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A GENERALIZED LEMOS–SOARES NORM INEQUALITY

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Abstract. In this short paper, we show a Lemos-Soares type norm inequality, which extends the relative result before.

1. Introduction

Thoughout this paper, a capital letter, such as T, stands for a bounded linear operator on a Hilbert space.

$A \geqslant 0$ means that A is positive and $A > 0$ means that A is positive and invertible.

In [3], F. Kubo and T. Ando introduced the α-power mean of A and B, where $\alpha \in [0,1]$, which is defined by

$$A\sharp_\alpha B = \begin{cases} A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, & A,B > 0; \\ \lim\limits_{\varepsilon\to 0^+}(A+\varepsilon I)\sharp_\alpha(B+\varepsilon I). & A,B \geqslant 0. \end{cases}$$

Similarly, if $t \notin [0,1]$, $A\natural_t B$ is defined by

$$A\natural_t B = \begin{cases} A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}, & A,B > 0; \\ \lim\limits_{\varepsilon\to 0^+}(A+\varepsilon I)\natural_t(B+\varepsilon I). & A,B \geqslant 0. \end{cases}$$

There are many perfect results on Kubo-Ando mean, such as [1, 2, 5].

Very recently, R. Lemos and G. Soares ([4]) introduced a notation which enlarge the definition of Kubo-Ando mean as follows,

$$A\natural_{s,t} B = \begin{cases} A^{\frac{s}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{s}{2}}, & A,B > 0; \\ \lim\limits_{\varepsilon\to 0^+}(A+\varepsilon I)\natural_{s,t}(B+\varepsilon I), & A,B \geqslant 0. \end{cases}$$

where $s,t \in (-\infty,+\infty)$.

Also, they obtain a beautiful norm inequality in [4] as follows.

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THEOREM 1.1. *(Lemos-Soares norm inequality,* [4]*). Let* $A,B>0$*, then*

$$\|(A\natural_{r,t}B)(A\natural_{s,1-t}B)\| \leqslant \|A^{r+s-1}B\| \tag{1.1}$$

holds for $r,s>0$ *and* $\frac{r}{r+s} \leqslant 2t \leqslant \frac{2r+s}{r+s}$.

In this short paper, we show an extension of Lemos-Soares norm inequality.

In order to prove the main result, we list a famous operator inequality first.

THEOREM 1.2. *(Löwner-Heinz inequality).* $A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for* $\alpha \in [0,1]$.

2. Main result and its proof

In this section, we will show the main result and prove it.

THEOREM 2.1. *Let* $A,B>0$*,* $r_1,r_2,\cdots,r_n>0$*, then*

$$\|(A\natural_{r_n,t_n}B)(A\natural_{r_{n-1},t_{n-1}}B)\cdots(A\natural_{r_2,t_2}B)(A\natural_{r_1,t_1}B)\| \leqslant \|A^{r_1+r_2\cdots+r_n-1}B\| \tag{2.1}$$

holds for $\frac{r_1}{\Sigma} \leqslant 2t_1 \leqslant \frac{2r_1+r_2}{\Sigma} \leqslant 2(t_1+t_2) \leqslant \frac{2r_1+2r_2+r_3}{\Sigma} \leqslant 2(t_1+t_2+t_3) \leqslant \frac{2r_1+2r_2+2r_3+r_4}{\Sigma} \leqslant \cdots \frac{2r_1+r_2+\cdots+2r_{n-2}+r_{n-1}}{\Sigma} \leqslant 2(t_1+t_2+\cdots+t_{n-1}) \leqslant \frac{2r_1+2r_2+\cdots+2r_{n-1}+r_n}{\Sigma}$*,* $t_1+t_2+\cdots+t_n = 1$*, where* $\Sigma \triangleq r_1+r_2+\cdots+r_n$.

Proof. We only need to prove that

$$A^{r_1+r_2\cdots+r_n-1}B^2A^{r_1+r_2\cdots+r_n-1} \leqslant I \tag{2.2}$$

ensures that

$$(A\natural_{r_n,t_n}B)\cdots(A\natural_{r_2,t_2}B)(A\natural_{r_1,t_1}B)\times(A\natural_{r_1,t_1}B)(A\natural_{r_2,t_2}B)\cdots(A\natural_{r_n,t_n}B) \leqslant I. \tag{2.3}$$

(2.2) is equivalent to

$$B^2 \leqslant A^{-2(r_1+r_2\cdots+r_n-1)}. \tag{2.4}$$

Applying Löwner-Heinz inequality to (2.4), we have

$$B \leqslant A^{1-(r_1+r_2\cdots+r_n)}. \tag{2.5}$$

It follows that

$$A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant A^{-(r_1+r_2+\cdots+r_n)}. \tag{2.6}$$

By the definition of the notation $\natural_{s,t}$, the left side of (2.3), denoted by $K(A,B)$, is just that

$$\begin{aligned}
&K(A,B) \qquad (2.7)\\
=&(A\natural_{r_n,t_n}B)(A\natural_{r_{n-1},t_{n-1}}B)\cdots(A\natural_{r_2,t_2}B)(A\natural_{r_1,t_1}B)\\
&\times(A\natural_{r_1,t_1}B)(A\natural_{r_2,t_2}B)\cdots(A\natural_{r_{n-1},t_{n-1}}B)(A\natural_{r_n,t_n}B)\\
=&A^{\frac{r_n}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_n}A^{\frac{r_n+r_{n-1}}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_{n-1}}A^{\frac{r_{n-1}+r_{n-2}}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_{n-2}}\\
&\cdots(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_2}A^{\frac{r_2+r_1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_1}A^{r_1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_1}A^{\frac{r_1+r_2}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_2}\\
&\cdots(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_{n-2}}A^{\frac{r_{n-1}+r_{n-2}}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_{n-1}}A^{\frac{r_n+r_{n-1}}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_n}A^{\frac{r_n}{2}}.
\end{aligned}$$

Put $A_1 \triangleq A^{-(r_1+r_2+\cdots+r_n)} = A^{-\Sigma},\ B_1 \triangleq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, thus we have that

$$\begin{aligned} &K(A,B) \\ =&A_1^{-\frac{r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-2}} \\ &\cdots A_1^{-\frac{r_2+r_3}{2\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_1}{2\Sigma}}B_1^{t_1}A_1^{-\frac{r_1}{\Sigma}}B_1^{t_1}A_1^{-\frac{r_1+r_2}{2\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_3}{2\Sigma}} \\ &\cdots B_1^{t_{n-2}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_n}{2\Sigma}}. \end{aligned} \tag{2.8}$$

Notice that $B_1 \leqslant A_1$ from (2.6) and $\frac{r_1}{\Sigma} \in [0,1]$. It is easy to obtain that $A_1^{-\frac{r_1}{\Sigma}} \leqslant B_1^{-\frac{r_1}{\Sigma}}$ according to Löwner-Heinz inequality. It follows that

$$\begin{aligned} &K(A,B) \\ \leqslant&A_1^{-\frac{r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-2}} \\ &\cdots A_1^{-\frac{r_2+r_3}{2\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_1}{2\Sigma}}B_1^{2t_1-\frac{r_1}{\Sigma}}A_1^{-\frac{r_1+r_2}{2\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_3}{2\Sigma}} \\ &\cdots B_1^{t_{n-2}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_n}{2\Sigma}}. \end{aligned} \tag{2.9}$$

Because that $\frac{r_1}{\Sigma} \leqslant 2t_1 \leqslant \frac{2r_1+r_2}{\Sigma}$, we can obtain that $0 \leqslant 2t_1 - \frac{r_1}{\Sigma} \leqslant \frac{r_1+r_2}{\Sigma} \leqslant 1$. It is easy to obtain that $B_1^{2t_1-\frac{r_1}{\Sigma}} \leqslant A_1^{2t_1-\frac{r_1}{\Sigma}}$. It follows that

$$\begin{aligned} &A_1^{-\frac{r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-2}} \\ &\cdots A_1^{-\frac{r_2+r_3}{2\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_1}{2\Sigma}}B_1^{2t_1-\frac{r_1}{\Sigma}}A_1^{-\frac{r_1+r_2}{2\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_3}{2\Sigma}} \\ &\cdots B_1^{t_{n-2}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_n}{2\Sigma}} \\ \leqslant&A_1^{-\frac{r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-2}}\cdots A_1^{-\frac{r_2+r_3}{2\Sigma}}B_1^{t_2}A_1^{2t_1-\frac{2r_1+r_2}{\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_3}{2\Sigma}} \\ &\cdots B_1^{t_{n-2}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_n}{2\Sigma}}. \end{aligned}$$

Similarly, because that $0 \leqslant \frac{2r_1+r_2}{\Sigma} - 2t_1 = \frac{r_1}{\Sigma} - 2t_1 + \frac{r_1+r_2}{\Sigma} \leqslant \frac{r_1+r_2}{\Sigma} \leqslant 1,$ we have $A_1^{2t_1-\frac{2r_1+r_2}{\Sigma}} \leqslant B_1^{2t_1-\frac{2r_1+r_2}{\Sigma}}$. Thus, we have that

$$\begin{aligned} &A_1^{-\frac{r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-2}}\cdots A_1^{-\frac{r_2+r_3}{2\Sigma}}B_1^{t_2}A_1^{2t_1-\frac{2r_1+r_2}{\Sigma}}B_1^{t_2}A_1^{-\frac{r_2+r_3}{2\Sigma}} \\ &\cdots B_1^{t_{n-2}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_n}{2\Sigma}} \\ \leqslant&A_1^{-\frac{r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-2}}\cdots A_1^{-\frac{r_2+r_3}{2\Sigma}}B_1^{2t_1+2t_2-\frac{2r_1+r_2}{\Sigma}}A_1^{-\frac{r_2+r_3}{2\Sigma}} \\ &\cdots B_1^{t_{n-2}}A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}}B_1^{t_{n-1}}A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}}B_1^{t_n}A_1^{-\frac{r_n}{2\Sigma}}. \end{aligned}$$

Next, because that $0 \leqslant 2t_1 + 2t_2 - \frac{2r_1+r_2}{\Sigma} \leqslant \frac{r_2+r_3}{\Sigma} \leqslant 1$, we have that

$$
\begin{aligned}
& A_1^{-\frac{r_n}{2\Sigma}} B_1^{t_n} A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}} B_1^{t_{n-1}} A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}} B_1^{t_{n-2}} \cdots A_1^{-\frac{r_2+r_3}{2\Sigma}} B_1^{2t_1+2t_2-\frac{2r_1+r_2}{\Sigma}} A_1^{-\frac{r_2+r_3}{2\Sigma}} \\
& \cdots B_1^{t_{n-2}} A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}} B_1^{t_{n-1}} A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}} B_1^{t_n} A_1^{-\frac{r_n}{2\Sigma}} \\
\leqslant & A_1^{-\frac{r_n}{2\Sigma}} B_1^{t_n} A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}} B_1^{t_{n-1}} A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}} B_1^{t_{n-2}} \cdots A_1^{2t_1+2t_2-\frac{2r_1+2r_2+r_3}{\Sigma}} \\
& \cdots B_1^{t_{n-2}} A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}} B_1^{t_{n-1}} A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}} B_1^{t_n} A_1^{-\frac{r_n}{2\Sigma}}.
\end{aligned}
$$

Continue to use the above method, we can obtain the following results.

$$
\begin{aligned}
& A_1^{-\frac{r_n}{2\Sigma}} B_1^{t_n} A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}} B_1^{t_{n-1}} A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}} B_1^{t_{n-2}} \cdots A_1^{2t_1+2t_2-\frac{2r_1+2r_2+r_3}{\Sigma}} \\
& \cdots B_1^{t_{n-2}} A_1^{-\frac{r_{n-2}+r_{n-1}}{2\Sigma}} B_1^{t_{n-1}} A_1^{-\frac{r_{n-1}+r_n}{2\Sigma}} B_1^{t_n} A_1^{-\frac{r_n}{2\Sigma}} \\
\leqslant & \cdots \\
\leqslant & A_1^{-\frac{r_n}{2\Sigma}} B_1^{t_n} A_1^{2(t_1+t_2+\cdots+t_{n-1})-\frac{2r_1+2r_2+\cdots+2r_{n-1}+r_n}{\Sigma}} B_1^{t_n} A_1^{-\frac{r_n}{2\Sigma}} \\
\leqslant & A_1^{-\frac{r_n}{2\Sigma}} B_1^{2(t_1+t_2+\cdots+t_n)-\frac{2r_1+2r_2+\cdots+2r_{n-1}+r_n}{\Sigma}} A_1^{-\frac{r_n}{2\Sigma}} \\
\leqslant & A_1^{2(t_1+t_2+\cdots+t_n)-\frac{2r_1+2r_2+\cdots+2r_{n-1}+2r_n}{\Sigma}} \\
= & A^{2-2} = A^0 = I.
\end{aligned}
$$

Together with above inequalities, we can obtain that $K(A,B) \leqslant I$. Then we complete the proof. □

REMARK 2.1. If we take $n = 2$ in the Theorem, it just is Theorem 1.1.

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