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GREEN'S FUNCTION OF THE PROBLEM OF BOUNDED SOLUTIONS IN THE CASE OF A BLOCK TRIANGULAR COEFFICIENT

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Abstract. It is well known that the equation $x'(t)=Ax(t)+f(t)$, $t\in\mathbb{R}$, where A is a bounded linear operator, has a unique bounded solution x for any bounded continuous free term f, provided the spectrum of the coefficient A does not intersect the imaginary axis. This solution can be represented in the form

$$x(t)=\int_{-\infty}^{\infty}\mathcal{G}(s)f(t-s)\,ds.$$

The kernel $\mathcal{G}$ is called Green's function. In this paper, the case when A admits a representation by a block triangular operator matrix is considered. It is shown that the blocks of $\mathcal{G}$ are sums of special convolutions of Green's functions of the diagonal blocks of A.

Introduction

Let us consider the equation

$$x'(t)-Ax(t)=f(t),\qquad t\in\mathbb{R},\tag{1}$$

where A is a linear bounded operator acting in a Banach space X. We assume that f is continuous. The *bounded solutions problem* is the problem of finding a bounded solution x that corresponds to a bounded free term f. The bounded solutions problem is closely connected with the problem of exponential dichotomy of solutions. For the discussion of the bounded solutions problem from different points of view and related questions, see [1, 2, 4, 9, 10, 21, 27, 28, 39, 43, 45, 47] and the references therein.

It is well known (see Theorem 5) that equation (1) has a unique bounded solution x for any bounded continuous free term f if and only if the spectrum of the coefficient A is disjoint from the imaginary axis. In this case, the solution can be represented in the form

$$x(t)=\int_{-\infty}^{\infty}\mathcal{G}(s)f(t-s)\,ds.$$

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The kernel $\mathscr{G}$ is called *Green's function*.

In this paper, we consider the case when A admits a representation in the form of a block triangular matrix (4). The simplest 2×2 matrix representation of the coefficient A is naturally induced by the decomposition of the space X into the direct sum $X_-\oplus X_+$ of two spectral subspaces related to the parts of $\sigma(A)$ that lie in the left and right complex half-planes. This matrix representation is diagonal, but it can be 'bad' in the sense that the corresponding projectors have large norms; in such a case it may be convenient to replace one of the subspaces by the orthogonal (or close to orthogonal) complement of the other; as a result one will arrive at a triangular matrix representation of A. Similarly, the spectrum of A may be divided into clusters; so, it is again natural to use a diagonal or triangular matrix representation; the phenomenon of clusterization is discussed, e.g., in [22, lecture 12], [11, 38]. Representation by triangular operator matrices is also natural for causal operators; in turn, causal operators are widely used in control theory [13, 16, 54] and functional differential equations [35, 36, 37]. For other aspects of the theory of triangular operator matrices, see [7, 8, 19, 20, 23, 24, 29, 32, 34, 46] and the references therein.

The main results of this paper are Theorems 19 and 23; see also Theorem 5. These theorems show that Green's function is also induced by a triangular matrix and its blocks can be represented as the sums of special convolutions of Green's functions of the diagonal blocks of A; see Example 2.

Similar representations and related formulas for the fundamental solution of equation (1) were proposed, discussed, and applied by many authors [6, 12, 14, 17, 25, 33, 42, 44, 48, 51, 52, 53]; such formulas are widely used in numerical methods and other applications. We repeat some of these results in this paper (i) for the convenience of their comparison with our results connected with Green's function, and because (ii) we propose a new proof for them (iii) and discuss the infinite-dimensional case, which requires some additional considerations in the proof; see Section 3.

The paper is organized as follows. In Section 1, we recall the definition of an analytic function with an operator argument. In Section 2, we describe the representation of the fundamental solution of initial value problem and Green's function of bounded solutions problem in the form of the analytic functions $\exp_{\pm,t}$ and g_t, respectively, of the coefficient A. In Section 3, we discuss the subalgebra of operators induced by block triangular matrices. This subalgebra is not full, which leads to some technical difficulties in the subsequent presentation. In Section 4, we describe a representation of blocks of an analytic function f of a triangular matrix via contour integrals (Theorem 10). In Section 5, the terms of the formula from Theorem 10 are represented as divided differences of f with operator arguments (Theorem 17). In Section 6, we show that the divided differences of $\exp_{\pm,t}$ and g_t can be represented as convolutions with respect to the variable t of functions of one variable (Theorem 22). In Section 7, we describe a representation of divided differences of $\exp_{\pm,t}$ and g_t with operator arguments (Theorem 23). The combination of Theorems 19 and 23 allows one to represent the blocks of the fundamental solution of initial value problem and Green's function of the bounded solutions problem as special convolutions of the functions $\exp_{\pm,t}$ and g_t applied to the diagonal blocks of A (Examples 1 and 2).

1. Functions of operators

Let X and Y be non-zero complex Banach spaces. We denote by $\mathbf{B}(X,Y)$ the set of all bounded linear operators $A: X \to Y$. If $X = Y$, we use the brief notation $\mathbf{B}(X)$. The symbol $\mathbf{1} = \mathbf{1}_X$ stands for the identity operator from $\mathbf{B}(X)$.

Let $\mathbf{B}$ be a non-zero complex Banach algebra [5, 30, 49] with the unit $\mathbf{1}$ (unital algebra). The main example of a unital Banach algebra is the algebra $\mathbf{B}(X)$; another important example is the algebra of all $n \times n$ matrices, $n \in \mathbb{N}$. A subset $\mathbf{R}$ of an algebra $\mathbf{B}$ is called a *subalgebra* if $A+B, \lambda A, AB \in \mathbf{R}$ for all $A, B \in \mathbf{R}$ and $\lambda \in \mathbb{C}$. If the unit $\mathbf{1}$ of an algebra $\mathbf{B}$ belongs to its subalgebra $\mathbf{R}$, then $\mathbf{R}$ is called a *subalgebra with a unit* or a *unital subalgebra*.

A unital subalgebra $\mathbf{R}$ of a unital algebra $\mathbf{B}$ is called [5, Ch. 1, § 3.6] *full* if it possesses the property: if for $B \in \mathbf{R}$ there exists $B^{-1} \in \mathbf{B}$ such that $BB^{-1} = B^{-1}B = \mathbf{1}$, then $B^{-1} \in \mathbf{R}$. In Remark 1 we will see that the subalgebra consisting of block triangular matrices is not always full.

Let $\mathbf{B}$ be a (nonzero) unital algebra and $A \in \mathbf{B}$. The set of all $\lambda \in \mathbb{C}$ such that the element $\lambda\mathbf{1} - A$ is not invertible is called the *spectrum* of the element A (in the algebra $\mathbf{B}$) and is denoted by the symbol $\sigma(A)$ or $\sigma_{\mathbf{B}}(A)$. The complement $\rho(A) = \rho_{\mathbf{B}}(A) = \mathbb{C} \setminus \sigma(A)$ is called the *resolvent set* of A. The function $R_\lambda = (\lambda\mathbf{1} - A)^{-1}$ is called the *resolvent* of the element A.

PROPOSITION 1. ([5, Ch. 1, Sec. 4, Theorem 3], [49, Theorem 10.18]) *Let $\mathbf{R}$ be a closed unital subalgebra of a unital algebra $\mathbf{B}$. Then the spectrum $\sigma_{\mathbf{R}}(A)$ of an element $A \in \mathbf{R}$ in the algebra $\mathbf{R}$ is the union of the spectrum $\sigma_{\mathbf{B}}(A)$ of A in the algebra $\mathbf{B}$ and (possibly empty) collection of bounded connected components of the resolvent set $\rho_{\mathbf{B}}(A)$.*

Let $A \in \mathbf{B}$ and let $U \subseteq \mathbb{C}$ be an open set that contains the spectrum $\sigma(A)$. The set U must not be connected. Let $f: U \to \mathbb{C}$ be an analytic function. The *function f of the element A* is defined [30, Ch. V, § 1], [10, p. 17] by the formula

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A)^{-1}\,d\lambda, \tag{2}$$

where the contour Γ surrounds the set $\sigma_{\mathbf{B}}(A)$ in the counterclockwise direction and the function f is analytic inside Γ.

PROPOSITION 2. ([30, Theorem 5.2.5], [49, Theorem 10.27]) *The mapping $f \mapsto f(A)$ preserves algebraic operations, i. e.,*

$$\begin{aligned}(f+g)(A) &= f(A) + g(A),\\ (\alpha f)(A) &= \alpha f(A),\\ (fg)(A) &= f(A)g(A),\end{aligned}$$

where $f+g$, αf, and fg are defined pointwise.

COROLLARY 3. *For the function* $r_{\lambda_0}(\lambda)=\frac{1}{\lambda_0-\lambda}$, $\lambda_0\in\rho(A)$, *we have*

$$r_{\lambda_0}(A)=(\lambda_0\mathbf{1}-A)^{-1}.$$

Proof. The proof follows from Proposition 2. □

2. The differential equation with a constant coefficient

In this Section, we describe three analytic functions that are closely related to the representation of solutions of linear ordinary differential equations with constant coefficients.

For $\lambda\in\mathbb{C}$ and $t\in\mathbb{R}$, we consider the functions

$$\exp_{+,t}(\lambda)=\begin{cases}e^{\lambda t}, & \text{if } t>0,\\ 0, & \text{if } t<0,\end{cases}$$

$$\exp_{-,t}(\lambda)=\begin{cases}0, & \text{if } t>0,\\ -e^{\lambda t}, & \text{if } t<0,\end{cases}$$

$$g_t(\lambda)=\begin{cases}\exp_{-,t}(\lambda), & \text{if } \operatorname{Re}\lambda>0,\\ \exp_{+,t}(\lambda), & \text{if } \operatorname{Re}\lambda<0.\end{cases}$$

These functions are undefined for $t=0$. The function g_t is also undefined for $\operatorname{Re}\lambda=0$. For any fixed $t\neq 0$, all three functions are analytic on their domains.

Let X be a Banach space and $A\in\mathbf{B}(X)$. We consider the differential equation

$$x'(t)=Ax(t)+f(t),\qquad t\in\mathbb{R}. \tag{3}$$

We recall two well-known theorems. The first theorem is connected with the initial value problems.

THEOREM 4. ([10, Ch. 1, § 4], [27, Ch. IV, Corollary 2.1]) *Let* $f:\mathbb{R}\to X$ *be a continuous function. The solution of the initial value problem*

$$\begin{aligned}&x'(t)=Ax(t)+f(t),\qquad t>0,\\&x(0)=0\end{aligned}$$

is the function

$$x(t)=\int_0^t\exp_{+,s}(A)\,f(t-s)\,ds,\qquad t>0.$$

The solution of the initial value problem

$$\begin{aligned}&x'(t)=Ax(t)+f(t),\qquad t<0,\\&x(0)=0\end{aligned}$$

is the function

$$x(t)=\int_t^0\exp_{-,s}(A)\,f(t-s)\,ds,\qquad t<0.$$

The function $t \mapsto \exp_{+,t}(A)$ is usually called [27] the *fundamental solution* of equation (3).

Now we turn to the *bounded solutions problem*, i. e. the problem of finding a bounded solution $x : \mathbb{R} \to X$ under the assumption that the free term $f : \mathbb{R} \to X$ is a bounded function.

THEOREM 5. ([10, Theorem 4.1, p. 81]) *Let $A \in \mathbf{B}(X)$. Equation* (3) *has a unique solution x bounded on $\mathbb{R}$ for any bounded continuous function f if and only if the spectrum $\sigma(A)$ of A does not intersect the imaginary axis. This solution admits the representation*

$$x(t) = \int_{-\infty}^{\infty} \mathscr{G}(s) f(t-s)\, ds,$$

where

$$\mathscr{G}(t) = g_t(A), \qquad t \neq 0.$$

The function $\mathscr{G}$ is called [10] *Green's function* of the bounded solutions problem for equation (3).

3. Causal spectrum of a block triangular matrix

Let a Banach space X be represented as the direct sum of its closed nonzero subspaces X_i, $i = 1, \dots, n$:

$$X = X_1 \oplus X_2 \oplus \ldots \oplus X_n.$$

This means that every $x \in X$ can be uniquely represented in the form

$$x = x_1 + x_2 + \ldots + x_n,$$

where $x_i \in X_i$, $i = 1, \dots, n$. It is easy to prove that the norm on X is equivalent to the norm

$$\|x\| = \|x_1\| + \|x_2\| + \ldots + \|x_n\|.$$

We denote by $\mathbf{M} = \mathbf{M}(X_1, X_2, \dots, X_n)$ the set of all operator matrices

$$\{\, T_{ij} \in \mathbf{B}(X_j, X_i) : i, j = 1, \dots, n \,\}.$$

We endow $\mathbf{M}$ with the norm $\|\{\, T_{ij} \,\}\| = \max_j \sum_{i=1}^{n} \|T_{ij}\|$. It is easy to show that $\mathbf{M}$ is a unital Banach algebra with respect to the usual matrix multiplication, and the Banach algebra $\mathbf{M}$ is isomorphic (not isometrically) to the algebra $\mathbf{B}(X)$. As usual, we do not distinguish very carefully matrices and operators induced by them.

We denote by $\mathbf{M}^+ = \mathbf{M}^+(X_1, X_2, \dots, X_n)$ the set of all lower triangular matrices

$$\begin{pmatrix} A_{1,1} & 0 & \dots & 0 & 0 \\ A_{2,1} & A_{2,2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n-1} & 0 \\ A_{n,1} & A_{n,2} & \dots & A_{n,n-1} & A_{n,n} \end{pmatrix}. \tag{4}$$

We denote by $\mathbf{B}^+(X)$ the class of operators induces by $\mathbf{M}^+$. Clearly, $\mathbf{M}^+$ is a closed subalgebra of the algebra $\mathbf{M}$. Therefore, $\mathbf{B}^+(X)$ is a closed subalgebra of the algebra $\mathbf{B}(X)$. We call operators from the class $\mathbf{B}^+(X)$ *causal* in analogy with a similar class of operators in the control theory [13, 16, 54] and in the theory of functional differential equations [35, 36, 37], see also the references therein. Namely, if one interprets the indices $i=1,\dots,n$ as successive instants of time, then the triangularity of a matrix A means that the value $(Ax)_i$ of the 'output' Ax at any instant i may depend only on values x_j of the 'input' x at the previous instants $j \leqslant i$.

REMARK 1. The subalgebra $\mathbf{M}^+$ (and consequently, the subalgebra $\mathbf{B}^+(X)$) may be not full if the space X is infinite-dimensional. We give a corresponding example. Let X be the space $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. We represent $X = L_p(\mathbb{R})$ as $L_p(-\infty,0] \oplus L_p[0,\infty)$, where $L_p(-\infty,0]$ and $L_p[0,\infty)$ are the subspaces of functions from $L_p(\mathbb{R})$ that are equal to zero outside $(-\infty,0]$ and $[0,\infty)$ respectively. Clearly, the operator of delay $\big(Sx\big)(t) = x(t-1)$ is induced by a lower triangular matrix (thus it is causal), but the inverse operator $\big(S^{-1}x\big)(t) = x(t+1)$ is induced by an upper triangular matrix (thus S^{-1} is not causal). Consequently, in contrast to the finite-dimensional case, the (ordinary) spectrum of a triangular matrix may be not the union of the spectra of its diagonal blocks, see Proposition 7. See a more detailed discussion of this phenomenon in [26].

If an operator $T \in \mathbf{B}^+(X)$ is invertible and the inverse operator belongs to $\mathbf{B}^+(X)$, we say that T is *causally invertible*. We call the spectrum of $T \in \mathbf{B}^+(X)$ in the algebra $\mathbf{B}^+(X)$ the *causal spectrum* and denote it by $\sigma^+(T)$. Clearly,

$$\sigma(T) \subseteq \sigma^+(T).$$

We denote by $\rho^+(T)$ the *causal resolvent set* $\mathbb{C} \setminus \sigma^+(T)$. The same terminology and notation will be used for matrices $M \in \mathbf{M}^+$.

We recall that an open set $D \subseteq \mathbb{C}$ is called *simply-connected* if any simple closed curve in D can be shrunk continuously to a point.

PROPOSITION 6. *Let the domain $D \subseteq \mathbb{C}$ of an analytic function f be simply-connected* (*examples of such functions are* $\exp_{\pm,t}$ *and* g_t). *Let $T \in \mathbf{B}^+(X)$. Then $\sigma^+(T) \subset D$ provided $\sigma(T) \subset D$. Thus the function $f(T)$ of a causal operator T is defined in algebras $\mathbf{B}(X)$ and $\mathbf{B}^+(X)$ simultaneously.*

Proof. A possible difficulty can occur when the spectrum $\sigma(T)$ is contained in the domain D of the definition of f, but $\sigma^+(T) \nsubseteq D$. Therefore the resolvent $(\lambda\mathbf{1} - A)^{-1}$ in integral (2) is defined in $\mathbf{B}(X)$, but it may not exist in $\mathbf{B}^+(X)$.

By Proposition 1, the causal spectrum $\sigma^+(T)$ is the union of the ordinary spectrum $\sigma(T)$ and (possibly) some bounded components of the resolvent set $\rho(A)$. Since the domain D of f is simply-connected, bounded components of the resolvent set $\rho(A)$ are contained in the domain D, provided the spectrum $\sigma(T)$ itself is contained in the domain D. $\square$

PROPOSITION 7. ([35], [37, Proposition 2.1.7]) *The causal spectrum of a lower triangular matrix* $\{T_{ij}\}$ (*and the causal spectrum of the corresponding operator*) *is the union of the* (*ordinary*) *spectra* $\sigma(T_{ii})$ *of the diagonal blocks* T_{ii}.

Proof. It suffices to prove that a lower triangular matrix has a lower triangular inverse if and only if all diagonal blocks T_{ii} are invertible.

Let the lower triangular matrix $\{B_{ij}\}$ be the inverse of the lower triangular matrix $\{T_{ij}\}$. Then it follows from the matrix multiplication rule that B_{ii} are inverses of T_{ii}.

Conversely, let the diagonal blocks T_{ii} be invertible. Then from the Gaussian elimination algorithm, it easily follows that the inverse matrix exists and is triangular. $\square$

4. Functions of block triangular matrices

THEOREM 8. *Let a causal matrix*

$$T=\begin{pmatrix} T_{1,1} & 0 & \dots & 0 & 0\\ T_{2,1} & T_{2,2} & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ T_{n-1,1} & T_{n-1,2} & \dots & T_{n-1,n-1} & 0\\ T_{n,1} & T_{n,2} & \dots & T_{n,n-1} & T_{n,n} \end{pmatrix}$$

be causally invertible. Then the elements of the inverse matrix

$$B=\begin{pmatrix} B_{1,1} & 0 & \dots & 0 & 0\\ B_{2,1} & B_{2,2} & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ B_{n-1,1} & B_{n-1,2} & \dots & B_{n-1,n-1} & 0\\ B_{n,1} & B_{n,2} & \dots & B_{n,n-1} & B_{n,n} \end{pmatrix}$$

have the form

$$B_{i,j}=\sum_{i=i_1>i_2\dots>i_m=j}(-1)^{m+1}T^{-1}_{i_1,i_1}T_{i_1,i_2}T^{-1}_{i_2,i_2}T_{i_2,i_3}\dots T_{i_{m-1},i_m}T^{-1}_{i_m,i_m},\qquad i\geqslant j.$$

Hereinafter, the sum $\sum\limits_{i=i_1>i_2\dots>i_m=j}$ *consists of one term if* $i=j$. *For example,* $\sum\limits_{i=i_1=j}A_{i_1}=A_i$. *In particular,*

$$\begin{aligned} B_{i,i}&=T^{-1}_{i,i},\\ B_{i+1,i}&=-T^{-1}_{i+1,i+1}T_{i+1,i}T^{-1}_{i,i},\\ B_{i+2,i}&=-T^{-1}_{i+2,i+2}T_{i+2,i}T^{-1}_{i,i}+T^{-1}_{i+2,i+2}T_{i+2,i+1}T^{-1}_{i+1,i+1}T_{i+1,i}T^{-1}_{i,i}. \end{aligned}$$

Proof. Let us verify that $TB=\mathbf{1}$. Clearly, $(TB)_{ii}=\mathbf{1}$. We calculate, for example,

$(TB)_{n,1}$:

$$\begin{aligned}
(TB)_{n,1} =& T_{n,1}T_{1,1}^{-1} - T_{n,2}T_{2,2}^{-1}T_{2,1}T_{1,1}^{-1} + T_{n,3}(-T_{3,3}^{-1}T_{3,1}T_{1,1}^{-1} + T_{3,3}^{-1}T_{3,2}T_{2,2}^{-1}T_{2,1}T_{1,1}^{-1}) + \dots \\
&+ T_{n,n} \sum_{n=i_1>i_2\dots>i_m=1} (-1)^{m+1} T_{n,n}^{-1} T_{n,i_2} T_{i_2,i_2}^{-1} T_{i_2,i_3} \dots T_{i_{m-1},1} T_{1,1}^{-1} \\
=& T_{n,1}T_{1,1}^{-1} - T_{n,2}T_{2,2}^{-1}T_{2,1}T_{1,1}^{-1} + T_{n,3}(-T_{3,3}^{-1}T_{3,1}T_{1,1}^{-1} + T_{3,3}^{-1}T_{3,2}T_{2,2}^{-1}T_{2,1}T_{1,1}^{-1}) + \dots \\
&+ \sum_{n=i_1>i_2\dots>i_m=1} (-1)^{m+1} T_{n,i_2} T_{i_2,i_2}^{-1} T_{i_2,i_3} \dots T_{i_{m-1},1} T_{1,1}^{-1} = 0.
\end{aligned}$$

In a similar way one establishes that $BT = \mathbf{1}$. $\square$

THEOREM 9. *Let $A \in \mathbf{M}^+$. Then the causal resolvent of A (i. e., the resolvent $(\lambda\mathbf{1} - A)^{-1}$ at points $\lambda \in \rho^+(A)$) has the form*

$$(\lambda\mathbf{1} - A)^{-1} = \begin{pmatrix} R_{1,1} & 0 & \dots & 0 & 0 \\ R_{2,1} & R_{2,2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ R_{n-1,1} & R_{n-1,2} & \dots & R_{n-1,n-1} & 0 \\ R_{n,1} & R_{n,2} & \dots & R_{n,n-1} & R_{n,n} \end{pmatrix},$$

where R_{ij} for $i \geqslant j$ are defined by the formula

$$R_{ij} = \sum_{i=i_1>i_2\dots>i_m=j} (\lambda\mathbf{1} - A_{i_1,i_1})^{-1} A_{i_1,i_2} (\lambda\mathbf{1} - A_{i_2,i_2})^{-1} A_{i_2,i_3} \dots A_{i_{m-1},i_m} (\lambda\mathbf{1} - A_{i_m,i_m})^{-1}.$$

In particular,

$$\begin{aligned}
R_{ii} =& (\lambda\mathbf{1} - A_{ii})^{-1}, \\
R_{i+1,i} =& (\lambda\mathbf{1} - A_{i+1,i+1})^{-1} A_{i+1,i} (\lambda\mathbf{1} - A_{ii})^{-1}, \\
R_{i+2,i} =& (\lambda\mathbf{1} - A_{i+2,i+2})^{-1} A_{i+2,i} (\lambda\mathbf{1} - A_{ii})^{-1} \\
&+ (\lambda\mathbf{1} - A_{i+2,i+2})^{-1} A_{i+2,i+1} (\lambda\mathbf{1} - A_{i+1,i+1})^{-1} A_{i+1,i} (\lambda\mathbf{1} - A_{ii})^{-1}.
\end{aligned}$$

Proof. The proof follows from Theorem 8. The sign $(-1)^{m+1}$ disappears because $(\lambda\mathbf{1} - A)_{i,j} = -A_{i,j}$ for $i > j$. $\square$

THEOREM 10. *Let a function f be analytic in a neighborhood of the causal spectrum $\sigma^+(A)$ of a matrix $A \in \mathbf{M}^+$. Then the matrix $F = f(A)$ has the form*

$$F = \begin{pmatrix} F_{1,1} & 0 & \dots & 0 & 0 \\ F_{2,1} & F_{2,2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ F_{n-1,1} & F_{n-1,2} & \dots & F_{n-1,n-1} & 0 \\ F_{n,1} & F_{n,2} & \dots & F_{n,n-1} & F_{n,n} \end{pmatrix},$$

where F_{ij} for $i \geqslant j$ are defined by the formula

$$F_{i,j} = \sum_{i=i_1>i_2\ldots>i_m=j} \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A_{i_1,i_1})^{-1} A_{i_1,i_2} (\lambda\mathbf{1} - A_{i_2,i_2})^{-1} A_{i_2,i_3} \ldots \times A_{i_{m-1},i_m} (\lambda\mathbf{1} - A_{i_m,i_m})^{-1}\, d\lambda,$$

where Γ surrounds the causal spectrum $\sigma^+(A)$ of the matrix A. In particular,

$$\begin{aligned}
F_{i,i} &= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A_{ii})^{-1}\, d\lambda,\\
F_{i+1,i} &= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A_{i+1,i+1})^{-1} A_{i+1,i} (\lambda\mathbf{1} - A_{ii})^{-1}\, d\lambda,\\
F_{i+2,i} &= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A_{i+2,i+2})^{-1} A_{i+2,i} (\lambda\mathbf{1} - A_{ii})^{-1}\, d\lambda\\
&\quad + \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A_{i+2,i+2})^{-1} A_{i+2,i+1} (\lambda\mathbf{1} - A_{i+1,i+1})^{-1} A_{i+1,i} (\lambda\mathbf{1} - A_{ii})^{-1}\, d\lambda.
\end{aligned}$$

Proof. Substituting the representation of $(\lambda\mathbf{1} - A)^{-1}$ from Theorem 9 into formula (2), we obtain the desired result. $\square$

REMARK 2. Let a matrix $A \in \mathbf{M}^+$ has only two non-zero diagonals:

$$\begin{pmatrix}
A_{1,1} & 0 & \ldots & 0 & 0\\
A_{2,1} & A_{2,2} & \ldots & 0 & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
0 & 0 & \ldots & A_{n-1,n-1} & 0\\
0 & 0 & \ldots & A_{n,n-1} & A_{n,n}
\end{pmatrix}.$$

Let a function f be analytic in a neighborhood of the causal spectrum $\sigma^+(A)$ of the matrix A. Then it follows from Theorem 10 that the elements F_{ij} with $i \geqslant j$ of the matrix $F = f(A)$ consist of exactly one summand:

$$F_{i,j} = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - A_{i,i})^{-1} A_{i,i+1} (\lambda\mathbf{1} - A_{i+1,i+1})^{-1} A_{i+1,i+2} \ldots \times A_{j-1,j} (\lambda\mathbf{1} - A_{j,j})^{-1}\, d\lambda.$$

For the function $f = \exp_{+,t}$, this phenomenon was described in [6] and applied in [25].

COROLLARY 11. *Let the domain $D \subseteq \mathbb{C}$ of an analytic function f be simply-connected (examples of such functions are $\exp_{\pm,t}$ and g_t). Then the conclusion of Theorem 10 is true if the function f is analytic in a neighborhood of the ordinary spectrum $\sigma(A)$ of the matrix $A \in \mathbf{M}^+$.*

Proof. The proof follows from Proposition 6. $\square$

REMARK 3. For scalar matrices, Theorem 10 goes back to [48]. For matrices consisting of finite-dimensional blocks, it was published in [12, Theorem 2]. More precisely, in [12] it was considered the case of polynomial functions f; but from the case of polynomials, it follows the case of general analytic functions, since (if A is a scalar matrix) any analytic function can be replaced by its interpolating polynomial.

5. Divided differences

Let f be an analytic function defined on an open (not obligatorily connected) subset of $\mathbb{C}$. Let μ_1, μ_2, ..., μ_n be arbitrary numbers from the domain of f (some of them may coincide with others); they are called *points of interpolation*. *Divided differences* of the function f with respect to the points μ_1, μ_2, ..., μ_n are defined (see, e.g., [18, 31]) by the recurrent relations

$$\begin{aligned}
f^{[0]}(\mu_i) &= f(\mu_i), && 1 \leqslant i \leqslant n,\\
f^{[1]}(\mu_i,\mu_{i+1}) &= \frac{f^{[0]}(\mu_{i+1}) - f^{[0]}(\mu_i)}{\mu_{i+1}-\mu_i}, && 1 \leqslant i \leqslant n-1,\\
f^{[m]}(\mu_i,\dots,\mu_{i+m}) &= \frac{f^{[m-1]}(\mu_{i+1},\dots,\mu_{i+m}) - f^{[m-1]}(\mu_i,\dots,\mu_{i+m-1})}{\mu_{i+m}-\mu_i}, && 1 \leqslant i \leqslant n-m.
\end{aligned} \tag{5}$$

In these formulas, if the denominator vanishes, then the quotient is understood as the derivative with respect to one of the arguments of the previous divided difference (the naturalness of this agreement can be derived by continuity from Corollary 13).

PROPOSITION 12. ([18, Ch. 1, Formula (54)]) *Let a function f be analytic in an open neighbourhood of the points of interpolation μ_1, μ_2, ..., μ_m. Then the divided difference $f^{[m-1]}$ admits the representation*

$$f^{[m-1]}(\mu_1,\dots,\mu_m) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{\Omega(z)}\,dz,$$

where the contour Γ encloses all the points of interpolation and

$$\Omega(z) = \prod_{k=1}^{m}(z-\mu_k).$$

COROLLARY 13. *Divided differences are differentiable functions of their arguments.*

Proof. The proof follows from Proposition 12. $\square$

COROLLARY 14. *If $D \subseteq \mathbb{C}$ is the domain of an analytic function f, then $f^{[m-1]}$ is defined in D^m.*

Proof. The proof follows from Proposition 12. $\square$

COROLLARY 15. *Divided differences $f^{[m-1]}(\mu_1,\dots,\mu_m)$ are symmetric functions, i. e., they do not depend on the order of their arguments $\mu_1,\dots,\mu_m$.*

Proof. The proof follows from Proposition 12. □

PROPOSITION 16. ([18, Ch. 1, formula (48)], [31, p. 19, formula (1)]) *Let us assume that the points of interpolation $\mu_1,\dots,\mu_m$ are distinct. Then the divided difference $f^{[m-1]}$ admits the representation*

$$f^{[m-1]}(\mu_1,\dots,\mu_m)=\sum_{j=1}^{m}\frac{f(\mu_j)}{\prod\limits_{\substack{k=1\\k\neq j}}^{m}(\mu_j-\mu_k)}.$$

Proof. The proof follows from Proposition 12. □

For a function f analytic in a neighborhood of the causal spectrum of a matrix $A\in\mathbf{M}^+$, we denote by $f^{[m-1]}(A;i_1,i_2,\dots,i_m)$ the summands from Theorem 10:

$$\begin{aligned}f^{[m-1]}(A;i_1,i_2,\dots,i_m)=\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1}-A_{i_1,i_1})^{-1}A_{i_1,i_2}(\lambda\mathbf{1}-A_{i_2,i_2})^{-1}A_{i_2,i_3}\dots\\ \times A_{i_{m-1},i_m}(\lambda\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda,\end{aligned}\tag{6}$$

where Γ surrounds the causal spectrum $\sigma^+(A)$ of the matrix A.

THEOREM 17. *Let a function f be analytic in a neighborhood of the causal spectrum $\sigma^+(A)$ of a matrix $A\in\mathbf{M}^+$. Then*

$$\begin{aligned}f^{[m-1]}(A;i_1,i_2,\dots,i_m)=\frac{1}{(2\pi i)^m}\int_{\Gamma_{i_1}}\dots\int_{\Gamma_{i_m}}f^{[m-1]}(\lambda_{i_1},\dots,\lambda_{i_m})(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}\\ \times A_{i_1,i_2}(\lambda_{i_2}\mathbf{1}-A_{i_2,i_2})^{-1}A_{i_2,i_3}\dots A_{i_{m-1},i_m}(\lambda_{i_m}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_1}\dots d\lambda_{i_m},\end{aligned}\tag{7}$$

where Γ_{i_k} surrounds the spectrum of A_{i_k,i_k}.

Proof. Since f is analytic in a neighborhood of $\sigma^+(A)=\bigcup_{i=1}^n\sigma(A_{ii})$ (see Proposition 7), we may assume without loss of generality that Γ_{i_k} in (7) surrounds the whole $\bigcup_{i=1}^n\sigma(A_{ii})$ and, moreover, Γ_{i_k} surrounds $\Gamma_{i_{k+1}}$, see Figure 1.

Since the contours Γ_{i_k} are disjoint, the points $\lambda_{i_1},\dots,\lambda_{i_m}$ in the integrand of (7) are distinct. Therefore we can substitute the representation of divided differences from Proposition 16 into definition (7):

$$\begin{aligned}f^{[m-1]}(A;i_1,i_2,\dots,i_m)=\frac{1}{(2\pi i)^m}\int_{\Gamma_{i_1}}\dots\int_{\Gamma_{i_m}}\sum_{j=1}^{m}\frac{f(\lambda_{i_j})}{\prod\limits_{\substack{k=1\\k\neq j}}^{m}(\lambda_{i_j}-\lambda_{i_k})}\\ \times(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}A_{i_1,i_2}\dots A_{i_{m-1},i_m}(\lambda_{i_m}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_1}\dots d\lambda_{i_m}.\end{aligned}\tag{8}$$

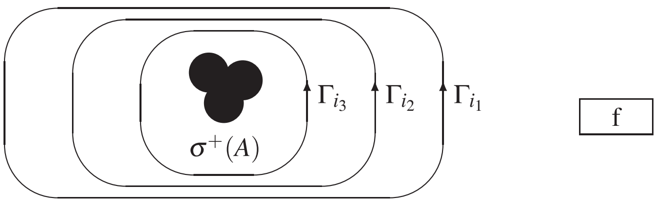

Figure 1: The choice of the contours Γ_{i_k} in the proof of Theorem 17. The symbol $\boxed{f}$ means the localization of singularities of f

Let us begin with the first summand. We have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_{i_{i_1}}} f(\lambda_{i_1})(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}A_{i_1,i_2}\\
&\quad\times\Bigg[\dots\frac{1}{2\pi i}\int_{\Gamma_{i_{m-2}}}\frac{1}{\lambda_{i_1}-\lambda_{i_{m-2}}}\times(\lambda_{i_{m-2}}\mathbf{1}-A_{i_{m-2},i_{m-2}})^{-1}A_{i_{m-2},i_{m-1}}\\
&\quad\times\Bigg[\frac{1}{2\pi i}\int_{\Gamma_{i_{m-1}}}\frac{1}{\lambda_{i_1}-\lambda_{i_{m-1}}}(\lambda_{i_{m-1}}\mathbf{1}-A_{i_{m-1},i_{m-1}})^{-1}A_{i_{m-1},i_m}\\
&\quad\times\Bigg[\frac{1}{2\pi i}\int_{\Gamma_{i_m}}\frac{1}{\lambda_{i_1}-\lambda_{i_m}}(\lambda_{i_m}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_m}\Bigg]\,d\lambda_{i_{m-1}}\Bigg]\,d\lambda_{i_{m-2}}\dots\Bigg]\,d\lambda_{i_1}.
\end{aligned}$$

By Corollary 3, for the internal integral, we have

$$\frac{1}{2\pi i}\int_{\Gamma_{i_m}}\frac{1}{\lambda_{i_1}-\lambda_{i_m}}(\lambda_{i_m}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_m}=(\lambda_{i_1}\mathbf{1}-A_{i_m,i_m})^{-1}.$$

Now we can calculate the next internal integral (again using Corollary 3):

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_{i_{m-1}}}\frac{1}{\lambda_{i_1}-\lambda_{i_{m-1}}}(\lambda_{i_{m-1}}\mathbf{1}-A_{i_{m-1},i_{m-1}})^{-1}A_{i_{m-1},i_m}(\lambda_{i_1}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_{m-1}}\\
=&\Bigg[\frac{1}{2\pi i}\int_{\Gamma_{i_{m-1}}}\frac{1}{\lambda_{i_1}-\lambda_{i_{m-1}}}(\lambda_{i_{m-1}}\mathbf{1}-A_{i_{m-1},i_{m-1}})^{-1}\,d\lambda_{i_{m-1}}\Bigg]A_{i_{m-1},i_m}(\lambda_{i_1}\mathbf{1}-A_{i_m,i_m})^{-1}\\
=&(\lambda_{i_1}\mathbf{1}-A_{i_{m-1},i_{m-1}})^{-1}A_{i_{m-1},i_m}(\lambda_{i_1}\mathbf{1}-A_{i_m,i_m})^{-1}.
\end{aligned}$$

And so on. Finally, we arrive at the representation (for the first summand in (8))

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma_{i_1}} f(\lambda_{i_1})(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}A_{i_1,i_2}(\lambda_{i_1}\mathbf{1}-A_{i_2,i_2})^{-1}A_{i_2,i_3}\dots&\\
\times A_{i_{m-1},i_m}(\lambda_{i_1}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_1}&,
\end{aligned}$$

which coincides with formula (6).

Next we show that the other summands in (8) are zero. Let us consider, for example, the second summand

$$\frac{1}{(2\pi i)^m}\int_{\Gamma_{i_1}}\cdots\int_{\Gamma_{i_m}}\frac{f(\lambda_{i_2})}{\prod\limits_{\substack{k=1\\k\neq 2}}^{m}(\lambda_{i_2}-\lambda_{i_k})}$$
$$\times(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}A_{i_1,i_2}\dots A_{i_{m-1},i_m}(\lambda_{i_m}\mathbf{1}-A_{i_m,i_m})^{-1}\,d\lambda_{i_1}\dots d\lambda_{i_m}.$$

Proceeding as above (i. e. successively calculating integrals over all variables except λ_{i_1}), at the final stage, we arrive at the integral

$$\frac{1}{2\pi i}\int_{\Gamma_{i_1}}\frac{1}{\lambda_{i_2}-\lambda_{i_1}}(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}\,d\lambda_{i_1}.$$

We notice that the singularity of the function $\lambda_{i_1}\mapsto\frac{1}{\lambda_{i_2}-\lambda_{i_1}}$ (i. e., the point $\lambda_{i_2}\in\Gamma_{i_2}$) lies inside the contour Γ_{i_1}. Hence the integrand $\lambda_{i_1}\mapsto\frac{1}{\lambda_{i_2}-\lambda_{i_1}}(\lambda_{i_1}\mathbf{1}-A_{i_1,i_1})^{-1}$ is analytic outside the contour Γ_{i_1} and decreases at infinity as $\frac{1}{\lambda_{i_1}^2}$. Therefore the integral equals zero. □

REMARK 4. For the case of the first divided difference, a more detailed discussion of formula (7) can be found in [40, Theorem 41], see also the references therein.

COROLLARY 18. *Let a function f be analytic in a neighborhood of the causal spectrum $\sigma^+(A)$ of a matrix $A\in\mathbf{M}^+$. Then the matrix $F=f(A)$ has the form*

$$F=\begin{pmatrix}F_{11}&0&\dots&0&0\\F_{21}&F_{22}&\dots&0&0\\\dots&\dots&\dots&\dots&\dots\\F_{n-1,1}&F_{n-1,2}&\dots&F_{n-1,n-1}&0\\F_{n,1}&F_{n,2}&\dots&F_{n,n-1}&F_{n,n}\end{pmatrix},$$

where $F_{i,j}$, $i\geqslant j$, admit the representation

$$F_{i,j}=\sum_{i=i_1>i_2\dots>i_m=j}f^{[m-1]}(A;i_1,i_2,\dots,i_m).$$

Proof. The proof follows from Theorems 10 and 17. □

THEOREM 19. *Let $A\in\mathbf{M}^+$. Then*

$$\exp_{\pm,t}(A)=\begin{pmatrix}E_{\pm,1,1}&0&\dots&0&0\\E_{\pm,2,1}&E_{\pm,2,2}&\dots&0&0\\\dots&\dots&\dots&\dots&\dots\\E_{\pm,n-1,1}&E_{\pm,n-1,2}&\dots&E_{\pm,n-1,n-1}&0\\E_{\pm,n,1}&E_{\pm,n,2}&\dots&E_{\pm,n,n-1}&E_{\pm,n,n}\end{pmatrix},$$

where $E_{\pm,i,j}$, $i \geqslant j$, *admit the representation*

$$E_{\pm,i,j} = \sum_{i=i_1>i_2\ldots>i_m=j} \exp_{\pm,t}^{[m-1]}(A;i_1,i_2,\ldots,i_m);$$

and (*provided the spectrum* $\sigma(A)$ *does not intersect the imaginary axis*)

$$g_t(A) = \begin{pmatrix} G_{1,1} & 0 & \ldots & 0 & 0 \\ G_{2,1} & G_{2,2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ G_{n-1,1} & G_{n-1,2} & \ldots & G_{n-1,n-1} & 0 \\ G_{n,1} & G_{n,2} & \ldots & G_{n,n-1} & G_{n,n} \end{pmatrix},$$

where $G_{i,j}$, $i \geqslant j$, *admit the representation*

$$G_{i,j} = \sum_{i=i_1>i_2\ldots>i_m=j} g_t^{[m-1]}(A;i_1,i_2,\ldots,i_m).$$

Proof. The proof follows from Corollary 11 and Theorem 17. □

6. Divided differences of the functions $\exp_{\pm,t}$ and g_t

LEMMA 20. *Let the points of interpolation* λ_1, λ_2, $\ldots$, λ_n *be distinct. Then for any* $\lambda \in \mathbb{C}$, *the divided differences of the function* $r_\lambda(\nu) = \frac{1}{\lambda-\nu}$ *admit the representation*

$$r_\lambda^{[n-1]}(\lambda_1,\ldots,\lambda_n) = \frac{1}{\prod_{j=1}^n(\lambda-\lambda_j)}.$$

Proof. The proof is by induction on n. For $n=1$ we have

$$r_\lambda^{[1]}(\lambda_1,\lambda_2) = \frac{\frac{1}{\lambda-\lambda_1}-\frac{1}{\lambda-\lambda_2}}{\lambda_1-\lambda_2} = \frac{1}{(\lambda-\lambda_1)(\lambda-\lambda_2)}.$$

Assuming that the formula holds for $n-2$, we prove it for $n-1$. We have

$$\begin{aligned} r_\lambda^{[n-1]}(\lambda_1,\ldots,\lambda_n) &= \frac{r_\lambda^{[n-1]}(\lambda_1,\ldots,\lambda_{n-2},\lambda_{n-1}) - r_\lambda^{[n-1]}(\lambda_1,\ldots,\lambda_{n-2},\lambda_n)}{\lambda_{n-1}-\lambda_n} \\ &= \frac{\frac{1}{\prod_{j=1}^{n-2}(\lambda-\lambda_j)}\frac{1}{\lambda-\lambda_{n-1}} - \frac{1}{\prod_{j=1}^{n-2}(\lambda-\lambda_j)}\frac{1}{\lambda-\lambda_n}}{\lambda_{n-1}-\lambda_n} \\ &= \frac{1}{\prod_{j=1}^{n-2}(\lambda-\lambda_j)} \frac{\frac{1}{\lambda-\lambda_{n-1}}-\frac{1}{\lambda-\lambda_n}}{\lambda_{n-1}-\lambda_n} = \frac{1}{\prod_{j=1}^{n}(\lambda-\lambda_j)}. \end{aligned}$$ □

We recall [15, 41, 50] that the *bilateral* (*or two-sided*) *Laplace transform* of a function $f:\mathbb{R}\to\mathbb{C}$ is the function

$$\big(\mathscr{B}f\big)(\lambda)=\int_{-\infty}^{\infty}e^{-\lambda t}f(t)\,dt.$$

The value $\big(\mathscr{B}f\big)(\lambda)$ at the point $\lambda\in\mathbb{C}$ is defined if the integral converges absolutely. If f equals zero on $(-\infty,0)$ (as the function $\exp_{+,(\cdot)}$), this definition takes the form

$$\big(\mathscr{B}f\big)(\lambda)=\int_{0}^{\infty}e^{-\lambda t}f(t)\,dt.$$

Usually in this case, the integral converges absolutely if $\operatorname{Re}\lambda$ is sufficiently large. If f equals zero on $(0,\infty)$ (as the function $\exp_{-,(\cdot)}$), the definition of the bilateral Laplace transform takes the form

$$\big(\mathscr{B}f\big)(\lambda)=\int_{-\infty}^{0}e^{-\lambda t}f(t)\,dt.$$

In this case, we assume that the integral converges absolutely if $\operatorname{Re}\lambda$ is sufficiently small.

We recall the following statement.

LEMMA 21. *Let* $\lambda_0\in\mathbb{C}$.

(a) *The bilateral Laplace transform of the function* $t\mapsto\exp_{+,t}(\lambda_0)$ *is the function*

$$\big(\mathscr{B}\exp_{+,(\cdot)}(\lambda_0)\big)(\lambda)=\frac{1}{\lambda-\lambda_0},\qquad \operatorname{Re}\lambda>\operatorname{Re}\lambda_0.$$

(b) *The bilateral Laplace transform of the function* $t\mapsto\exp_{-,t}(\lambda_0)$ *is the function*

$$\big(\mathscr{B}\exp_{-,(\cdot)}(\lambda_0)\big)(\lambda)=\frac{1}{\lambda-\lambda_0},\qquad \operatorname{Re}\lambda<\operatorname{Re}\lambda_0.$$

(c) *The bilateral Laplace transform of the function* $t\mapsto g_t(\lambda_0)$, $\operatorname{Re}\lambda_0\neq 0$, *is the function* (*the complete domain of the function of* $\mathscr{B}g_{(\cdot)}(\lambda_0)$ *is* $\operatorname{Re}\lambda>\operatorname{Re}\lambda_0$ *if* $\operatorname{Re}\lambda_0<0$ *and is* $\operatorname{Re}\lambda<\operatorname{Re}\lambda_0$ *if* $\operatorname{Re}\lambda_0>0$)

$$\big(\mathscr{B}g_{(\cdot)}(\lambda_0)\big)(\lambda)=\frac{1}{\lambda-\lambda_0},\qquad |\operatorname{Re}\lambda|<|\operatorname{Re}\lambda_0|.$$

Proof. Assertion (a) is widely known [41, pp. 300, 305]. The proofs of all assertions are reduced to straightforward calculations. The proof of assertion (c) can also be obtained from the definition of g_t and (a) and (b). □

We recall [49, Ch. 6] that the *convolution* of two summable functions $f,g:\mathbb{R}\to\mathbb{C}$ is the function

$$\big(f*g\big)(t)=\int_{-\infty}^{\infty}f(s)g(t-s)\,ds.$$

If $f(t)=0$ and $g(t)=0$ for $t<0$, then this formula takes the form

$$(f*g)(t)=\begin{cases}\int_0^t f(s)g(t-s)\,ds, & \text{for } t>0,\\ 0, & \text{for } t<0.\end{cases}$$

If $f(t)=0$ and $g(t)=0$ for $t>0$, then the definition of convolution takes the form

$$(f*g)(t)=\begin{cases}0, & \text{for } t>0,\\ \int_t^0 f(s)g(t-s)\,ds, & \text{for } t<0.\end{cases}$$

THEOREM 22.

(a) *The divided differences of the function $t\mapsto \exp_{+,t}$ admit the representation*

$$\exp^{[n-1]}_{+,(\cdot)}(\lambda_1,\dots,\lambda_n)=\exp_{+,(\cdot)}(\lambda_1)*\dots*\exp_{+,(\cdot)}(\lambda_n).$$

For example, for $t>0$, we have

$$\exp^{[1]}_{+,t}(\lambda_1,\lambda_2)=\int_0^t \exp_{+,s}(\lambda_1)\exp_{+,t-s}(\lambda_2)\,ds,$$
$$\exp^{[2]}_{+,t}(\lambda_1,\lambda_2,\lambda_3)=\int_0^t\int_0^r \exp_{+,s}(\lambda_1)\exp_{+,r-s}(\lambda_2)\exp_{+,t-r}(\lambda_3)\,ds\,dr.$$

(b) *The divided differences of the function $t\mapsto \exp_{-,t}$ admit the representation*

$$\exp^{[n-1]}_{-,(\cdot)}(\lambda_1,\dots,\lambda_n)=\exp_{-,(\cdot)}(\lambda_1)*\dots*\exp_{-,(\cdot)}(\lambda_n).$$

For example, for $t<0$, we have

$$\exp^{[1]}_{-,t}(\lambda_1,\lambda_2)=\int_t^0 \exp_{-,s}(\lambda_1)\exp_{-,t-s}(\lambda_2)\,ds,$$
$$\exp^{[2]}_{-,t}(\lambda_1,\lambda_2,\lambda_3)=\int_t^0\int_r^0 \exp_{-,s}(\lambda_1)\exp_{-,r-s}(\lambda_2)\exp_{-,t-r}(\lambda_3)\,ds\,dr.$$

(c) *The divided differences of the function $t\mapsto g_t$ admit the representation*

$$g^{[n-1]}_{(\cdot)}(\lambda_1,\dots,\lambda_n)=g_{(\cdot)}(\lambda_1)*\dots*g_{(\cdot)}(\lambda_n),\qquad \operatorname{Re}\lambda_1,\dots,\operatorname{Re}\lambda_n\neq 0.$$

For example, for $t\neq 0$, we have

$$g^{[1]}_t(\lambda_1,\lambda_2)=\int_{-\infty}^{\infty} g_s(\lambda_1)g_{t-s}(\lambda_2)\,ds,$$
$$g^{[2]}_t(\lambda_1,\lambda_2,\lambda_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_s(\lambda_1)g_{r-s}(\lambda_2)g_{t-r}(\lambda_3)\,ds\,dr.$$

REMARK 5. Assertion (a) is established in [52]. Assertion (b) is proved in a similar way. Assertion (c) is proved in [21, p. 53] for other aims. For completeness, we give here an independent proof of (c).

Proof. Suppose that the points of interpolation λ_1, λ_2, …, λ_n are distinct. By Proposition 16, we have

$$g_t^{[n-1]}(\lambda_1,\dots,\lambda_n)=\sum_{j=1}^{n}\frac{g_t(\lambda_j)}{\prod\limits_{\substack{k=1\\k\neq j}}^{n}(\lambda_j-\lambda_k)}.$$

Form this representation and Lemma 21, it easily follows that the bilateral Laplace transform of the function $t\mapsto g_t^{[n-1]}(\lambda_1,\dots,\lambda_n)$ is

$$\big(\mathscr{B}g_{(\cdot)}^{[n-1]}(\lambda_1,\dots,\lambda_n)\big)(\lambda)=\sum_{j=1}^{n}\frac{1}{\lambda-\lambda_j}\frac{1}{\prod\limits_{\substack{k=1\\k\neq j}}^{n}(\lambda_j-\lambda_k)}.$$

By Proposition 16, the last expression is the $(n-1)$-th divided difference of the function $r_\lambda(\nu)=\frac{1}{\lambda-\nu}$. By Lemma 20,

$$r_\lambda^{[n-1]}(\lambda_1,\dots,\lambda_n)=\frac{1}{\prod_{j=1}^{n}(\lambda-\lambda_j)}.$$

We apply the inverse Laplace transform to the function $\lambda\mapsto r_\lambda^{[n-1]}(\lambda_1,\dots,\lambda_n)$. Clearly, the restriction of the bilatiral Laplace transform to the imaginary axis is the Fourier transform. The Fourier transform maps the convolution of functions to the product of their images [41, p. 337], which implies assertion (c).

The case of coinciding points of interpolation λ_j follows from continuity. □

7. The divided differences $\exp_{\pm,t}^{[m]}$ and $g_t^{[m]}$ with operator arguments

In this Section, we apply previous results to the representation of the functions $\exp_{\pm,t}^{[m]}$ and $g_t^{[m]}$ with operator arguments.

THEOREM 23. *Let $A\in\mathbf{M}^+$. Then for $t>0$, we have*

$$\exp_{+,t}^{[m-1]}(A;i_1,i_2,\dots,i_{m-1},i_m)=\int_0^t\int_0^{s_{m-1}}\dots\int_0^{s_2}\exp_{+,s_1}(A_{i_1,i_1})A_{i_1,i_2}\exp_{+,s_2-s_1}(A_{i_2,i_2})$$
$$\times A_{i_2,i_3}\dots\exp_{+,s_{m-1}-s_{m-2}}(A_{i_{m-1},i_{m-1}})A_{i_{m-1},i_m}\exp_{+,t-s_{m-1}}(A_{i_m,i_m})\,ds_1\dots ds_{m-1};$$

for $t<0$, we have

$$\exp_{-,t}^{[m-1]}(A;i_1,i_2,\dots,i_{m-1},i_m)=\int_t^0\int_{s_{m-1}}^0\dots\int_{s_2}^0\exp_{-,s_1}(A_{i_1,i_1})A_{i_1,i_2}\exp_{-,s_2-s_1}(A_{i_2,i_2})$$
$$\times A_{i_2,i_3}\dots\exp_{-,s_{m-1}-s_{m-2}}(A_{i_{m-1},i_{m-1}})A_{i_{m-1},i_m}\exp_{-,t-s_{m-1}}(A_{i_m,i_m})\,ds_1\dots ds_{m-1};$$

and (if the spectrum $\sigma(A)$ is disjoint from the imaginary axis) for $t\neq 0$, we have

$$g_t^{[m-1]}(A;i_1,i_2,\dots,i_{m-1},i_m)=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}g_{s_1}(A_{i_1,i_1})A_{i_1,i_2}\,g_{s_2-s_1}(A_{i_2,i_2})\\
\times A_{i_2,i_3}\dots g_{s_{m-1}-s_{m-2}}(A_{i_{m-1},i_{m-1}})A_{i_{m-1},i_m}\,g_{t-s_{m-1}}(A_{i_m,i_m})\,ds_1\,\dots\,ds_{m-1}.$$

Proof. For simplicity of notation, we prove only the formula

$$g_t^{[2]}(A;3,2,1)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_{s_1}(A_{3,3})A_{3,2}\,g_{s_2-s_1}(A_{2,2})A_{2,1}\,g_{t-s_2}(A_{1,1})\,ds_1\,ds_2.$$

By Theorem 17, we have

$$g_t^{[2]}(A;3,2,1)=\frac{1}{(2\pi i)^3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3}g_t^{[2]}(\lambda_1,\lambda_2,\lambda_3)\\
\times(\lambda_3\mathbf{1}-A_{3,3})^{-1}A_{3,2}(\lambda_2\mathbf{1}-A_{2,2})^{-1}A_{2,1}(\lambda_1\mathbf{1}-A_{1,1})^{-1}\,d\lambda_1\,d\lambda_2\,d\lambda_3.$$

By Theorem 22, we have

$$g_t^{[2]}(\lambda_1,\lambda_2,\lambda_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_s(\lambda_1)\,g_{r-s}(\lambda_2)\,g_{t-r}(\lambda_3)\,ds\,dr.$$

Substituting the latter formula into the former one and performing the integration over λ_1, λ_2, and λ_3, we arrive at the desired formula. $\square$

Combining Theorems 19 and 23, we obtain the following examples.

EXAMPLE 1. Let A be the block matrix

$$A=\begin{pmatrix}A_{1,1}&0&0\\A_{2,1}&A_{2,2}&0\\A_{3,1}&A_{3,2}&A_{3,3}\end{pmatrix}. \tag{9}$$

Then for $t>0$, we have

$$\exp_{+,t}(A)=\begin{pmatrix}\exp_{+,t}(A_{1,1})&0&0\\\exp_{+,t}^{[1]}(A;2,1)&\exp_{+,t}(A_{2,2})&0\\\exp_{+,t}^{[1]}(A;3,1)+\exp_{+,t}^{[2]}(A;3,2,1)&\exp_{+,t}^{[1]}(A;3,2)&\exp_{+,t}(A_{3,3})\end{pmatrix}, \tag{10}$$

where

$$\exp_{+,t}^{[1]}(A;i_1,i_2)=\int_0^t\exp_{+,s}(A_{i_1,i_1})A_{i_1,i_2}\exp_{+,t-s}(A_{i_2,i_2})\,ds, \tag{11}$$

$$\exp_{+,t}^{[2]}(A;3,2,1)=\int_0^t\int_0^r\exp_{+,s}(A_{3,3})A_{3,2}\exp_{+,r-s}(A_{2,2})A_{2,1}\exp_{+,t-r}(A_{1,1})\,ds\,dr.$$

REMARK 6. Integral (11) was first obtained in [3, Ch. 10, § 14, Formula (5)] in a different context. Formula (10) for a triangular block matrix (with blocks consisting of scalars) of the size less than or equal to 4×4 has appeared in [52, theorem 1] and for a triangular block matrix of any size in [6].

EXAMPLE 2. Let A be block matrix (9), whose spectrum is disjoint from the imaginary axis. Then for $t \neq 0$, we have the representation of Green's function

$$\mathscr{G}(t) = g_t(A) = \begin{pmatrix} g_t(A_{1,1}) & 0 & 0 \\ g_t^{[1]}(A;2,1) & g_t(A_{2,2}) & 0 \\ g_t^{[1]}(A;3,1) + g_t^{[2]}(A;3,2,1) & g_t^{[1]}(A;3,2) & g_t(A_{3,3}) \end{pmatrix},$$

where

$$g_t^{[1]}(A;i_1,i_2) = \int_{-\infty}^{\infty} g_s(A_{i_1,i_1}) A_{i_1,i_2} g_{t-s}(A_{i_2,i_2})\, ds,$$
$$g_t^{[2]}(A;3,2,1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_s(A_{3,3}) A_{3,2} g_{r-s}(A_{2,2}) A_{2,1} g_{t-r}(A_{1,1})\, ds\, dr.$$

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