt{a} + 1)e) \quad (y \in Y_0),$$

and consequently,

$$\|(\varphi(y) - \mathtt{a} + 1)J(e)\| = \|(\varphi_f(y) - \mathtt{a} + 1)J_f(e)\| \quad (y \in Y_0).$$

Since J and J_f are isometries, we infer that $\varphi(y) - \mathtt{a} + 1 = \varphi_f(y) - \mathtt{a} + 1$, whence

$$\varphi(y) = \varphi_f(y) \quad (y \in Y_0). \qquad (\star)$$

Suppose, on the contrary, that there is a point $y_0 \in Y \setminus Y_0$. Set $x_0 = \varphi_f(y_0)$. Since φ is surjective, there exists a point y in Y_0 such that $x_0 = \varphi(y)$. Now from $(\star)$ we have $\varphi_f(y) = x_0$, and so $\varphi_f(y) = \varphi_f(y_0)$, which contradicts the injectivity of φ_f. Hence, we conclude that $Y_0 = Y$.

It should be noted, as observed in the above part, $\varphi = \varphi_f$, and in consequence, φ is a monotonic absolutely continuous homeomorphism. Therefore, T is of the form mentioned in the statement.

Furthermore, by [14, Theorem 4.1], it is immediately seen that T is surjective if and only if J is surjective. $\square$

Here we determine the structure of isometric reflections of absolutely continuous function spaces. First let us recall that an *isometric reflection* of a normed space E is a linear isometry $T : E \longrightarrow E$ satisfying $T^2 = I$. Clearly, any isometric reflection is surjective. Let $\mathrm{Iso}^2(E)$ be the set of all isometric reflections of E.

It should be noted that isometric reflections of $AC(X,E)$ are not necessarily weighted composition operators. For instance, let $X = Y = \{1,2\}$ and define $T : AC(X) \longrightarrow AC(Y)$ by $Tf(1) = f(1)$ and $Tf(2) = f(1) - f(2)$. It is easy to check that T is an isometric reflection which is not a weighted composition operator. However, as the following theorem shows, an isometric reflection of $AC(X,E)$ is a weighted composition operator if and only if it belongs to $\mathscr{G}(AC(X,E))$.

Before stating the next theorem, let us denote by $\mathscr{G}^2(AC(X,E))$ the set of all isometric reflections of $AC(X,E)$ which belong to $\mathscr{G}(AC(X,E))$. In fact, $\mathscr{G}^2(AC(X,E)) = \{T \in \mathscr{G}(AC(X,E)) : T^2 = I\}$.

THEOREM 2.2. *(1) $T \in \mathscr{G}^2(AC(X,E))$ if and only if there exist a monotonic absolutely continuous homeomorphism $\varphi : X \longrightarrow X$ with $\varphi^2 = I$, and $J \in \mathrm{Iso}^2(E)$ such that $Tf(x) = J(f(\varphi(x)))$ for all $f \in AC(X,E)$ and $x \in X$.*

In particular, $T \in \mathscr{G}^2(AC(X))$ if and only if there exist a monotonic absolutely continuous homeomorphism of X with $\varphi^2 = I$, and $\lambda \in \{1,-1\}$ such that $Tf(x) = \lambda f(\varphi(x))$ for all $f \in AC(X)$ and $x \in X$.

(2) If T belongs locally to $\mathscr{G}^2(AC(X,E))$, then there exist a monotonic absolutely continuous homeomorphism $\varphi : X \longrightarrow X$ with $\varphi^2 = I$, and a linear isometry $J : E \longrightarrow E$ belonging locally to $\mathrm{Iso}^2(E)$ such that $Tf(x) = J(f(\varphi(x)))$ for all $f \in AC(X,E)$ and $x \in X$. Furthermore, T is surjective if and only if J is surjective.

Proof.

(1) To prove the "only if" part, let $T \in \mathscr{G}^2(AC(X,E))$. According to [14, Theorem 4.1], there exist a monotonic absolutely continuous homeomorphism $\varphi : X \longrightarrow X$, and $J \in \mathrm{Iso}(E)$ such that

$$Tf(x) = J(f(\varphi(x))) \quad (f \in AC(X,E), x \in X).$$

For any $e \in E$ we have

$$e = T^2\hat{e}(x) = T(T\hat{e})(x) = J^2(e) \quad (x \in X),$$

which yields $J^2 = I$. Now, fix a norm one vector $e \in E$ and define $f(x) = xe$ ($x \in X$). Then we get

$$xe = f(x) = (T^2 f)(x) = T(Tf)(x) = J((Tf)(\varphi(x))) = J^2(f(\varphi^2(x))) = \varphi^2(x)e$$
$(x \in X)$.

Hence $\varphi^2(x) = x$ for al $x \in X$, as claimed. Moreover, taking into account [14, Theorem 4.1], it is easy to obtain the "if" part.

(2) From the proof of Theorem 2.1 and the above part (1), one can derive immediately the result. □

Thanks to the above theorems, we can easily deduce our results on the algebraic reflexivity of the sets of isometries of absolutely continuous function spaces as follows.

THEOREM 2.3. *(1) If* $\mathrm{Iso}(E,F)$ *is algebraically reflexive, then* $\mathscr{G}(AC(X,E),AC(Y,F))$ *is algebraically reflexive. In particular, the group* $\mathscr{G}(AC(X))$ *and the group of all unital surjective linear isometries of* $AC(X)$ *are algebraically reflexive.*

(2) If $\mathrm{Iso}^2(E)$ *is algebraically reflexive, then so is* $\mathscr{G}^2(AC(X,E))$*. In particular,* $\mathscr{G}^2(AC(X))$ *is algebraically reflexive.*

Proof.

(1) It is a direct consequence of Theorem 2.1.

(2) Assume that T belongs locally to $\mathscr{G}^2(AC(X,E))$. By Theorem 2.2(2), there exist a monotonic absolutely continuous homeomorphism $\varphi : X \longrightarrow X$ with $\varphi^2 = I$, and a linear isometry $J : E \longrightarrow E$ belonging locally to $\mathrm{Iso}^2(E)$ such that $Tf(x) = J(f(\varphi(x)))$ for all $f \in AC(X,E)$ and $x \in X$. Since $\mathrm{Iso}^2(E)$ is algebraically reflexive, J is surjective, which again taking into account Theorem 2.2(2), yields the surjectivity of T. Therefore, $T \in \mathscr{G}^2(AC(X,E))$, by Theorem 2.2(1). □

COROLLARY 2.4. $\mathrm{Iso}(AC[0,1])$ and $\mathrm{Iso}^2(AC[0,1])$ are algebraically reflexive.

Proof. From [14, Corollary 4.4], $\mathrm{Iso}(AC[0,1]) = \mathscr{G}(AC[0,1])$ and $\mathrm{Iso}^2(AC[0,1]) = \mathscr{G}^2(AC[0,1])$. Now the result is obtained by Theorem 2.3. □

REMARK 2.5. *The above result holds when* $AC[0,1]$ *is equipped with the norm* $\|\cdot\|_\infty + \mathscr{V}(\cdot)$*. Indeed, from* [9, Theorem 2.7] *we deduce that the isometry group of* $(AC[0,1], \|\cdot\|_\infty + \mathscr{V}(\cdot))$ *coincide with* $\mathscr{G}(AC[0,1])$*, and so the claim follows immediately from Corollary 2.4.*

3. Algebraic reflexivity of generalized bi-circular projections on $AC(X,E)$-spaces

DEFINITION 3.1. For a normed space E, a linear projection $P : E \longrightarrow E$ is said to be *generalized bi-circular projection* (gbp) if $T = P + \lambda(I - P)$ is an isometry for some unimodular scalar λ not equal to one. Note that if E is a real normed space, generalized bi-circular projections are those projections P such that $P - (I - P) = 2P - I$ is an isometry.

It is clear that if P is a gbp, then so is its complementary projection $I-P$. Meantime, it is an easy consequence of the definition that T must be a surjective linear isometry (see [11, Lemma 1.1]). Hence the characterization of the surjective linear isometries of E is the key tool to study such operators.

Here by $\mathscr{GBP}(AC(X,E))$ we mean the set of all generalized bi-circular projections of $AC(X,E)$ for which there exists a unimodular scalar $\lambda \neq 1$ such that $T = P+\lambda(I-P) \in \mathscr{G}(AC(X,E))$.

THEOREM 3.2. *For a strictly convex complex normed space E, $P \in \mathscr{GBP}(AC(X,E))$ if and only if one of the following statements holds:*

(i) $P = \frac{I+T}{2}$ *for a unique* $T \in \mathscr{G}^2(AC(X,E))$.

(ii) There exists a gbp P_0 on E such that $Pf(x) = P_0((f(x))$ for all $f \in AC(X,E)$ and $x \in X$.

Meantime, for a strictly convex real normed space E, $P \in \mathscr{GBP}(AC(X,E))$ if and only if (i) holds.

Proof. We prove this theorem by arguments similar to the proof [3, Theorem 3.2]. Let E be a strictly convex complex normed space and $P \in \mathscr{GBP}(AC(X,E))$. Hence there exists a scalar $\lambda \in \mathbb{T}\setminus\{1\}$ such that $T = P+\lambda(I-P)$ belongs to $\mathscr{G}(AC(X,E))$. It is easy to check that $T^2-(\lambda+1)T+\lambda I = 0$ (see, e.g., [11, Lemma 1.1]). We consider two cases as follows:

Case 1. $\lambda = -1$. In this case, from the above relations it follows easily that $P = \frac{I+T}{2}$ and $T^2 = I$, as stated in item (i).

Case 2. $\lambda \neq -1$. According to Theorem 2.2, there exist a monotonic absolutely continuous homeomorphism $\varphi : X \longrightarrow X$, and $J \in \mathrm{Iso}(E)$ such that

$$Tf(x) = J(f(\varphi(x))) \quad (f \in AC(X,E), x \in X).$$

We claim that $\varphi = I$. Contrary to what we claim, assume that, there exists an $x_0 \in X$ so that $x_0 \neq \varphi(x_0)$. Clearly, $\varphi^2(x_0) \neq \varphi(x_0)$ since φ is injective. We choose $f \in AC(X,E)$ with $f(\varphi^2(x_0)) = f(x_0) = 0$ and $f(\varphi(x_0)) = e$ for some $e \in \mathscr{S}_E$. Thus from the representation of T it follows that

$$T^2 f(x_0) = T(Tf)(x_0) = J(Tf(\varphi(x_0))) = J^2(f(\varphi^2(x_0))) = 0,$$

on the other side,

$$\begin{aligned} T^2 f(x_0) &= ((\lambda+1)T-\lambda I)f(x_0) = (\lambda+1)Tf(x_0)-\lambda f(x_0) = (\lambda+1)J(f(\varphi(x_0))) \\ &= (\lambda+1)J(e) \neq 0. \end{aligned}$$

This contradiction shows that $\varphi(x) = x$ for each $x \in X$. Therefore, we get

$$Tf(x) = J(f(x)) \quad (f \in AC(X,E), x \in X).$$

Put $P_0 = \frac{J-\lambda I}{1-\lambda}$. It is easily seen that

$$Pf(x) = \frac{Tf(x) - \lambda f(x)}{1-\lambda} = \frac{J(f(x)) - \lambda f(x)}{1-\lambda} = P_0(f(x)) \quad (f \in AC(X,E), x \in X).$$

Notice that P_0 is a projection since $P^2(\hat{e}) = P(\hat{e})$ for all $e \in E$. Moreover, for any $e \in E$ we have $T^2\hat{e} - (\lambda+1)T\hat{e} + \lambda\hat{e} = 0$, which implies that

$$J^2 - (\lambda+1)J + \lambda I = 0,$$

and in consequence, taking into account [11, Lemma 1.1], we conclude that P_0 is a gbp on E. Whence P is of the form stated in item (ii).

The converse can be verified by simple computations.

Furthermore, from the above discussion it is clear that for a strictly convex real normed space E, $P \in \mathcal{GBP}(AC(X,E))$ if and only if the statement (i) holds. $\square$

Below, we state the scalar-valued version of the above theorem.

COROLLARY 3.3. *$P \in \mathcal{GBP}(AC(X))$ if and only if there exist a monotonic absolutely continuous homeomorphism $\varphi : X \longrightarrow X$ with $\varphi^2 = I$, and $\lambda \in \{1,-1\}$ such that $Pf(x) = \frac{1}{2}[f(x) + \lambda f(\varphi(x))]$ for all $f \in AC(X)$ and $x \in X$.*

Now, taking into account Theorem 3.2, we obtain the following result immediately.

THEOREM 3.4. *If* $\mathrm{Iso}(E)$ or $\mathrm{Iso}^2(E)$ is algebraically reflexive, then the set of all gbp's of $AC(X,E)$ described as the average of the identity with an isometric reflection, is algebraically reflexive. In particular, if E is a real normed space, or E is the scalar field then $\mathcal{GBP}(AC(X,E))$ is algebraically reflexive.

Proof. Let P belongs locally to $\mathcal{GBP}(AC(X,E))$. Then for any $f \in AC(X,E)$, there is an isometric reflection T_f such that $Pf = \frac{(I+T_f)f}{2}$, which yields $(2P-I)f = T_f f$. Hence we observe that $2P-I$ belongs locally to $\mathcal{G}^2(AC(X,E))$. If $\mathrm{Iso}(E)$ or $\mathrm{Iso}^2(E)$ is algebraically reflexive, then $2P-I$ is surjective by Theorem 2.3. Meantime, from this fact that P is a projection, we get $(2P-I)^2 = I$. Consequently, it is deduced that $(2P-I) \in \mathcal{G}^2(AC(X,E))$, and so $P \in \mathcal{GBP}(AC(X,E))$, as desired. Furthermore, the particular case is obtained immediately from Theorem 3.2 and Corollary 3.3. $\square$

REMARK 3.5. *The set of all gbp's of $AC[0,1]$ equipped with either of the norms $\|\cdot\|$ or $\|\cdot\|_\infty + \mathcal{V}(\cdot)$, is algebraically reflexive (combine Remark 2.5 and Theorem 3.4).*

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