Operators and Matrices
Volume 13, Number 4 (2019), 921–936 doi:10.7153/oam-2019-13-65

$\alpha-$FREDHOLM OPERATORS RELATIVE TO INVARIANT SUBSPACES

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(*Communicated by R. Curto*)

Abstract. Let T be a bounded linear operator on a Hilbert space H and let W be a closed $T-$invariant subspace of H. Then T has a matrix representation on the space $W \oplus W^{\perp}$ by $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. In this paper, the relationships between the $\alpha-$Fredholm properties of T and those of the pair of operators A and B are studied.

1. Introduction

Let H be a complex Hilbert space of dimension $h > \aleph_0$ and let α be a cardinal number such that $1 \leqslant \alpha \leqslant h$. A linear subspace K of H is called $\alpha-$closed if there is a closed linear subspace E of H such that $E \subseteq K$ and

$$\dim(\overline{K \cap E^{\perp}}) < \alpha.$$

This concept, introduced by G. Edgar et al. in [8], allowed to generalize the definition of a Fredholm operator. For a bounded linear operator $T \in B(H)$, let $N(T)$ and $R(T)$ the null space and the range, respectively, of the mapping T. Also, let $n(T) = \dim N(T)$ and $d(T) = \dim R(T)^{\perp}$. If the range $R(T)$ of $T \in B(H)$ is $\alpha-$closed and $n(T) < \alpha$ (respectively, $d(T) < \alpha$), then T is said to be an *upper semi* $\alpha-$*Fredholm* (respectively, a *lower semi* $\alpha-$*Fredholm*) operator and we denote $T \in \Phi_{\alpha}^{+}(H)$ (respectively $T \in \Phi_{\alpha}^{-}(H)$). If $T \in \Phi_{\alpha}^{-}(H) \cap \Phi_{\alpha}^{+}(H)$ then we say that T is an $\alpha-$*Fredholm* operator (in notation $T \in \Phi_{\alpha}(H)$). This notion is of interest only when $\alpha > \aleph_0$, since $\aleph_0-$Fredholm operators are Fredholm operators.

For each α, $\aleph_0 \leqslant \alpha \leqslant h$, let $\mathscr{F}_{\alpha}$ denote the two-sided ideal in $B(H)$ of all bounded linear operators such that $\dim \overline{R(T)} < \alpha$ and let $\mathscr{I}_{\alpha}$ denote the norm closure of $\mathscr{F}_{\alpha}$ in $B(H)$. The closed two-sided ideal $\mathscr{I}_{\alpha}$ of $B(H)$ permits consider the quotient space $B(H)/\mathscr{I}_{\alpha}$ as a complex unital Banach algebra. The operators which are left (resp. right) invertible modulo $\mathscr{I}_{\alpha}$ are precisely the upper (resp. lower) semi $\alpha-$Fredholm operators. See [8],[9]. This implies that $\Phi_{\alpha}^{+}(H)$ and $\Phi_{\alpha}^{-}(H)$ are open sets in $B(H)$ for all $\alpha \geqslant \aleph_0$. See, for example, Theorem 2.7.

Corresponding spectra of an operator $T \in B(H)$ are defined as:

Mathematics subject classification (2010): 47A53.
Keywords and phrases: $\alpha-$closed subspaces, $\alpha-$Fredholm operators, invariant subspaces.
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the upper semi $\alpha-$Fredholm spectrum:

$$\sigma_{\alpha u}(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \notin \Phi_{\alpha}^{+}(H)\},$$

the lower semi $\alpha-$Fredholm spectrum:

$$\sigma_{\alpha l}(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \notin \Phi_{\alpha}^{-}(H)\},$$

the $\alpha-$Fredholm spectrum:

$$\sigma_{\alpha}(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \notin \Phi_{\alpha}(H)\}.$$

All of these spectra are non-empty compact subsets of the complex plane.

Let W be a closed subspace of H. We shall use $\mathscr{F}_W(H)$ to denote the set of all bounded operators $T : H \to H$ for which W is $T-$invariant. If $T \in \mathscr{F}_W(H)$ then T has on $W \oplus W^{\perp}$ the matrix representation

$$T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where $A = T|_W$, $B = QT|_{W^{\perp}}$ and $C = PT|_{W^{\perp}}$; here P is the projection of H on W and Q is the projection of H on $W^{\perp}$. In the present paper the relationships between the $\alpha-$Fredholm properties of T and those of the pair of operators A and B are studied. This work has been influenced by the work of Bruce A. Barnes in [4].

The results obtained are applied to show that the $\alpha-$Fredholm spectrum of T, A and B form ([10]) a “love knot”, namely each is a subset of union of the other two. Also, we make a similar observation about the continuity of the $\alpha-$Fredholm spectrum $\sigma_{\alpha} : a \to \sigma_{\alpha}(a)$, from $B(Y)$ to the collection of all non-empty compact subsets of $\mathbb{C}$, for each $a \in \{T, A, B\}$ and each $Y \in \{H, W, W^{\perp}\}$.

2. Preliminary results

The goal of this section consists in establishing some preliminary results which will be needed in the sequel.

PROPOSITION 2.1. [11, Lemma 2.4]. *If H, K are Hilbert spaces and $T \in B(H,K)$ then* $\dim \overline{R(T)} \leqslant \dim H$.

PROPOSITION 2.2. *Let H, K be Hilbert spaces. If there exists an injective bounded linear operator $T : H \to K$ then* $\dim H \leqslant \dim K$.

Proof. Let $\{v_j\}_{j\in J}$ be an orthonormal basis for K. Observe that if $\langle x, T^* v_j\rangle = 0$ for all $j \in J$, then $x = 0$. Indeed, suppose that $x \neq 0$, then since T is injective, $Tx \neq 0$. Thus there exists $j \in J$ such that $\langle Tx, v_j\rangle \neq 0$ and hence $\langle x, T^* v_j\rangle \neq 0$ which is a contradiction. Consequently, $\{T^* v_j\}_{j\in J}$ is a complete system in H. This implies that $H = \overline{\operatorname{span}(\{T^* v_j\}_{j\in J})}$. On the other hand, $R(T^*) = \operatorname{span}(\{T^* v_j\}_{j\in J})$, thus by Proposition 2.1, $\dim H = \dim \overline{\operatorname{span}(\{T^* v_j\}_{j\in J})} = \dim \overline{R(T^*)} \leqslant \dim K$. $\square$

PROPOSITION 2.3. *If L and Y are closed subspaces of H such that $H = L\oplus^{\perp} Y$ then* $\dim L^{\perp} = \dim Y$.

Proof. For each $l \in L^{\perp}$, there exist unique $s_l \in L$ and $t_l \in Y$ such that $l = s_l + t_l$. Define the linear operator $U : L^{\perp} \to Y$ as $U(l) = t_l$. Since $L \perp Y$ it follows that $\|U(l)\|^2 = \|t_l\|^2 \leqslant \|s_l\|^2 + \|t_l\|^2 = \|l\|^2$, therefore U is bounded. Let $l_1, l_2 \in L^{\perp}$ such that $U(l_1) = U(l_2)$, then $l_1 - s_{l_1} = l_2 - s_{l_2}$ and so $l_1 - l_2 = s_{l_2} - s_{l_1} \in L \cap L^{\perp}$, hence $l_1 = l_2$. Now, let $y \in Y$ then there exist unique $u_y \in L$ and $w_y \in L^{\perp}$ such that $y = u_y + w_y$. This implies that $0 \oplus y = y = u_y + w_y = (u_y + s_{w_y}) \oplus t_{w_y}$ and hence $y = t_{w_y}$. Thus $U(w_y) = t_{w_y} = y$. Consequently U is bijective.

From Proposition 2.1, $\dim Y = \dim \overline{U(L^{\perp})} \leqslant \dim L^{\perp}$. And by Proposition 2.2, $\dim L^{\perp} \leqslant \dim Y$. $\square$

PROPOSITION 2.4. *If E, F, Y are closed subspaces of H such that E, F are contained in Y then*

$$\dim[(E\cap F)^{\perp} \cap F] \leqslant \dim(Y \cap E^{\perp}).$$

Proof. Since $E = (E^{\perp} \cap Y)^{\perp} \cap Y$, it follows that

$$\begin{aligned}(E\cap F)^{\perp} \cap F &= [((E^{\perp}\cap Y)^{\perp} \cap Y)\cap F]^{\perp} \cap F = [(E^{\perp}\cap Y)^{\perp}\cap F]^{\perp}\cap F \\ &= [E^{\perp}\cap Y + F^{\perp}]^{\perp\perp} \cap F = \overline{E^{\perp}\cap Y + F^{\perp}} \cap F^{\perp\perp}.\end{aligned}$$

Moreover, since $F^{\perp} \subseteq F^{\perp} + E^{\perp}\cap Y$, from [8, Lemma 2.2] we obtain that

$$\overline{E^{\perp}\cap Y + F^{\perp}} \cap F^{\perp\perp} = \overline{[E^{\perp}\cap Y + F^{\perp}]\cap F^{\perp\perp}}.$$

Consequently,

$$(E\cap F)^{\perp} \cap F = \overline{[E^{\perp}\cap Y + F^{\perp}]\cap F}. \tag{2.1}$$

On the other hand, observe that

$$H = F \oplus F^{\perp}$$

and

$$F = (E\cap F) \oplus [(E\cap F)^{\perp} \cap F].$$

This implies that for each $z \in Y\cap E^{\perp}$, there exist unique $u_z \in E\cap F$, $v_z \in (E\cap F)^{\perp}\cap F$ and $w_z \in F^{\perp}$ such that $z = u_z \oplus v_z \oplus w_z$. Define $S : Y\cap E^{\perp} \to (E\cap F)^{\perp}\cap F$ as $S(z) = v_z$. Clearly S is a bounded linear operator. Let $f \in [E^{\perp}\cap Y + F^{\perp}]\cap F$, then by (2.1), $f \in (E\cap F)^{\perp}\cap F$, also there exist $e^* \in E^{\perp}\cap Y$ and $w^* \in F^{\perp}$ such that $f = e^* + w^*$. Therefore $e^* = 0 \oplus f \oplus (-w^*) \in [E\cap F] \oplus [(E\cap F)^{\perp}\cap F] \oplus F^{\perp}$ and so $S(e^*) = f$. Consequently, $[E^{\perp}\cap Y + F^{\perp}]\cap F \subseteq R(S)$. Thus by (2.1),

$$\overline{R(S)} = (E\cap F)^{\perp}\cap F.$$

Finally, by Proposition 2.1, $\dim[(E\cap F)^{\perp}\cap F] = \dim\overline{R(S)} \leqslant \dim Y\cap E^{\perp}$. $\square$

It is well known that if $T \in B(H)$ and $S \in B(H)$ are $\alpha-$Fredholm operators then ST is an $\alpha-$Fredholm operator, see [3, Lemma 3.1]. The following theorem shows a similar result for upper and lower semi $\alpha-$Fredholm operators.

THEOREM 2.5. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. For every S,T operators in $B(H)$ the following statements hold:*

(1) if $T \in \Phi_\alpha^+(H)$ and $S \in \Phi_\alpha^+(H)$, then $TS \in \Phi_\alpha^+(H)$;

(2) if $T \in \Phi_\alpha^-(H)$ and $S \in \Phi_\alpha^-(H)$, then $TS \in \Phi_\alpha^-(H)$;

(3) if $ST \in \Phi_\alpha^+(H)$, then $T \in \Phi_\alpha^+(H)$;

(4) if $ST \in \Phi_\alpha^-(H)$, then $S \in \Phi_\alpha^-(H)$.

Proof. We only prove *(1)* and *(4)*.

(1) By [8, Theorem 2.6], the operators T,S are left invertible modulo $\mathscr{I}_\alpha$, hence there exist $U,V \in B(H)$ such that $(U+\mathscr{I}_\alpha)(T+\mathscr{I}_\alpha) = I+\mathscr{I}_\alpha$ and $(V+\mathscr{I}_\alpha)(S+\mathscr{I}_\alpha) = I+\mathscr{I}_\alpha$. This implies that $UT-I, VS-I \in \mathscr{I}_\alpha$. Now, since $\mathscr{I}_\alpha$ is a two-sided ideal of $B(H)$, it follows that $VUTS-VS \in \mathscr{I}_\alpha$. Thus

$$[VUTS-I-(VS-I)]+(VS-I) \in \mathscr{I}_\alpha,$$

hence $VUTS-I \in \mathscr{I}_\alpha$, i.e.,

$$(VU+\mathscr{I}_\alpha)(TS+\mathscr{I}_\alpha) = I+\mathscr{I}_\alpha.$$

Therefore, by [8, Theorem 2.6], $TS \in \Phi_\alpha^+(H)$.

(4) Since $ST \in \Phi_\alpha^-(H)$, by [9, Theorem 4], it follows that ST is right invertible modulo $\mathscr{F}_\alpha$, i.e., there exists $U \in B(H)$ such that $(ST+\mathscr{F}_\alpha)(U+\mathscr{F}_\alpha) = I+\mathscr{F}_\alpha$. Therefore $(S+\mathscr{F}_\alpha)(TU+\mathscr{F}_\alpha) = I+\mathscr{F}_\alpha$ i.e. S is right invertible modulo $\mathscr{F}_\alpha$. Thus, again by [9, Theorem 4], $S \in \Phi_\alpha^-(H)$. □

PROPOSITION 2.6. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. For every operator $T \in B(H)$ the following assertions hold:*

(1) $T \in \Phi_\alpha^+(H)$ if and only if $T^* \in \Phi_\alpha^-(H)$;

(2) $T \in \Phi_\alpha^-(H)$ if and only if $T^* \in \Phi_\alpha^+(H)$.

Proof. By [9, Theorem 2], $R(T)$ is $\alpha-$closed if and only if $R(T^*)$ is $\alpha-$closed. Thus the conclusion of the proposition holds, because $n(T) = \dim N(T) = \dim R(T^*)^\perp = d(T^*)$ and $d(T) = \dim R(T)^\perp = \dim \overline{R(T)}^\perp = \dim N(T^*) = n(T^*)$. □

In [3, Lemma 2.1] was observed that $\Phi_\alpha(H)$ is an open set. We show in the next theorem that $\Phi_\alpha^+(H)$ and $\Phi_\alpha^-(H)$ are also open sets.

THEOREM 2.7. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. Then $\Phi_\alpha^+(H)$, $\Phi_\alpha^-(H)$ and $\Phi_\alpha(H)$ are open sets in $B(H)$.*

Proof. Let $\mathscr{G}_l$ the set of all left invertible elements in $B(H)/\mathscr{I}_\alpha$. From [6, Theorem], $\mathscr{G}_l$ is an open set in $B(H)/\mathscr{I}_\alpha$. Take $T \in \Phi_\alpha^+(H)$, then by [8, Theorem 2.6], $T+\mathscr{I}_\alpha \in \mathscr{G}_l$. Thus, there exists $r>0$ such that if $\|U+\mathscr{I}_\alpha-(T+\mathscr{I}_\alpha)\|<r$ then $U+\mathscr{I}_\alpha \in \mathscr{G}_l$. Let $S \in B(H)$ such that $\|S-T\|<r$. Since $\|S+\mathscr{I}_\alpha-(T+\mathscr{I}_\alpha)\| \leqslant \|S-T\|$, it follows that $S+\mathscr{I}_\alpha \in \mathscr{G}_l$, and so by [8, Theorem 2.6], $S \in \Phi_\alpha^+(H)$. The other cases are analogous. $\square$

3. $\alpha-$ Fredholm properties of T involving its diagonal

Throughout this paper, given a bounded operator $T \in \mathscr{F}_W(H)$ we shall denote by A the restriction $T|_W$, by B the operator $QT|_{W^\perp}$ and by C the operator $PT|_{W^\perp}$, where P is the projection of H on W and Q is the projection of H on $W^\perp$.

PROPOSITION 3.1. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. Let $U \in B(W)$, $V \in B(W^\perp)$ and U_1, V_1 be bounded operators defined on $W \oplus W^\perp$ as*

$$U_1 = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \quad \textit{and} \quad V_1 = \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}.$$

Then, the following conditions hold:

(1) $R(U)$ is $\alpha-$ closed if and only if $R(U_1)$ is $\alpha-$ closed;

(2) $n(U_1) = n(U)$ and $d(U_1) = d(U)$.

A similar statements hold if we replace U, U_1 by V, V_1.

Proof.

(1) Suppose that $R(U)$ is $\alpha-$ closed, namely there exists a closed linear subspace Z of W such that $Z \subseteq R(U)$ and $\dim[\overline{R(U) \cap (Z^\perp \cap W)}] < \alpha$. We set $E = Z \oplus W^\perp$, then E is a closed linear subspace of H such that $E \subseteq R(U_1)$ and $R(U_1) \cap E^\perp = R(U_1) \cap (Z \oplus W^\perp)^\perp = R(U_1) \cap Z^\perp \cap W = R(U) \cap (Z^\perp \cap W)$. Therefore $\dim(\overline{R(U_1) \cap E^\perp}) = \dim(\overline{R(U) \cap Z^\perp \cap W}) < \alpha$, thus $R(U_1)$ is $\alpha-$ closed.

Now, suppose that $R(U_1)$ is $\alpha-$ closed. Then there exists a closed linear subspace E of H such that $E \subseteq R(U_1)$ and $\dim \overline{R(U_1) \cap E^\perp} < \alpha$. Let $D = E \cap \overline{R(U)}$, so D is a closed linear subspace of W and

$$D = E \cap \overline{R(U)} \subseteq R(U_1) \cap W = R(U).$$

By [8, Lemma 2.2], $\overline{R(U) \cap D^\perp} = \overline{R(U)} \cap D^\perp$. Then by Proposition 2.4,

$$\begin{aligned} \dim \overline{R(U)} \cap D^\perp &= \dim[(E \cap \overline{R(U)})^\perp \cap \overline{R(U)}] \leqslant \dim[\overline{R(U_1)} \cap E^\perp] \\ &= \dim \overline{R(U_1) \cap E^\perp} \ < \ \alpha. \end{aligned}$$

(2) It is clear that $n(U_1) = \dim N(U_1) = \dim[N(U) \oplus \{0\}] = \dim N(U) = n(U)$. Moreover, $d(U_1) = \dim R(U_1)^{\perp} = \dim[R(U) \oplus W^{\perp}]^{\perp} = \dim[R(U)^{\perp} \cap W] = d(U)$.

The statements with respect to the operators V and V_1 are proved in a similar way. □

REMARK 3.2. *As consequence of Proposition 3.1, we have that:*

(1) $U_1 \in \Phi_{\alpha}^{-}(H)$ *if and only if* $U \in \Phi_{\alpha}^{-}(W)$*;*

(2) $U_1 \in \Phi_{\alpha}^{+}(H)$ *if and only if* $U \in \Phi_{\alpha}^{+}(W)$*.*

Also,

(3) $V_1 \in \Phi_{\alpha}^{-}(H)$ *if and only if* $V \in \Phi_{\alpha}^{-}(W^{\perp})$*;*

(4) $V_1 \in \Phi_{\alpha}^{+}(H)$ *if and only if* $V \in \Phi_{\alpha}^{+}(W^{\perp})$*.*

THEOREM 3.3. *Let* α *be a cardinal number such that* $\aleph_0 \leqslant \alpha \leqslant h$*. For every* $T \in \mathscr{F}_W(H)$ *we have:*

(1) if $A \in \Phi_{\alpha}^{+}(W)$ *and* $B \in \Phi_{\alpha}^{+}(W^{\perp})$*, then* $T \in \Phi_{\alpha}^{+}(H)$*;*

(2) if $A \in \Phi_{\alpha}^{-}(W)$ *and* $B \in \Phi_{\alpha}^{-}(W^{\perp})$*, then* $T \in \Phi_{\alpha}^{-}(H)$*;*

(3) if $T \in \Phi_{\alpha}^{+}(H)$*, then* $A \in \Phi_{\alpha}^{+}(W)$*;*

(4) if $T \in \Phi_{\alpha}^{-}(H)$*, then* $B \in \Phi_{\alpha}^{-}(W^{\perp})$*.*

Proof. We only prove *(1)* and *(3)*. Let A_1, B_1 and C_1 be bounded operators defined on $W \oplus W^{\perp}$ as

$$A_1 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad B_1 = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}.$$

Then $T = B_1 C_1 A_1$ and C_1 is invertible. Assume first that $A \in \Phi_{\alpha}^{+}(W)$ and $B \in \Phi_{\alpha}^{+}(W^{\perp})$. From Remark 3.2 *(2)* and *(4)*, we have $A_1, B_1 \in \Phi_{\alpha}^{+}(H)$, so by Theorem 2.5 *(1)*, $T = B_1 C_1 A_1 \in \Phi_{\alpha}^{+}(H)$.

Now, suppose that $(B_1 C_1) A_1 = T \in \Phi_{\alpha}^{+}(H)$. From Theorem 2.5 *(3)*, we have that $A_1 \in \Phi_{\alpha}^{+}(H)$ and, this implies by Remark 3.2 *(2)*, that $A \in \Phi_{\alpha}^{+}(W)$. □

As an immediate consequence of parts *(1)* and *(2)* of Theorem 3.3 we have the following corollary.

COROLLARY 3.4. *Let* α *be a cardinal number such that* $\aleph_0 \leqslant \alpha \leqslant h$*. For every* $T \in \mathscr{F}_W(H)$ *we have:*

(1) $\sigma_{\alpha u}(T) \subseteq \sigma_{\alpha u}(A) \cup \sigma_{\alpha u}(B)$*;*

(2) $\sigma_{\alpha l}(T) \subseteq \sigma_{\alpha l}(A) \cup \sigma_{\alpha l}(B)$*.*

LEMMA 3.5. *Let $T \in \mathscr{F}_W(H)$. If there exists a closed linear subspace E of H such that $E \subseteq R(T)$, then*

$$d(A) \leqslant n(B) + \dim[\overline{R(T)} \cap E^{\perp}] + d(T).$$

Proof. Suppose that E is a closed linear subspace of H such that $E \subseteq R(T)$. From $R(A)^{\perp} \subseteq (\overline{R(A)} \cap E)^{\perp}$, it follows that

$$d(A) = \dim[R(A)^{\perp} \cap W] \leqslant \dim[(\overline{R(A)} \cap E)^{\perp} \cap W]. \tag{3.1}$$

Consider the decompositions

$$W = (E \cap W) \oplus [(E \cap W)^{\perp} \cap W]$$

and

$$E \cap W = (E \cap \overline{R(A)}) \oplus [(E \cap \overline{R(A)})^{\perp} \cap (E \cap W)].$$

Then

$$W = (E \cap \overline{R(A)}) \oplus \Big[[(E \cap \overline{R(A)})^{\perp} \cap (E \cap W)] \oplus [(E \cap W)^{\perp} \cap W]\Big],$$

so by Proposition 2.3,

$$\dim[(E \cap \overline{R(A)})^{\perp} \cap W] = \dim[(E \cap \overline{R(A)})^{\perp} \cap (E \cap W)] + \dim[(E \cap W)^{\perp} \cap W]. \tag{3.2}$$

From Proposition 2.4, $\dim[(E \cap W)^{\perp} \cap W] \leqslant \dim(H \cap E^{\perp}) = \dim E^{\perp}$. Moreover, since $\overline{R(T)} = E \oplus [E^{\perp} \cap \overline{R(T)}]$, it follows that $H = \overline{R(T)} \oplus R(T)^{\perp} = E \oplus [(\overline{R(T)} \cap E^{\perp}) \oplus R(T)^{\perp}]$. Consequently by Proposition 2.3, $\dim E^{\perp} = \dim(\overline{R(T)} \cap E^{\perp}) + \dim R(T)^{\perp} = \dim(\overline{R(T)} \cap E^{\perp}) + d(T)$.

Therefore

$$\dim[(E \cap W)^{\perp} \cap W] \leqslant \dim(\overline{R(T)} \cap E^{\perp}) + d(T). \tag{3.3}$$

We prove that $\dim[(E \cap \overline{R(A)})^{\perp} \cap (E \cap W)] \leqslant \dim N(B)$. Let

$$Y = N(T)^{\perp} \cap T^{-1}(E). \tag{3.4}$$

Then, Y is a closed linear subspace of H and $T|_Y$ is bounded below. Indeed, take $u \in E$, so there exists $x \in H$ such that $u = Tx$. Consider the representation $x = x_1 \oplus x_2$, where $x_1 \in N(T)$ and $x_2 \in N(T)^{\perp}$, thus $Tx_2 = Tx = u$ which implies that $x_2 \in T^{-1}(E) \cap N(T)^{\perp} (= Y)$ and hence $u \in T(Y)$. This shows that $E \subseteq T(Y) (\subseteq E)$. On the other hand, since $N(T) \cap Y = \{0\}$, it follows that $N(T|_Y) = \{0\}$. Therefore, $T|_Y$ is bounded below.

Then, for each $y \in E \cap W$, there exists an unique $x_y \in Y$ such that $y = Tx_y$. Also, there are unique $w_y \in W$ and $v_y \in W^{\perp}$ such that $x_y = w_y \oplus v_y$. From $[Aw_y + Cv_y] \oplus Bv_y = Tx_y = y \in W$, it follows that $v_y \in N(B)$. Define $U : (E \cap \overline{R(A)})^{\perp} \cap (E \cap W) \to$

$N(B)$ as $U(y) = v_y$. This operator is linear and bounded. Indeed, since $T|_Y$ is bounded below, there exists $M > 0$ such that $\|Tx\| \geqslant M\|x\|$ for all $x \in Y$. Then

$$\|Uy\| = \|v_y\| \leqslant \|w_y + v_y\| = \|x_y\| \leqslant \frac{1}{M}\|Tx_y\| = \frac{1}{M}\|y\|.$$

Let $y_1, y_2 \in (E \cap \overline{R(A)})^\perp \cap (E \cap W)$ be such that $U(y_1) = U(y_2)$. Then $x_{y_1} - w_{y_1} = v_{y_1} = v_{y_2} = x_{y_2} - w_{y_2}$ which implies that $x_{y_1} - x_{y_2} = w_{y_1} - w_{y_2}$ and hence $y_1 - y_2 = T(x_{y_1} - x_{y_2}) = T(w_{y_1} - w_{y_2}) \in [\overline{R(A)} \cap E] \cap (E \cap \overline{R(A)})^\perp = \{0\}$. Thus $y_1 = y_2$, i.e. U is injective. Therefore, by Proposition 2.2,

$$\dim[(E \cap \overline{R(A)})^\perp \cap (E \cap W)] \leqslant \dim N(B). \tag{3.5}$$

Consequently, by (3.1), (3.2), (3.3) and (3.5),

$$d(A) \leqslant n(B) + \dim[\overline{R(T)} \cap E^\perp] + d(T). \quad \square$$

LEMMA 3.6. *Let $T \in \mathscr{F}_W(H)$. If there exists a closed linear subspace F of W such that $F \subseteq R(A)$, then*

$$n(B) \leqslant n(T) + \dim[\overline{R(A)} \cap F^\perp] + d(A).$$

Proof. Take a closed linear subspace F of W such that $F \subseteq R(A)$. Note that $N(B)$ is contained in the pre-image $T^{-1}(W) = \{h \in H \mid Th \in W\}$, thus

$$\dim N(B) \leqslant \dim T^{-1}(W) \cap W^\perp. \tag{3.6}$$

In similar way to (3.4), it follows that A is bounded below on $Y = [N(A)^\perp \cap W] \cap A^{-1}(F)$. Let $x \in T^{-1}(W) \cap W^\perp$ be arbitrary, then $Tx \in W$ and so there exist unique $f_x \in F(\subseteq R(A))$ and $g_x \in F^\perp \cap W$ such that $Tx = f_x \oplus g_x$. Take an unique $y_x \in Y$ such that $f_x = Ty_x$, then $T(x - y_x) = g_x \in F^\perp \cap W$. Define $V : T^{-1}(W) \cap W^\perp \to T^{-1}(F^\perp \cap W)$ as $V(x) = x - y_x$. It is clear that V is a linear operator. In order to prove that V is bounded, consider $M > 0$ such that $\|Ax\| \geqslant M\|x\|$ for all $x \in Y$. Then for every $x \in T^{-1}(W) \cap W^\perp$,

$$\begin{aligned}\|V(x)\| = \|x - y_x\| \leqslant \|x\| + \|y_x\| &\leqslant \|x\| + \frac{1}{M}\|Ay_x\| = \|x\| + \frac{1}{M}\|f_x\| \\ &\leqslant \|x\| + \frac{1}{M}\|f_x + g_x\| = \|x\| + \frac{1}{M}\|Tx\| \leqslant (1 + \frac{\|T\|}{M})\|x\|.\end{aligned}$$

Thus V is bounded. Moreover V is injective, because if $x_1, x_2 \in T^{-1}(W) \cap W^\perp$ are such that $V(x_1) = V(x_2)$ then $x_1 - y_{x_1} = x_2 - y_{x_2}$ and so $x_1 - x_2 = y_{x_1} - y_{x_2} \in W^\perp \cap W$ i.e. $x_1 = x_2$. Therefore, by Proposition 2.2,

$$\dim T^{-1}(W) \cap W^\perp \ \leqslant\ \dim T^{-1}(F^\perp \cap W). \tag{3.7}$$

On the other hand, since $N(T) \subseteq T^{-1}(F^\perp \cap W)$ it follows that

$$\dim T^{-1}(F^\perp \cap W) = \dim N(T) + \dim[N(T)^\perp \cap T^{-1}(F^\perp \cap W)] \leqslant n(T) + \dim(F^\perp \cap W), \tag{3.8}$$

where the last inequality is because the application $T: N(T)^{\perp}\cap T^{-1}(F^{\perp}\cap W)\to F^{\perp}\cap W$ is bounded and injective. From the equalities

$$W=\overline{R(A)}\oplus[R(A)^{\perp}\cap W]$$

and

$$\overline{R(A)}=F\oplus[F^{\perp}\cap\overline{R(A)}]$$

we obtain that $W=F\oplus[(\overline{R(A)}\cap F^{\perp})\oplus(R(A)^{\perp}\cap W)]$ and so by Proposition 2.3,

$$\dim[F^{\perp}\cap W]=\dim[\overline{R(A)}\cap F^{\perp}]+d(A). \tag{3.9}$$

Consequently by (3.6), (3.7), (3.8) and (3.9),

$$n(B)\leqslant n(T)+\dim[\overline{R(A)}\cap F^{\perp}]+d(A). \quad \square$$

As an immediate consequence of Lemmas 3.5 and 3.6 we obtain the next theorem.

THEOREM 3.7. *Let α be a cardinal number such that $\aleph_0\leqslant\alpha\leqslant h$ and let $T\in\mathscr{F}_W(H)$. The following conditions hold:*

(1) if $T\in\Phi_{\alpha}^{-}(H)$ and $n(B)<\alpha$, then $d(A)<\alpha$;

(2) if $T\in\Phi_{\alpha}^{+}(H)$ and $d(A)<\alpha$, then $n(B)<\alpha$.

Proof.

(1) It is an immediately consequence of Lemma 3.5.

(2) If $T\in\Phi_{\alpha}^{+}(H)$ then $n(T)<\alpha$, and from Theorem 3.3 *(3)*, $A\in\Phi_{\alpha}^{+}(W)$. Thus there exists a closed linear subspace F of W such that $F\subseteq R(A)$ and dim $[\overline{R(A)\cap(F^{\perp}\cap W)}]<\alpha$. From [8, Lemma 2.2], $\overline{R(A)\cap(F^{\perp}\cap W)}=\overline{R(A)}\cap(F^{\perp}\cap W)$. Consequently by Lemma 3.6,

$$n(B)\leqslant n(T)+\dim[\overline{R(A)}\cap F^{\perp}]+d(A)=n(T)+\overline{R(A)\cap(F^{\perp}\cap W)}+d(A)<\alpha. \quad \square$$

The following corollary is a version of [4, Theorem 8] for $\alpha-$ Fredholm operators.

COROLLARY 3.8. *Let $T\in\mathscr{F}_W(H)$ and let α be a cardinal number such that $\aleph_0\leqslant\alpha\leqslant h$. The following statements are equivalent:*

(1) $T\in\Phi_{\alpha}(H)$ and $n(B)<\alpha$;

(2) $T\in\Phi_{\alpha}(H)$ and $d(A)<\alpha$;

(3) $A\in\Phi_{\alpha}(W)$ and $B\in\Phi_{\alpha}(W^{\perp})$.

Proof. *(1)* $\Rightarrow$ *(2)* It follows from Theorem 3.7 *(1)*.
(2) $\Rightarrow$ *(3)* From Theorem 3.3 *(3)* and *(4)*, we have that $A \in \Phi_\alpha^+(W)$ and $B \in \Phi_\alpha^-(W^\perp)$. Since $d(A) < \alpha$, it follows by Theorem 3.7 *(2)*, that $n(B) < \alpha$. Therefore $A \in \Phi_\alpha(W)$ and $B \in \Phi_\alpha(W^\perp)$.
(3) $\Rightarrow$ *(1)* By Theorem 3.3 *(1)* and *(2)*, $T \in \Phi_\alpha(H)$. Obviously, by hypothesis, $n(B) < \alpha$. $\square$

Of this corollary it follows that the $\alpha-$Fredholm spectrum of T, A and B form a "love knot".

COROLLARY 3.9. *If $T \in \mathscr{F}_W(H)$ then:*

(1) $\sigma_\alpha(T) \subseteq \sigma_\alpha(A) \cup \sigma_\alpha(B)$*;*

(2) $\sigma_\alpha(A) \subseteq \sigma_\alpha(T) \cup \sigma_\alpha(B)$*;*

(3) $\sigma_\alpha(B) \subseteq \sigma_\alpha(T) \cup \sigma_\alpha(A)$.

Moreover,

(4) $(\sigma_\alpha(A) \cup \sigma_\alpha(B)) \setminus \sigma_\alpha(T) \subseteq \sigma_\alpha(A) \cap \sigma_\alpha(B)$*;*

(5) $(\sigma_\alpha(T) \cup \sigma_\alpha(B)) \setminus \sigma_\alpha(A) \subseteq \sigma_\alpha(T) \cap \sigma_\alpha(B)$*;*

(6) $(\sigma_\alpha(T) \cup \sigma_\alpha(A)) \setminus \sigma_\alpha(B) \subseteq \sigma_\alpha(T) \cap \sigma_\alpha(A)$.

THEOREM 3.10. *Let $D_1 \in B(W)$, $D_2 \in B(W^\perp)$ and α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. If for every $D \in B(W^\perp, W)$, M_D is defined on $W \oplus W^\perp$ by*

$$M_D = \begin{bmatrix} D_1 & D \\ 0 & D_2 \end{bmatrix},$$

then

$$\bigcap_{D \in B(W^\perp, W)} \sigma_\alpha(M_D) \supseteq \sigma_{\alpha u}(D_1) \cup \sigma_{\alpha l}(D_2) \cup \mathscr{W},$$

where $\mathscr{W} = \{\lambda \in \mathbb{C} \mid n(\lambda - D_2) \neq d(\lambda - D_1)$ and at least one these cardinals is greater than or equal to $\alpha\}$.

Proof. From Theorem 3.3 *(3)* and *(4)*, it follows that for every $D \in B(W^\perp, W)$,

$$\mathbb{C} \setminus \sigma_\alpha(M_D) \subseteq \mathbb{C} \setminus (\sigma_{\alpha u}(D_1) \cup \sigma_{\alpha l}(D_2)).$$

Consequently, $\sigma_{\alpha u}(D_1) \cup \sigma_{\alpha l}(D_2)) \subseteq \sigma_\alpha(M_D)$ for all $D \in B(W^\perp, W)$. Let $\lambda \in \mathscr{W}$ and suppose that $\lambda \notin \sigma_\alpha(M_D)$ for some $D \in B(W^\perp, W)$. Then $\lambda - M_D \in \Phi_\alpha(H)$. This implies that $n(\lambda - M_D) < \alpha$, $d(\lambda - M_D) < \alpha$ and there exists a closed linear subspace E of H such that $E \subseteq R(\lambda - M_D)$ and $\dim \overline{R(\lambda - M_D) \cap E^\perp} < \alpha$. Also, by Theorem 3.3 *(3)*, there exists a closed linear subspace F of W such that $F \subseteq R(\lambda - D_1)$ and $\dim \overline{R(\lambda - D_1) \cap (F^\perp \cap W)} < \alpha$. Therefore by Lemmas 3.5 and 3.6, we have that

$$d(\lambda - D_1) \leqslant n(\lambda - D_2) + \dim[\overline{R(\lambda - M_D)} \cap E^\perp] + d(\lambda - M_D)$$

and
$$n(\lambda-D_2)\leqslant n(\lambda-M_D)+\dim[\overline{R(\lambda-D_1)}\cap(F^{\perp}\cap W)]+d(\lambda-D_1).$$
Consequently,
$$d(\lambda-D_1)\leqslant n(\lambda-D_2)+\alpha \tag{3.10}$$
and
$$n(\lambda-D_2)\leqslant d(\lambda-D_1)+\alpha. \tag{3.11}$$

If $n(\lambda-D_2)\geqslant\alpha$ and $d(\lambda-D_1)\geqslant\alpha$ then, by inequalities (3.10) and (3.11), $n(\lambda-D_2)=d(\lambda-D_1)$. This contradicts the fact that $n(\lambda-D_2)\neq d(\lambda-D_1)$. Now, if $n(\lambda-D_2)<\alpha$ then by inequality (3.10), $d(\lambda-D_1)<\alpha$ which is a contradiction, because at least one the cardinals $n(\lambda-D_2)$ or $d(\lambda-D_1)$ is greater than or equal to α. Finally, if $d(\lambda-D_1)<\alpha$ then by inequality (3.11), $n(\lambda-D_2)<\alpha$, again a contradiction. In any case we have a contradiction. Therefore $\lambda\in\sigma_\alpha(M_D)$ for all $D\in B(W^{\perp},W)$. $\square$

In similar way to [4, Proposition 7] we have the following theorem for arbitrary dimensions.

THEOREM 3.11. *Let $T\in\mathscr{F}_W(H)$, then the following assertions hold:*

(1) $n(T)\leqslant n(A)+n(B)$; moreover, if $R(A)=R(T)\cap W$, then $n(T)=n(A)+n(B)$;

(2) $d(T)\leqslant d(A)+d(B)$; moreover, if $\overline{R(A)}=\overline{R(T)}\cap W$, then $d(T)=d(A)+d(B)$.

Proof.

(1) Consider the decomposition
$$N(T)=N(A)\oplus[N(A)^{\perp}\cap N(T)]. \tag{3.12}$$
Let $Y=N(A)^{\perp}\cap N(T)$. For each $y\in Y$, there exist unique $w_y\in W$ and $v_y\in W^{\perp}$ such that $y=w_y\oplus v_y$. Observe that
$$\begin{bmatrix}0\\0\end{bmatrix}=Ty=\begin{bmatrix}A&C\\0&B\end{bmatrix}\begin{bmatrix}w_y\\v_y\end{bmatrix}=\begin{bmatrix}Aw_y+Cv_y\\Bv_y\end{bmatrix},$$
thus $v_y\in N(B)$. Define $U:Y\to N(B)$ by $U(y)=v_y$. It is clear that U is a continuous linear operator. Let $y_1,y_2\in Y$ be such that $U(y_1)=U(y_2)$, then $v_{y_1}=v_{y_2}$. This implies that $y_1-y_2=w_{y_1}-w_{y_2}+v_{y_1}-v_{y_2}=w_{y_1}-w_{y_2}$ and hence $y_1-y_2\in W$. Thus $A(y_1-y_2)=T(y_1-y_2)=Ty_1-Ty_2=0$. Therefore $y_1-y_2\in N(A)\cap N(A)^{\perp}(=\{0\})$, i.e. $y_1=y_2$, which implies that U is injective. Consequently, by (3.12) and Proposition 2.2,
$$n(T)=n(A)+\dim Y\leqslant n(A)+n(B).$$
Now, suppose that $R(A)=R(T)\cap W$. Take $z\in N(B)$, then
$$Tz=\begin{bmatrix}A&C\\0&B\end{bmatrix}\begin{bmatrix}0\\z\end{bmatrix}=\begin{bmatrix}Cz\\Bz\end{bmatrix}=\begin{bmatrix}Cz\\0\end{bmatrix}=Cz\oplus 0=Cz\in W.$$

Therefore $Tz \in R(T) \cap W (= T(W))$, thus $Tz = Tw$ for some $w \in W$. This implies that $z - w \in N(T)$, and so by (3.12), there exist $x \in N(A)$ and $y \in Y$ such that $z - w = x \oplus y$. Finally, since $y = -w - x + z$, $-w - x \in W$ and $z \in W^\perp$, it follows that $U(y) = z$, which implies that U is surjective. Consequently, by Proposition 2.1, $\dim N(B) = \dim \overline{U(Y)} \leqslant \dim Y$, and hence

$$n(T) = n(A) + \dim Y \geqslant \dim N(A) + n(B).$$

(2) From the inclusion $R(T) \subseteq W \oplus R(B)$, it follows that $W^\perp \cap R(B)^\perp \subseteq R(T)^\perp$. Thus,

$$R(T)^\perp = (W^\perp \cap R(B)^\perp) \oplus [(W^\perp \cap R(B)^\perp)^\perp \cap R(T)^\perp].$$

Moreover, observe that $(W^\perp \cap R(B)^\perp)^\perp \cap R(T)^\perp = (W + \overline{R(B)}) \cap R(T)^\perp$. Therefore

$$d(T) = \dim[W^\perp \cap R(B)^\perp] + \dim[(W + \overline{R(B)}) \cap R(T)^\perp]. \tag{3.13}$$

For each $y \in R(A)^\perp \cap W$, there exist unique $r_y \in \overline{R(T)}$ and $s_y \in R(T)^\perp$ such that $y = r_y \oplus s_y$. Let us consider the operator S defined on $R(A)^\perp \cap W$ as $S(y) = s_y$. Clearly S is linear and bounded. We prove that

$$\overline{R(S)} = (W + \overline{R(B)}) \cap R(T)^\perp.$$

First note that $(W + \overline{R(B)}) \cap R(T)^\perp = \overline{W + \overline{R(T)}} \cap R(T)^\perp$, and by [8, Lemma 2.2], $\overline{W + \overline{R(T)}} \cap R(T)^\perp = \overline{(W + \overline{R(T)}) \cap R(T)^\perp}$. Thus

$$(W + \overline{R(B)}) \cap R(T)^\perp = \overline{(W + \overline{R(T)}) \cap R(T)^\perp}. \tag{3.14}$$

Let $y \in R(A)^\perp \cap W$, then $S(y) = s_y = y - r_y \in [W + \overline{R(T)}] \cap R(T)^\perp$. Therefore, $R(S) \subseteq (W + \overline{R(T)}) \cap R(T)^\perp$. On the other hand, let $s \in (W + \overline{R(T)}) \cap R(T)^\perp$, then there exist $w \in W$ and $r \in \overline{R(T)}$ such that $s = w + r$. Also, there exist $u \in \overline{R(A)}$ and $v \in R(A)^\perp \cap W$ such that $w = u + v$. Thus, $v = (-u - r) + s \in \overline{R(T)} \oplus R(T)^\perp$ and so $S(v) = s$. Therefore $(W + \overline{R(T)}) \cap R(T)^\perp \subseteq R(S)$, which implies that

$$R(S) = (W + \overline{R(T)}) \cap R(T)^\perp.$$

Consequently by (3.14), $\overline{R(S)} = \overline{(W + \overline{R(T)}) \cap R(T)^\perp} = (W + \overline{R(B)}) \cap R(T)^\perp$. Thus by Proposition 2.1,

$$\dim[(W + \overline{R(B)}) \cap R(T)^\perp] = \dim \overline{R(S)} \leqslant \dim[R(A)^\perp \cap W].$$

Therefore by (3.13), $d(T) \leqslant d(B) + d(A)$.

Now, suppose that $\overline{R(A)} = \overline{R(T)} \cap W$. Let $y_1, y_2 \in R(A)^\perp \cap W$ such that $S(y_1) = S(y_2)$. Then $y_1 - y_2 = r_{y_1} - r_{y_2} \in W \cap \overline{R(T)}$ $(= \overline{R(A)})$. So that $y_1 - y_2 \in \overline{R(A)} \cap R(A)^\perp$, i.e. $y_1 = y_2$, which proves that S is injective. Consequently, by Proposition 2.2,

$$\dim[R(A)^\perp \cap W] \leqslant \dim[(W + \overline{R(B)}) \cap R(T)^\perp].$$

Thus by (3.13),

$$d(A) + d(B) \leqslant \dim[(W + \overline{R(B)}) \cap R(T)^\perp] + d(B) = d(T). \quad \square$$

In the same way that L. A. Coburn defined the Weyl spectrum, B. S. Yadav and S. C. Arora in [14] did it for the $\alpha-$Weyl spectrum of a weight α, $\aleph_0 < \alpha < h$, for an operator $T \in B(H)$, as

$$\omega_\alpha(T) = \bigcap_{K \in \mathscr{I}_\alpha} \sigma(T+K). \tag{3.15}$$

L. Burlando in [5] defined the $\beta-$index of an operator $T : H \to H$ for $\aleph_0 \leqslant \beta \leqslant h$ as

$$\operatorname{ind}_\beta(T) = \begin{cases} n(T) - d(T), & \text{if either } \beta = \aleph_0 \text{ or } \beta > \aleph_0 \text{ and} \\ & \max\{n(T), d(T)\} \geqslant \beta; \\ 0, & \text{if } \beta > \aleph_0 \text{ and } \max\{n(T), d(T)\} < \beta. \end{cases}$$

With this index S. V. Djordjević and F. Hernández-Díaz in [7] presented a Schechter's manner to introduce $\alpha-$Weyl operators. An operator $T \in B(H)$ is said $\alpha-$Weyl operator, for some cardinal α, $\aleph_0 \leqslant \alpha < h$, if T is an $\alpha-$Fredholm operator with $\operatorname{ind}_\beta(T) = 0$, for all cardinals β, $\aleph_0 \leqslant \beta < \alpha$. They proved, see [7, Theorem 3], that the Weyl spectrum of a weight α may be characterized as the following set

$$\begin{aligned} \omega_\alpha(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not an } \alpha - \text{Weyl operator}\} \\ &= \{\lambda \in \mathbb{C} \mid \lambda \in \sigma_\alpha(T) \text{ or } \operatorname{ind}_\beta(\lambda - T) \neq 0, \text{ for some } \aleph_0 \leqslant \beta < \alpha\}. \end{aligned}$$

Let us now consider the set

$$\begin{aligned} \mathscr{N}_W(H) = \Big\{ & T \in \mathscr{F}_W(H) \mid R(\lambda - A) = R(\lambda - T) \cap W \text{ and} \\ & \overline{R(\lambda - A)} = \overline{R(\lambda - T)} \cap W \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\} \Big\}. \end{aligned}$$

THEOREM 3.12. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. If $T \in \mathscr{N}_W(H)$, then*

$$\omega_\alpha(T) \subseteq \omega_\alpha(A) \cup \omega_\alpha(B).$$

Proof. Take $\lambda \notin (\omega_\alpha(A) \cup \omega_\alpha(B))$, then $\lambda - A$ and $\lambda - B$ are $\alpha-$Weyl operators, so by [7, Theorem 5], $d(\lambda - A) = n((\lambda - A)^*) = n(\lambda - A) < \alpha$ and $d(\lambda - B) = n((\lambda - B)^*) = n(\lambda - B) < \alpha$. Since $T \in \mathscr{N}_W(H)$, it follows by Theorem 3.11, that

$$n(\lambda - T) = n(\lambda - A) + n(\lambda - B)$$

and

$$d(\lambda - T) = d(\lambda - A) + d(\lambda - B).$$

Therefore $n(\lambda - T) = n(\lambda - A) + n(\lambda - B) = d(\lambda - A) + d(\lambda - B) = d(\lambda - T) = n((\lambda - T)^*)$. On the other hand, $\lambda - A$ and $\lambda - B$ are $\alpha-$Fredholm operators, hence by Corollary 3.8, $\lambda - T$ is an $\alpha-$Fredholm operator, consequently by [7, Theorem 5], $\lambda - T$ is an $\alpha-$Weyl operator. Thus $\lambda \notin \omega_\alpha(T)$. $\square$

4. Application to spectral $\nu-$continuity

Let $\mathscr{A}$ be a complex Banach algebra with identity e. A sequence (a_n) in $\mathscr{A}$ is said to be norm convergent to a (in notation $a_n \to a$), if $\|a_n - a\| \to 0$. Recently, M. Ahues in [1] introduced a new mode of convergence on $B(X)$, named $\nu-$convergence. This type of convergence can be generalized in the same way to complex unital Banach algebras. Indeed, a sequence (a_n) in $\mathscr{A}$ is said to be $\nu-$convergent to a, denoted by $a_n \xrightarrow{\nu} a$, if $(\|a_n\|)$ is bounded, $\|(a_n - a)a\| \to 0$ and $\|(a_n - a)a_n\| \to 0$. This convergence is a pseudo-convergence in the sense that it is possible to have $a_n \xrightarrow{\nu} a$ and $a_n \xrightarrow{\nu} b$ but $a \neq b$, see for instance [12, Example 1]. There is a connection between norm convergence and $\nu-$convergence as follows: if $a_n \to a$ then $a_n \xrightarrow{\nu} a$, also, if $a_n \xrightarrow{\nu} a$ and a is right invertible then $a_n \to a$. Investigation of the $\nu-$continuity of the spectrum on the space $B(X)$ is relatively new, some results on this topic we can find for example in [1], [2], [12] and [13].

A function τ, defined on $\mathscr{A}$, whose values are non-empty compact subsets of $\mathbb{C}$ is said to be $\nu-$upper (resp. $\nu-$lower) semi-continuous at a, if $a_n \xrightarrow{\nu} a$ implies that $\limsup \tau(a_n) \subseteq \tau(a)$ (resp. $\tau(a) \subseteq \liminf \tau(a_n)$). If τ is both $\nu-$upper and $\nu-$lower semi-continuous at a, then τ is said to be $\nu-$continuous at a.

For $a \in \mathscr{A}$, let $\sigma(a) := \{\lambda \in \mathbb{C} \mid \lambda e - a \text{ is not invertible in } \mathscr{A}\}$, the spectrum of a. It is well known that $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$ and $\sigma(a) \subseteq B(0, \|a\|)$. From this it follows the next proposition.

PROPOSITION 4.1. *σ is $\nu-$continuous at a if and only if $\sigma(a_n) \to \sigma(a)$ in the Hausdorff metric for every $a_n \xrightarrow{\nu} a$.*

Proceeding exactly as in the proof of [1, Corollary 2.7] we obtain the next result.

THEOREM 4.2. *For each $a \in \mathscr{A}$, σ is $\nu-$upper semi-continuous at a.*

As an immediate consequence of the previous theorem for $\mathscr{A} = B(H)/\mathscr{I}_\alpha$ is that the $\alpha-$Fredholm spectrum, viewed as a function from $B(H)$ into the space of non-empty compact sets, is $\nu-$upper semi-continuous.

COROLLARY 4.3. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$. For each $T \in B(H)$, σ_α is $\nu-$upper semi-continuous at T.*

Proof. Let (T_n) be a sequence in $B(H)$ such that $T_n \xrightarrow{\nu} T$. Consider the natural homomorphism $\pi : H \to B(H)/\mathscr{I}_\alpha$ defined by $\pi(T) = T + \mathscr{I}_\alpha$. Then $\pi(T_n) \xrightarrow{\nu} \pi(T)$ and so by Theorem 4.2, $\limsup \sigma(\pi(T_n)) \subseteq \sigma(\pi(T))$. On the other hand, for each $n \in \mathbb{N}$, $\sigma_\alpha(T_n) = \sigma(\pi(T_n))$, and $\sigma_\alpha(T) = \sigma(\pi(T))$. Thus $\limsup \sigma_\alpha(T_n) \subseteq \sigma_\alpha(T)$. $\square$

THEOREM 4.4. *Let α be a cardinal number such that $\aleph_0 \leqslant \alpha \leqslant h$ and let $T \in \mathscr{F}_W(H)$. Suppose that one of the following conditions holds:*

(i) $\sigma_\alpha(A) \cap \sigma_\alpha(B) = \emptyset$;

(ii) $\sigma_\alpha(T)\cap\sigma_\alpha(A)=\emptyset$;

(iii) $\sigma_\alpha(T)\cap\sigma_\alpha(B)=\emptyset$.

Then:

(1) if σ_α is $\nu-$continuous at A and B, then σ_α is $\nu-$continuous at T;

(2) if σ_α is $\nu-$continuous at T and A, then σ_α is $\nu-$continuous at B;

(3) if for each $\{A_n\}$ in $B(W)$ with $A_n\xrightarrow{\nu}A$, $A_nC\to AC$, and if σ_α is $\nu-$continuous at T and B, then σ_α is $\nu-$continuous at A.

Proof. We suppose that $\sigma_\alpha(A)\cap\sigma_\alpha(B)=\emptyset$.

(1) Let $\{T_n\}$ be a sequence in $\mathscr{F}_W(H)$ such that $T_n\xrightarrow{\nu}T$. Each T_n has the following 2×2 upper triangular operator matrix representation: $T_n=\begin{bmatrix}A_n & C_n\\ 0 & B_n\end{bmatrix}$. Since $\|A_n\|\leqslant\|T_n\|$ and $\|B_n\|\leqslant\|T_n\|$, it follows that $A_n\xrightarrow{\nu}A$ and $B_n\xrightarrow{\nu}B$. Let $\lambda\in\sigma_\alpha(T)$, from Corollary 3.9 *(1)*, $\lambda\in\sigma_\alpha(A)\cup\sigma_\alpha(B)$.

We may suppose without loss of generality that $\lambda\in\sigma_\alpha(A)$. Since that σ_α is $\nu-$lower semi continuous at A, $\lambda\in\liminf\ \sigma_\alpha(A_n)$. Thus there exists a sequence $\{\lambda_n\}$ in $\mathbb{C}$ such that $\lambda_n\to\lambda$ and $\lambda_n\in\sigma_\alpha(A_n)$ for all $n\in\mathbb{N}$. Suppose that there exists a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_k}\notin\sigma_\alpha(T_{n_k})$. Since $\lambda_{n_k}\in[\sigma_\alpha(A_{n_k})\cup\sigma_\alpha(B_{n_k})]\setminus\sigma_\alpha(T_{n_k})$ it follows by Corollary 3.9 *(4)* that $\lambda_{n_k}\in\sigma_\alpha(A_{n_k})\cap\sigma_\alpha(B_{n_k})$. Therefore $\lambda\in\limsup\sigma_\alpha(B_n)$ and so, by $\nu-$upper semi continuity of σ_α at B, $\lambda\in\sigma_\alpha(B)$, which implies that $\lambda\in\sigma_\alpha(A)\cap\sigma_\alpha(B)$, a contradiction. Consequently, there exists a natural number n_0 such that for every $n\geqslant n_0$, $\lambda_n\in\sigma_\alpha(T_n)$, thus $\lambda\in\liminf\sigma_\alpha(T_n)$.

(2) Let $\{B_n\}$ be a sequence in $B(W^\perp)$ such that $B_n\xrightarrow{\nu}B$. Consider the sequence $\{T_n\}$ where each operator is defined by $T_n=\begin{bmatrix}A & C\\ 0 & B_n\end{bmatrix}$. It is clear that $\{T_n\}$ is a sequence in $\mathscr{F}_W(H)$, moreover, observe that $\|(T_n-T)T\|=\|(B_n-B)B\|$, $\|(T_n-T)T_n\|=\|(B_n-B)B_n\|$ and $\|T_n\|\leqslant[\max\{2\max\{\|A\|^2,\|C\|^2\},\|B_n\|^2\}]^{1/2}$. Thus $T_n\xrightarrow{\nu}T$. Let $\lambda\in\sigma_\alpha(B)$, then by Corollary 3.9 *(3)*, $\lambda\in\sigma_\alpha(T)$ and so $\lambda\in\liminf\sigma_\alpha(T_n)$, on the other hand, $\liminf\sigma_\alpha(T_n)\subseteq\liminf[\sigma_\alpha(A)\cup\sigma_\alpha(B_n)]\subseteq\sigma_\alpha(A)\cup[\liminf\sigma_\alpha(B_n)]$, hence $\lambda\in\liminf\sigma_\alpha(B_n)$.

(3) Let $\{A_n\}$ be a sequence in $B(W)$ such that $A_n\xrightarrow{\nu}A$. By hypothesis, $A_nC\to AC$. Consider $T_n=\begin{bmatrix}A_n & C\\ 0 & B\end{bmatrix}$, $n\in\mathbb{N}$. Then

$$\|(T_n-T)T\|=[2\max\{\|(A_n-A)A\|^2,\|(A_n-A)C\|^2\}]^{1/2}$$

and

$$\|(T_n-T)T_n\|=[2\max\{\|(A_n-A)A_n\|^2,\|(A_n-A)C\|^2\}]^{1/2}.$$

Therefore $T_n \xrightarrow{\nu} T$, thus

$$\sigma_\alpha(A) \subseteq \sigma_\alpha(T) \subseteq \liminf \sigma_\alpha(T_n) \subseteq [\liminf \sigma_\alpha(A_n)] \cup \sigma_\alpha(B).$$

Consequently, $\sigma_\alpha(A) \subseteq \liminf \sigma_\alpha(A_n)$. $\square$

Acknowledgement. The first author was supported by CONACYT.

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(Received May 31, 2017)

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