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THE QUINTIC COMPLEX MOMENT PROBLEM

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Abstract. Let $\gamma^{(m)} \equiv \{\gamma_{ij}\}_{0\leqslant i+j\leqslant m}$ be a given complex-valued sequence. The truncated complex moment problem (TCMP in short) involves determining necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j d\mu$ for $0 \leqslant i+j \leqslant m$. The TCMP has been completely solved only when $m = 1,2,3$ and 4.

We provide in this paper a concrete solution to the, almost all, quintic TCMP (that is, when $m = 5$). We also study the cardinality of the minimal representing measure. Based on the bi-variate recurrence sequences properties with some Curto-Fialkow's results. Our method intended to be useful for all odd-degree moment problems.

1. Introduction

Given a doubly indexed finite sequence of complex numbers

$$\gamma \equiv \gamma^{(m)} = \{\gamma_{ij}\}_{0\leqslant i+j\leqslant m} = \{\gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0m}, \dots, \gamma_{m0}\},$$

with $\gamma_{00} > 0$ and $\overline{\gamma}_{ij} = \gamma_{ji}$ for $0 \leqslant i+j \leqslant m$. The truncated complex moment problem (in short, TCMP) associated with γ entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu, \qquad 0 \leqslant i+j \leqslant m. \tag{1.1}$$

A sequence $\{\gamma_{ij}\}_{0\leqslant i+j\leqslant m}$ satisfying (1.1) will be called a truncated moment sequence and the solution μ is said to be a representing measure associated to the sequence $\{\gamma_{ij}\}_{0\leqslant i+j\leqslant m}$.

In [34] J. Stochel has shown that solving TCMP solves the widely studied Full Moment Problem (see, for example, [1, 2, 3, 17, 29, 30, 33, 36]). More precisely, a full moment sequence $\{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ admits a representing measure if and only if each of its truncation $\gamma^{(m)}$ admits a representing measure.

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The truncated complex moment problem serves as a prototype for several other moment problems to which it is closely related. Its application can be found in subnormal operator theory [31, 24, 35], polynomial hyponormality [12] and joint hyponormality [4, 5]. It is also related to the optimization theory [26, 25, 27, 28, 29] and arises in pure and applied mathematics, in physics and in several other domains.

For the even case $m = 2n$, Curto and Fialkow developed in a series of papers an approach for TCMP based on positivity and flat extensions of the moment matrix, see Section 2. This allowed them to find solutions for various particular cases of truncated moment problems (see, for instance, [6, 8, 7, 10, 11, 21, 20]). However, only the cases $m = 2$ and $m = 4$ are completely solved (cf. [6, 9, 19, 14]).

In the odd case $m = 2n+1$, a general solution to some partial cases of the TCMP can be found in [22] and [23] as well as a solution to the truncated matrix moment problem; a solution to the cubic complex moment problem (when m = 3) was given in [23], see also [16]. The solution is based on commutativity conditions of matrices determined by $\{\gamma_{ij}\}_{0\leqslant i+j\leqslant 2n+1}$.

Therefore, only the cases $m = 1, 2, 3$ and 4 (the quadratic, the cubic and the quartic moment problem) have been completely achieved. All the other cases (quintic, sixtic, ...) are open and interest several authors; as indicated in many recent papers (see, for instance, [13, 15, 16, 37, 38]).

In this paper, we provide a concrete solution to the, almost all, quintic moment problem (i.e. $m = 5$) when one desires a minimal representing measure. To this aim, we investigate the structure of recursive complex-valued bi-indexed sequences and we combine the obtained observations with some results due to R.Curto and L. Fialkow, to provide a new technique for solving the odd-degree TCMP. We notice that our techniques furnish a short solution to the cubic moment problem (we omit the proof because the cubic moment problem is already solved, see [16, 23]) and expected to be useful for higher odd-degree truncated moment problems.

Let $\gamma^{(5)} = \{\gamma_{ij}\}_{0\leqslant i+j\leqslant 5}$ be a given complex valued bi-sequence. We associate with $\gamma^{(5)}$ the next two matrices that will play a crucial role in our approach.

$$M(2) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix} B := \begin{pmatrix} \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \end{pmatrix}. \tag{1.2}$$

Let us recall that thanks to Douglas factorization theorem, we have $Rang\ B \subseteq Rang\ M(2)$ if, and only if, there exists a matrix W such that $B = M(2)W$. We will show, in Section 2, that the Hermitian matrix $W^*M(2)W$ is symmetric with respect to the counter diagonal (persymetric), then one can set

$$W^*M(2)W = \begin{pmatrix} a & b & c & d \\ \overline{b} & e & f & c \\ \overline{c} & \overline{f} & e & b \\ \overline{d} & \overline{c} & \overline{b} & a \end{pmatrix} \tag{1.3}$$

As we will see in the sequel, the entries a, b, e and f in the matrix $W^*M(2)W$ encode the complete information on the cardinal of the support of the minimal representing measure.

THEOREM 1.1. *Let* $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 5}$ *be a given finite sequence, such that*

$$M(2) \geqslant 0,\ Rang\ \ B \subseteq Rang\ \ M(2)\ \text{and}\ a \neq e\ \text{or}\ b = f.$$

Then the quintic moment problem, associated with $\gamma^{(5)}$*, admits a solution* μ. *Moreover, the smallest cardinality of supp* μ *is:*

- *card supp* $\mu = r \iff a = e$ *and* $b = f$;
- *card supp* $\mu = r+1 \iff \quad a \neq e$ *and* $\frac{a-e}{2} <| b-f |$;
- *card supp* $\mu = r+2 \iff \quad a > e$ *and* $\frac{a-e}{2} \geqslant| b-f |$;

where r *is the rank of* $M(2)$ *and the numbers* a,b,e *and* f *are given by* (1.3).

Since (as we will show in Section 2) $M(2) \geqslant 0$ and $Rang\ \ B \subseteq Rang\ \ M(2)$ are two necessary conditions for the quintic TCMP, associated with $\gamma^{(5)}$ to own a solution, then Theorem 1.1 provides a concrete solution to the quintic complex moment problem, except for the case $a = e$ and $b \neq f$. The difficulty that we encountered in solving the remaining case ($a = e$ and $b \neq f$) is technical, not a failure in the method, see Section 5.

This paper is organized as follows. In Section 2, we will give useful tools and results usually used in the treatment of the truncated complex moment problems. We will investigate in Section 3 the complex-valued recursive bi-sequences and we will present in section 4 a solution for the quintic TCMP associated with each, as well as a solution to the minimal support problem. Finally, in Section 5, we give several examples illustrating the different cases arising in the quintic complex moment problem.

2. Preliminaries

First, we recall a fundamental necessary condition. To this end, let us assume that $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 2n}$ is a given moment sequence and let μ be the associated representing measure. Then, for every $p \equiv \sum\limits_{h,k} a_{hk}\overline{z}^h z^k \in \mathbb{C}[\overline{z}, z]$, we have

$$0 \leqslant \int | p |^2 \, d\mu = \sum_{h,k,h',k'} a_{hk}\overline{a_{h'k'}} \int \overline{z}^{h+k'} z^{k+h'} = \sum_{h,k,h',k'} a_{hk}\overline{a_{h'k'}} \gamma_{h+k',k+h'},$$

or equivalently, the moment matrix $M(n) \equiv M(n)(\gamma^{(2n)})$, defined below, is positive semi-definite.

$$M(n) := \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix}, \tag{2.1}$$

where

$$M[i,j] = \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{j,i} \end{pmatrix}.$$

Considering the lexicographic order,

$$1, Z, \overline{Z}, Z^2, Z\overline{Z}, \overline{Z}^2, \dots, Z^n, Z^{n-1}\overline{Z}, \dots, Z\overline{Z}^{n-1}, \overline{Z}^n, \tag{2.2}$$

to denote rows and columns of the moment matrix $M(n)$. For example, the $M(3)$ matrix is

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & Z & \overline{Z} & Z^2 & Z\overline{Z} & \overline{Z}^2 & Z^3 & Z^2\overline{Z} & Z\overline{Z}^2 & \overline{Z}^3 \end{array} \\
\begin{array}{c} 1 \\ Z \\ \overline{Z} \\ Z^2 \\ Z\overline{Z} \\ \overline{Z}^2 \\ Z^3 \\ Z^2\overline{Z} \\ Z\overline{Z}^2 \\ \overline{Z}^3 \end{array} &
\left(\begin{array}{c|cc|ccc|cccc}
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\hline
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\hline
\gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\
\hline
\gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\
\gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\
\gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\
\gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33}
\end{array}\right)
\end{array}. \tag{2.3}$$

Observe in passing that each block $M[i,j]$ has a Toeplitz form. That is each of its diagonals contains constant entries. On the other hand, it is easy to see that the matrix $M(n)$ detects the positivity of the Riesz functional given by

$$\Lambda_{\gamma^{(2n)}} : p(\overline{z},z) \equiv \sum_{0 \leqslant i+j \leqslant 2n} a_{ij}\overline{z}^i z^j \longrightarrow \sum_{0 \leqslant i+j \leqslant 2n} a_{ij}\gamma_{ij}$$

on the cone generated by the collection $\{p\overline{p} : p \in \mathbb{C}_n[\overline{z},z]\}$, where $\mathbb{C}_n[Z,\overline{Z}]$ is the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to n.

It is an immediate observation that the row $\overline{Z}^k Z^l$, column $\overline{Z}^i Z^j$ entry of the matrix $M(n)$ is equal to $\Lambda_{\gamma^{(2n)}}(\overline{z}^{i+l}z^{j+k}) = \gamma_{i+l,j+k}$. For reason of simplicity, we identify a polynomial $p(\overline{z},z) \equiv \sum a_{ij}\overline{z}^i z^j$ with its coefficient vector $p = (a_{ij})$ with respect to the basis of monomials of $\mathbb{C}_n[\overline{z},z]$ in degree-lexicographic order. Clearly, $M(n)$ acts on these coefficient vectors as follows:

$$\langle M(n)p,q\rangle = \Lambda_{\gamma^{(2n)}}(p\overline{q}). \tag{2.4}$$

A theorem of Smul'jan [32] shows that a block matrix

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geqslant 0, \tag{2.5}$$

if and only if:

(i) $A \geqslant 0$;

(ii) there exists a matrix W such that $B = AW$;

(iii) $C \geqslant W^*AW$.

Since $A = A^*$, we obtain that W^*AW is independent of the choice of W provided that $B = AW$. Moreover, $rank\ \ M = rank\ \ A \Leftrightarrow C = W^*AW$ for some W such that $B = AW$. Conversely, if $A \geqslant 0$, any extension M satisfying $rank\ \ M = rank\ \ A$ (if this condition is satisfied, we say that M is a flat extension of A) is necessarily positive. Notice also that from the expression

$$\begin{pmatrix} I & 0 \\ -W^* & I' \end{pmatrix} M \begin{pmatrix} I & -W \\ 0 & I' \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - W^*AW \end{pmatrix},$$

where I and I' denote the unit matrices, we deduce that

$$rank\ M = rank\ A + rank\ (C - W^*AW). \tag{2.6}$$

By Smul'jan's theorem, $M(n) \geqslant 0$ admits a (necessarily positive) flat extension

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \tag{2.7}$$

in the form of a moment matrix $M(n+1)$ if and only if:

(i) $B = M(n)W$ for some W;

(ii) $C = W^*M(n)W$ is a Toeplitz matrix.

Furthermore, we have the next result due to Curto and Fialkow.

THEOREM 2.1. *[6, Theorem 5.13] The finite sequence $\gamma^{(2n)}$ has a rank $M(n)$-atomic representing measure if and only if $M(n) \geqslant 0$ and $M(n)$ admits a flat extension $M(n+1)$. That is, $M(n)$ can be extended to a positive moment matrix $M(n+1)$ satisfying rank $M(n+1) =$ rank $M(n)$.*

An important step in our approach is to show that the Hermitian matrix $W^*M(n)W$ is persymmetric, that is, it is symmetric across its lower-left to upper-right diagonals. For this purpose, we introduce first some additional notations.

We denote the successive columns of W and B (given as in Expression (2.7)) by $W_{|Z^{n+1}}, W_{|\overline{Z}Z^n}, \dots, W_{|\overline{Z}^{n+1}}$ and $B_{|Z^{n+1}}, B_{|\overline{Z}Z^n}, \dots, B_{|\overline{Z}^{n+1}}$, respectively.

Let us consider the $\frac{(n+1)(n+2)}{2}$-matrix built as follows,

$$M_\varphi(n) := J_0 \oplus J_1 \oplus \dots \oplus J_n;$$

where $J_p = (\delta_{i+j,p+1})_{1\leqslant i,j\leqslant p}$ with $\delta_{i,j}$ is the Kronecker symbol given by $\delta_{l,k} = 1$ for $k = l$ and zero otherwise. For example

$$J_0 = (1), \quad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

LEMMA 2.2. *Let $M_\varphi(n)$, $M(n)$ and $B_{\overline{Z}^{n-i}Z^i}$ $(i = 0,\dots,n)$ be as above, then:*

1. $(M_\varphi(n))^2 = I$;
2. $(M_\varphi(n))^* = M_\varphi(n)$;
3. $M_\varphi(n)B_{\overline{Z}^iZ^{n-i}} = B_{\overline{Z}^{n-i}Z^i}, \qquad i = 0,\dots,n$;
4. $M_\varphi(n)M(n) = \overline{M(n)}M_\varphi(n)$.

Proof. The assertions (1), (2) and (3) are obvious. Only the fourth assertion requires a proof. To this aim, we recall that $M(n) = [M(i,j)]_{0\leqslant i,j\leqslant n}$, see (2.1). Therefore

$$\begin{aligned}[M_\varphi(n)]M(n) &= \left[\bigoplus_{i=0}^n J_i\right][M(i,j)]_{i,j\leqslant n} = [J_iM(i,j)]_{i,j\leqslant n} = [\overline{M(i,j)}J_j]_{i,j\leqslant n} \\ &= [\overline{M(i,j)}]_{i,j\leqslant n}\left[\bigoplus_{i=0}^n J_i\right] = \overline{M(n)}M_\varphi(n). \quad \square\end{aligned}$$

PROPOSITION 2.3. *Let n be a given integer and let $M(n)$ and W be as above, then $W^*M(n)W$ is an Hermitian Persymmetric matrix.*

Proof. Setting $W^*M(n)W = (c_{ij})_{0\leqslant i,j\leqslant n}$, then we have

$$c_{n-j,n-i} = W^*_{\overline{Z}^{n-j}Z^j} M(n) W_{\overline{Z}^{n-i}Z^i}. \tag{2.8}$$

By multiplying left sides of the fourth equation in Lemma 2.2 by $M_\varphi(n)$ we obtain

$$M_\varphi(n)M_\varphi(n)M(n) = M_\varphi(n)\overline{M(n)}M_\varphi(n). \tag{2.9}$$

Hence, by applying Lemma 2.2-(1), we get

$$M(n) = M_\varphi(n)\overline{M(n)}M_\varphi(n). \tag{2.10}$$

It follows, from (2.8) and (2.10), that

$$c_{n-j,n-i} = W^*_{\overline{Z}^{n-j}Z^j} M_\varphi(n)\overline{M(n)}M_\varphi(n) W_{\overline{Z}^{n-i}Z^i}. \tag{2.11}$$

The fact that $M_\varphi(n)$ is self-adjoint allows to write

$$c_{n-j,n-i} = \left(M_\varphi(n)W_{\overline{Z}^{n-j}Z^j}\right)^* \overline{M(n)} \left(M_\varphi(n)W_{\overline{Z}^{n-i}Z^i}\right). \tag{2.12}$$

By using the assertions (3) and (4), in Lemma 2.2, we deduce that:

$$\overline{M(n)}M_\varphi(n)W_{\overline{Z}^{n-i}Z^i} = M_\varphi(n)M(n)W_{\overline{Z}^{n-i}Z^i} = M_\varphi(n)B_{\overline{Z}^{n-i}Z^i} = B_{\overline{Z}^iZ^{n-i}}.$$

Therefore, (2.12) implies that

$$\begin{aligned}
c_{n-j,n-i} &= (M_\varphi(n)W_{\overline{Z}^{n-j}Z^j})^* B_{\overline{Z}^iZ^{n-i}} = W^*_{\overline{Z}^{n-j}Z^j} M_\varphi(n)M(n)W_{\overline{Z}^iZ^{n-i}}\\
&= ((M(n)M_\varphi(n))^* W_{\overline{Z}^{n-j}Z^j})^* W_{\overline{Z}^iZ^{n-i}} = (M_\varphi(n)M(n)W_{\overline{Z}^{n-j}Z^j})^* W_{\overline{Z}^iZ^{n-i}}\\
&= (\overline{M(n)}M_\varphi(n)W_{\overline{Z}^{n-j}Z^j})^* W_{\overline{Z}^iZ^{n-i}} = (M(n)W_{\overline{Z}^jZ^{n-j}})^* W_{\overline{Z}^iZ^{n-i}} = W^*_{\overline{Z}^jZ^{n-j}} M(n) W_{\overline{Z}^iZ^{n-i}}\\
&= c_{i,j}.
\end{aligned}$$

This concludes the proof of the Proposition 2.3. □

3. Complex-valued recursive bi-sequences

Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant n}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$ and $n \in \mathbb{N}\cup\{+\infty\}$, be a given complex-valued sequence and let $P_{\overline{z}^e z^{d-e}} = \sum\limits_{\substack{l+k\leqslant d\\(l,k)\neq(e,d-e)}} a_{lk}\overline{z}^l z^k$ be in $\mathbb{C}_d[\overline{z},z]$, the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to d (we assume that $d\leqslant n$). The sequence $\gamma^{(n)}$ is said to be recursive, associated with a generating polynomial $\overline{z}^e z^{d-e} - P_{\overline{z}^e z^{d-e}}$, if

$$\gamma_{e+i,d-e+j} = \Lambda_{\gamma^{(n)}}(\overline{z}^i z^j P_{\overline{z}^e z^{d-e}}), \qquad \text{for all } i+j\leqslant n-d, \tag{3.1}$$

or, equivalently, if

$$\gamma_{e+i,d-e+j} = \sum_{\substack{l+k\leqslant d\\(l,k)\neq(e,d-e)}} a_{lk}\gamma_{l+i,k+j}, \qquad i+j\leqslant n-d. \tag{3.2}$$

We notice that, because of the equality $\overline{\gamma_{ij}} = \gamma_{ji}$, Equation (3.2) is equivalent to the following one:

$$\gamma_{d-e+i,e+j} = \sum_{\substack{l+k\leqslant d\\(l,k)\neq(e,d-e)}} \overline{a_{lk}}\gamma_{k+i,l+j}, \tag{3.3}$$

for all integers i and j, with $i+j \leqslant n-d$.
Therefore, $\overline{z}^{d-e}z^e - P_{\overline{z}^{d-e}z^e}$ (where $P_{\overline{z}^{d-e}z^e} = \overline{P_{\overline{z}^e z^{d-e}}}$) is, also, a generating polynomial, associated with $\gamma^{(n)}$; that is,

$$\gamma_{d-e+i,e+j} = \Lambda_{\gamma^{(n)}}(\overline{z}^i z^j P_{\overline{z}^{d-e}z^e}), \qquad i+j \leqslant n-d. \tag{3.4}$$

The following proposition provides a connection, via Λ, between the polynomials $P_{\overline{z}^f z^{f+1}}$ and $P_{\overline{z}^{f+1}z^f}$.

PROPOSITION 3.1. *Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant n}$ be a recursive bi-sequence and let $\overline{z}^f z^{f+1} - P_{\overline{z}^f z^{f+1}}$ be an associated generating polynomial, then*

$$\Lambda_{\gamma^{(n)}}(\overline{z}^{l+1}z^k P_{\overline{z}^f z^{f+1}}) = \Lambda_{\gamma^{(n)}}(\overline{z}^l z^{k+1} P_{\overline{z}^{f+1}z^f}), \qquad l+k \leqslant n-2f-2.$$

Proof. For all integers l and k, with $l+k \leqslant n-2f-2$, we have

$$\begin{aligned}\Lambda_{\gamma^{(n)}}(\overline{z}^{l+1}z^k P_{\overline{z}^f z^{f+1}}) &= \gamma_{f+l+1,f+k+1} = \overline{\gamma_{f+k+1,f+l+1}} = \overline{\Lambda_{\gamma^{(n)}}(\overline{z}^{k+1}z^l P_{\overline{z}^f z^{f+1}})}\\ &= \Lambda_{\gamma^{(n)}}(\overline{z}^l z^{k+1} P_{\overline{z}^{f+1}z^f}). \quad \square\end{aligned}$$

It is well known that the classical singly indexed recursive sequence can be defined by the initial data and the associated recurrence relation (or, characteristic polynomial), see [18]. In a similar way, one can define recursive bi-sequences as observed below.

REMARK 3.2. (i) A generating polynomial $z^e - P_{z^e}$ (or, equivalently, $\overline{z}^e - P_{\overline{z}^e}$), where $\deg P_{z^e} < e$, with the initial data $\{\gamma_{ij}\}_{i,j<e}$ verifying $\overline{\gamma_{ij}} = \gamma_{ji}$, stand together to furnish the sequence $\gamma^{(n)}$.

(ii) For a generating polynomial $\overline{z}z^{e-1} - P_{\overline{z}z^{e-1}}$, with $\deg P_{\overline{z}z^{e-1}} < e$, we need (all) the data $\{\gamma_{ij}\}_{i,j<e} \cup \{\gamma_{0j}\}_{j=e,\dots,n}$ and the equality $\overline{\gamma_{ij}} = \gamma_{ji}$ to get the recursive bi-sequence $\gamma^{(n)}$.

In the next lemma, we provide useful results for solving the quintic moment problem.

LEMMA 3.3. *Let $\gamma^{(8)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 8}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$, be a truncated bi-sequence and let $z^4 - P_{z^4}$ (where $P_{z^4} = \beta z^3 + R_{z^4}$ and $R_{z^4} \in \mathbb{C}_2[\overline{z},z]$) be an associated generating polynomial. Assume that $\overline{z}z^2 - P_{\overline{z}z^2}$ (where $P_{\overline{z}z^2} = \alpha z^3 + R_{\overline{z}z^2}$, $\alpha \neq 0$ and $R_{\overline{z}z^2} \in \mathbb{C}_2[\overline{z},z]$) is a generating polynomial for $\gamma^{(6)} \cup \{\gamma_{34},\gamma_{43}\}$, then $\overline{z}z^2 - P_{\overline{z}z^2}$ is a generating polynomial for $\gamma^{(8)}$.*

Proof. We have $z^4 - P_{z^4}$ is a generating polynomial for $\gamma^{(8)}$, that is,

$$\gamma_{i,j+4} = \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j P_{z^4}) = \beta\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j R_{z^4}), \qquad i+j \leqslant 4. \tag{3.5}$$

As shown in (3.4), the last equality (3.7) is equivalent to

$$\gamma_{i+4,j} = \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j P_{\overline{z}^4}) = \overline{\beta}\gamma_{i+3,j} + \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j R_{\overline{z}^4}), \qquad i+j \leqslant 4; \tag{3.6}$$

where $\overline{P_{z^4}} := P_{\overline{z}^4} = \overline{\beta}\overline{z}^3 + \overline{R_{\overline{z}z^2}}$.
Also, the polynomial $\overline{z}z^2 - P_{\overline{z}z^2}$ is a generating one for $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$; that is, for all $i+j \leqslant 3$ and $(i,j) = (2,2),(3,1)$:

$$\gamma_{i+1,j+2} = \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j P_{\overline{z}z^2}) = \alpha\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j R_{\overline{z}z^2}), \tag{3.7}$$

or, equivalently, for $i+j \leqslant 3$ and $(i,j) = (2,2),(1,3)$:

$$\gamma_{i+2,j+1} = \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j P_{\overline{z}^2 z^1}) = \overline{\alpha}\gamma_{i+3,j} + \Lambda_{\gamma^{(8)}}(\overline{z}^i z^j R_{\overline{z}^2 z}), \tag{3.8}$$

where $P_{\overline{z}^2 z} := \overline{P_{\overline{z}z^2}} = \overline{\beta}\overline{z}^3 + R_{\overline{z}^2 z}$, see (3.4).
We have to show that (3.7) remains valid for all integers i and j, verifying $i+j \leqslant 5$. To this end, we consider the recursive bi-sequence $\widehat{\gamma}^{(8)} \equiv \{\widehat{\gamma}_{ij}\}_{i+j\leqslant 8}$ defined by

$$\begin{cases} \widehat{\gamma}_{i+1,j+2} & = \Lambda_{\widehat{\gamma}^{(8)}}(\overline{z}^i z^j P_{\overline{z}z^2}), \qquad i+j \leqslant 5, \\ \widehat{\gamma}_{i,j} & = \gamma_{i,j}, \qquad \text{otherwise;} \end{cases} \tag{3.9}$$

and we will show that $\widehat{\gamma}^{(8)} = \gamma^{(8)}$. Notice that since $\overline{z}z^2 - P_{\overline{z}z^2}$ is a generating polynomial for $\widehat{\gamma}^{(8)}$, then $\overline{z}^2 z - P_{\overline{z}^2 z}$ is an other one. Thus

$$\widehat{\gamma}_{i+2,j+1} = \Lambda_{\widehat{\gamma}^{(8)}}(\overline{z}^i z^j P_{\overline{z}^2 z}), \qquad i+j \leqslant 5. \tag{3.10}$$

It follows from (3.7) and (3.9) that, for $n+m \leqslant 6$, $n = 0$ and $(n,m) = (3,4),(4,3)$:

$$\gamma_{nm} = \Lambda_{\gamma^{(8)}}(\overline{z}^n z^m) = \Lambda_{\widehat{\gamma}^{(8)}}(\overline{z}^n z^m) := \widehat{\gamma}_{nm}. \tag{3.11}$$

Remark that if $\widehat{\gamma}_{nm} = \gamma_{nm}$ then $\widehat{\gamma}_{mn} = \overline{\widehat{\gamma}_{nm}} = \overline{\gamma_{nm}} = \gamma_{mn}$.

Therefore, it remains to show (3.11), only, for $(n,m) = (2,5),(1,6);(1,7),(2,6),(3,5),(4,4)$. We start with γ_{25} and γ_{16}

$$\begin{aligned}
\gamma_{25} &= \Lambda_{\gamma^{(8)}}(\overline{z}^2 z P_{z^4}) && \text{utilizing (3.5)} \\
&= \Lambda_{\gamma^{(8)}}(P_{\overline{z}^2 z} P_{z^4}) && \text{employ (3.8) and } \deg P_{z^4} \leqslant 3 \\
&= \overline{\alpha}\Lambda_{\gamma^{(8)}}(\overline{z}^3 P_{z^4}) + \Lambda_{\gamma^{(8)}}(R_{\overline{z}^2 z} P_{z^4}) && \\
&= \overline{\alpha}\gamma_{34} + \Lambda_{\gamma^{(8)}}(z^4 R_{\overline{z}^2 z}) && \text{applying (3.5)} \\
&= \overline{\alpha}\widehat{\gamma}_{34} + \Lambda_{\widehat{\gamma}^{(8)}}(z^4 R_{\overline{z}^2 z}) && \text{use } \deg z^4 R_{\overline{z}^2 z} \leqslant 6 \text{ and (3.11)} \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\overline{\alpha}\overline{z}^3 z^4 + z^4 R_{\overline{z}^2 z}) && \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^4 P_{\overline{z}^2 z}) && \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\overline{z}z^3 P_{z^2\overline{z}}) && \text{according to Proposition 3.1} \\
&= \widehat{\gamma}_{25} && \text{from (3.10).}
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\gamma_{16} &= \Lambda_{\gamma^{(8)}}(\bar{z}z^2 P_{z^4}) && \text{use(3.5)} \\
&= \Lambda_{\gamma^{(8)}}(P_{\bar{z}z^2} P_{z^4}) && \text{employ (3.7) and } \deg P_{z^4} \leqslant 3 \\
&= \Lambda_{\gamma^{(8)}}(\alpha z^3 P_{z^4} + R_{\bar{z}z^2} P_{z^4}) \\
&= \alpha\gamma_{07} + \Lambda_{\gamma^{(8)}}(z^4 R_{\bar{z}z^2}) && \text{utilizing (3.5)} \\
&= \alpha\widehat{\gamma}_{07} + \Lambda_{\widehat{\gamma}^{(8)}}(z^4 R_{\bar{z}z^2}) && \text{using (3.11) and } \deg z^4 R_{\bar{z}z^2} \leqslant 6 \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\alpha z^7 + z^4 R_{\bar{z}z^2}) \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^4 P_{\bar{z}z^2}) \\
&= \widehat{\gamma}_{16} && \text{according to (3.9).}
\end{aligned} \tag{3.13}$$

Thus, the equality (3.11) is valid for every integer n and m with $n+m \leqslant 7$. In other words,

$$\gamma_{nm} = \Lambda_{\gamma^{(8)}}(\bar{z}^n z^m) = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^n z^m) := \widehat{\gamma}_{nm}, \qquad n+m \leqslant 7. \tag{3.14}$$

Hence one can generalize relation (3.7) as follows

$$\gamma_{i+1,j+2} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) = \alpha\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}z^2}), \qquad i+j \leqslant 4. \tag{3.15}$$

Now, let us show (3.11) for the remaining cases $(n+m=8)$.

$$\gamma_{08} = \widehat{\gamma}_{08}, \qquad \text{by the construction of } \widehat{\gamma}^{(8)}, \text{ see (3.9).} \tag{3.16}$$

$$\begin{aligned}
\gamma_{17} &= \Lambda_{\gamma^{(8)}}(\bar{z}z^3 P_{z^4}) && \text{according to (3.5)} \\
&= \Lambda_{\gamma^{(8)}}(z P_{z^4} P_{\bar{z}z^2}) && \text{because } \deg z P_{z^4} \leqslant 4 \text{ and (3.15)} \\
&= \Lambda_{\gamma^{(8)}}(z^5 P_{\bar{z}z^2}) && \text{utilizing (3.5)} \\
&= \alpha\Lambda_{\gamma^{(8)}}(z^8) + \Lambda_{\gamma^{(8)}}(z^5 R_{\bar{z}z^2}) \\
&= \alpha\Lambda_{\widehat{\gamma}^{(8)}}(z^8) + \Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}z^2}) && \text{using (3.16) and (3.14)} \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^5 P_{\bar{z}z^2}) \\
&= \widehat{\gamma}_{17} && \text{applying (3.9).}
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\gamma_{26} &= \Lambda_{\gamma^{(8)}}(\bar{z}^2 z^2 P_{z^4}) && \text{according to (3.5)} \\
&= \Lambda_{\gamma^{(8)}}(\bar{z} P_{z^4} P_{\bar{z}z^2}) && \text{use } \deg \bar{z} P_{z^4} \leqslant 4 \text{ and (3.15)} \\
&= \Lambda_{\gamma^{(8)}}(\bar{z}z^4 P_{\bar{z}z^2}) && \text{utilizing (3.5)} \\
&= \alpha\Lambda_{\gamma^{(8)}}(\bar{z}z^7) + \Lambda_{\gamma^{(8)}}(\bar{z}z^4 R_{\bar{z}z^2}) \\
&= \alpha\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^7) + \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^4 R_{\bar{z}z^2}) && \text{by using (3.17) and (3.14)} \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^4 P_{\bar{z}z^2}) \\
&= \widehat{\gamma}_{26} && \text{according to (3.9).}
\end{aligned} \tag{3.18}$$

Before we continue the proof of our lemma, let us remark that the relation (3.14) implies that, for all $i+j \leqslant 5$,

$$\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^{i+1} z^{j+2}) = \widehat{\gamma}_{i+1,j+2} = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j (\alpha z^3 + R_{\bar{z}z^2})),$$

therefore

$$\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^{j+3}) = \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j(\bar{z}z^2 - R_{\bar{z}z^2})), \qquad i+j \leqslant 5. \tag{3.19}$$

Now, we write

$$\begin{aligned}
\gamma_{35} &= \Lambda_{\gamma^{(8)}}(\bar{z}^3 z P_{z^4}) && \text{according to (3.5)}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z P_{z^4}) && \text{because } \deg \bar{z}^3 z P_{z^4} \leqslant 7\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) z P_{z^4})\\
&= \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}((\bar{z}^2 z - R_{\bar{z}^2 z}) z P_{z^4})\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}((\bar{z}^2 z - R_{\bar{z}^2 z}) z P_{z^4}) && \text{applying (3.14)}\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^2 z^6) - \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(z^5 R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\gamma_{26} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}^2 z}) && \text{remark that } \deg z^5 R_{\bar{z}^2} \leqslant 7\\
&= \frac{1}{\alpha}\widehat{\gamma}_{26} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}^2 z}) && \text{from (3.18)}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) z^5)\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z^5)\\
&= \widehat{\gamma}_{35}.
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\gamma_{44} &= \Lambda_{\gamma^{(8)}}(\bar{z}^4 P_{z^4})\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 \bar{z} P_{z^4}) && \text{using (3.14)}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) \bar{z} P_{z^4}) && \text{by (3.19)}\\
&= \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z P_{z^4}) - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^3 z P_{z^4}) - \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\gamma_{35} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\widehat{\gamma}_{35} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z}) && \text{applying (3.20)}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) \bar{z} P_{z^4})\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^4 P_{z^4})\\
&= \widehat{\gamma}_{44}.
\end{aligned} \tag{3.21}$$

This finishes the proof of Lemma 3.3. $\square$

4. Solving the quintic moment problem

Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 5}$ be a given complex-valued bi-sequence, with $\gamma_{00} > 0$ and $\overline{\gamma_{ij}} = \gamma_{ji}$ for $i+j \leqslant 5$. The quintic complex moment problem involves determining necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{C}$ (called a representing measure for $\gamma^{(5)}$) such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu, \qquad \text{for } i+j \leqslant 5.$$

In this section we will show that in almost all cases the classical necessary conditions $M(2) \geqslant 0$ and $B = M(2)W$, for some W, (with $M(2)$ and B are as in (2.7)) guarantee the existence of at most $(r+2)$-atomic (here $r := rank\ M(2)$) representing measure for $\gamma^{(5)}$.

According to Proposition 2.3, the Hermitian 4×4-matrix $W^*M(2)W$ is symmetric with respect to the counter diagonal, then one can set

$$W^*M(2)W = \begin{pmatrix} a & b & c & d \\ \bar{b} & e & f & c \\ \bar{c} & \bar{f} & e & b \\ \bar{d} & \bar{c} & \bar{b} & a \end{pmatrix}. \tag{4.1}$$

The next theorem gives a concrete solution to the quintic complex moment problem, except for the case $a = e$ and $b \neq f$.

THEOREM 4.1. *Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 5}$ be a given sequence, we assume that $M(2) \geqslant 0$ and Rang $B \subseteq$ Rang $M(2)$, and $a \neq e$ or $b = f$.*

Then the quintic moment problem, associated with $\gamma^{(5)}$, admits a solution μ. Moreover, the smallest cardinality of supp μ is given by:

- *card supp $\mu = r \iff a = e$ and $b = f$;*
- *card supp $\mu = r+1 \iff \quad a \neq e$ and $\frac{a-e}{2} <| b-f |$;*
- *card supp $\mu = r+2 \iff \quad a > e$ and $\frac{a-e}{2} \geqslant| b-f |$;*

where a, b, e and f are as in (4.1).

Before we develop the proof of our theorem, let us introduce some notations. For $n \in \{3,4\}$; let $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 2n}$ be a truncated complex bi-sequence and let $M(n)$ be the associated moment matrix. As before, we denote by $B(n)$ and $C(n)$ the $(n-1)\times n$-matrix and the $n \times n$-matrix, respectively, such that

$$M(n) = \begin{pmatrix} M(n-1) & B(n) \\ B^*(n) & C(n) \end{pmatrix}. \tag{4.2}$$

Let $\mathfrak{B} \equiv \mathfrak{B}(n) \equiv \{\overline{Z}^i Z^j\}_{(i,j)\in\mathfrak{R}}$ (where $\mathfrak{R} \equiv \mathfrak{R}(n) \subseteq \{0,1,\ldots,n\}\times\{0,1,\ldots,n\}$) be a basis for the column space of $M(n)$. Let us remark that the $r\times r$-matrix $M(n)_{|\mathfrak{B}}$, where $r \equiv r(n) :=$ *card* $\mathfrak{R}(n)$, the restriction of the moment matrix $M(n)$ to the basis $\mathfrak{B}$, is invertible.

Proof of Theorem 4.1. The main idea is to extend the initial data $\gamma^{(5)}$ to an even-degree $\gamma^{(6)}$ (by adding the sixtic moments $\gamma_{60} = \overline{\gamma_{06}}$, $\gamma_{51} = \overline{\gamma_{15}}$, $\gamma_{42} = \overline{\gamma_{24}}$ and $\gamma_{33} \in \mathbb{R}$) such that the associated moment matrix $M(3)$, for an appropriate choice of the missing moments, is either a flat extension of $M(2)$ or admits a flat extension $M(4)$. Theorem 2.1 yields that $M(3)$ has a representing measure; and as a consequence, $\gamma^{(5)}$ also admits a representing measure μ. It is also proved that the smallest cardinality of *supp* μ will be $r := rank\ M(2)$ or $r+1$ or $r+2$.

By virtue of the Smul'jan's Theorem, we need to find a Toeplitz square matrix $C(3)$, built with the new, sixtic, moments as entries and such that $C(3) - W^*M(2)W \geqslant 0$. Setting

$$C(3) - W^*M(2)W = \begin{pmatrix} \gamma_{33}-a & \gamma_{42}-b & \gamma_{51}-c & \gamma_{60}-d \\ \gamma_{24}-\bar{b} & \gamma_{33}-e & \gamma_{42}-f & \gamma_{51}-c \\ \gamma_{15}-\bar{c} & \gamma_{24}-\bar{f} & \gamma_{33}-e & \gamma_{42}-b \\ \gamma_{06}-\bar{d} & \gamma_{15}-\bar{c} & \gamma_{24}-\bar{b} & \gamma_{33}-a \end{pmatrix}, \tag{4.3}$$

in the sequel, we distinguish two cases:

Case I: $a = e$ **and** $b = f$**.** In this case the matrix $W^*M(2)W$ is a Toeplitz one, then it suffices to consider that $C(3) = W^*M(2)W$. According to (2.6), the matrix $M(3)$ is a flat extension of $M(2)$ and thus $\gamma^{(6)}$ (and in force $\gamma^{(5)}$) has a r-representing measure.

Case II: $a \neq e$**.** We proceed in two steps for this case. Obviously, the matrix $W^*M(2)W$ is not a Toeplitz one. Therefore, for every choice of a Toeplitz 4×4-matrix $C(3)$, we have *rank* $(C(3) - W^*M(2)W) \geqslant 1$. We will show, in first step, that the smallest possible *rank* of $C(3) - W^*M(2)W$ will be either 1 or 2. In the second step, we will show that the moment matrix $M(3)$, obtained by extending $\gamma^{(5)}$ with the entries of some suitable $C(3)$, has a flat extension and thus admits a *rank* $M(3)$-atomic representing measure, see Theorem 2.1.

Step 1: Construction of $C(3)$. Firstly, let us observe that

$$rank(C(3) - W^*M(2)W) = 1 \text{ and } C(3) - W^*M(2)W \geqslant 0 \tag{4.4}$$

if and only if we have:

$$\begin{aligned}
&(0)\ \gamma_{33} > max(a,e);\\
&(i)\ |\gamma_{42} - b| = \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)} \quad \text{and} \quad |\gamma_{42} - f| = \gamma_{33} - e;\\
&(ii)\ (\gamma_{15} - \overline{c})(\gamma_{42} - b) = (\gamma_{33} - a)(\gamma_{24} - \overline{f});\\
&(iii)\ (\gamma_{06} - \overline{d})(\gamma_{42} - b)^2 = (\gamma_{33} - a)^2(\gamma_{24} - \overline{f}) \quad \text{and} \quad |\gamma_{06} - \overline{d}|^2 = (\gamma_{33} - a)^2.
\end{aligned} \tag{4.5}$$

Remark that the equalities in (i) provide the compatibility of the two equalities in (iii) and vice versa.

The condition (i) means that γ_{42} belongs to the intersection of the circles $\mathscr{C} = \mathscr{C}(b, \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)})$, of radius $\sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}$ and centered at b, and $\mathscr{C}' = \mathscr{C}(f, \gamma_{33} - e)$.

It is geometricaly easy to see that, the two circles $\mathscr{C}$ and $\mathscr{C}'$ have a nonempty intersection if, and only if, there exists $\gamma_{33} > \max(a,e)$, such that

$$|(\gamma_{33}-e) - \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}| \leqslant |b-f| \leqslant \gamma_{33} - e + \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}. \tag{4.6}$$

Since the function $x \to (x-e) - \sqrt{(x-a)(x-e)}$ is decreasing (on $[\max(a,e); +\infty[$), then $(x-e) - \sqrt{(x-a)(x-e)} \xrightarrow[x\to+\infty]{} \frac{a-e}{2}$ and $(x-e) + \sqrt{(x-a)(x-e)} \xrightarrow[x\to+\infty]{} +\infty$. Therefore (4.6) is verified if and only if $a = e$ and $b \neq f$ or $a < e$ or $a > e$ and $|b-f| > \frac{a-e}{2}$.

Sub-case II-1: $a < e$ **or** $a > e$ **and** $|b-f| > \frac{a-e}{2}$**.** It suffices to choose γ_{33} verifying (4.6), and thus γ_{42} exists (as an intersection point of the two circles $\mathscr{C}$ and $\mathscr{C}'$). Furthermore , from (0) and (i) we derive that

$$(\gamma_{42} - b)(\gamma_{42} - f) \neq 0. \tag{4.7}$$

Equality (ii) gives the moment γ_{15} and (iii) supplies γ_{06}, and this completes the construction of a Toeplitz matrix $C(3)$ for which $rank(C(3) - W^*M(2)W) = 1$. Note that,

$$rank\ \ M(3)_{|\mathfrak{B}(2)\cup\{Z^3\}} = rank\ \ M(2) + 1 = rank\ \ M(3).$$

Hence, in $M(3)$, the columns $\overline{Z}Z^2, \overline{Z}^2Z$ and $\overline{Z}^3$ are linear combinations of the columns $\mathfrak{B}(2) \cup \{Z^3\}$. In particular, we can set

$$\overline{Z}Z^2 = P_{\overline{Z}Z^2}(Z,\overline{Z}) = \alpha Z^3 + R_{\overline{Z}Z^2}(Z,\overline{Z}), \tag{4.8}$$

with

$$\alpha = \frac{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & \overline{Z}Z^2{}_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{42} \end{vmatrix}}{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33} \end{vmatrix}} = \frac{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & \overline{Z}Z^2{}_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & b \end{vmatrix} + (\gamma_{42}-b) \quad \det | M(2)_{|\mathfrak{B}(2)} |}{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & a \end{vmatrix} + (\gamma_{33}-a) \quad \det | M(2)_{|\mathfrak{B}(2)} |}$$

$$= \frac{(\gamma_{42}-b) \quad \det | M(2)_{|\mathfrak{B}(2)} |}{(\gamma_{33}-a) \quad \det | M(2)_{|\mathfrak{B}(2)} |} = \frac{\gamma_{42}-b}{\gamma_{33}-a} \neq 0; \quad \text{by virtue of (4.7)}. \tag{4.9}$$

Sub-case II-2: $a > e$ **and** $\frac{a-e}{2} \geqslant | b-f |$. Then $rank(C(3)-W^*M(2)W) \geqslant 2$ for every 4×4-Toeplitz matrix $C(3)$. Let us choose the sixtic moments as follows

$$\begin{cases} \gamma_{33} > \max(a,e), \\ | \gamma_{42}-b |= \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}, \\ \gamma_{15}-\overline{c} = \frac{\gamma_{33}-a}{\gamma_{42}-b}(\gamma_{24}-\overline{f}), \\ \gamma_{06}-\overline{d} = (\frac{\gamma_{33}-a}{\gamma_{42}-b})^2(\gamma_{24}-\overline{f}). \end{cases} \tag{4.10}$$

We notice that as the first sub-case II-1, we have

$$\frac{\gamma_{42}-b}{\gamma_{33}-a} = \sqrt{\frac{\gamma_{33}-e}{\gamma_{33}-a}} \neq 0. \tag{4.11}$$

The moments defined in (4.10) construct a Toeplitz matrix $C(3)$ for which $rank(C(3)-W^*M(2)W) = 2$. Indeed, it suffices to observe that:

- $(C(3)-W^*M(2)W)(Z^3) = \frac{\gamma_{33}-a}{\gamma_{42}-b}(C(3)-W^*M(2)W)(\overline{Z}Z^2)$;
- $(C(3)-W^*M(2)W)(\overline{Z}^3) = \frac{\gamma_{33}-a}{\gamma_{24}-\overline{b}}(C(3)-W^*M(2)W)(\overline{Z}^2Z)$;
- the columns $(C(3)-W^*M(2)W)(Z^3)$ and $(C(3)-W^*M(2)W)(\overline{Z}^3)$ are nonlinear (because (i) can not be verified).

Therefore, in $M(3)$, the column $\overline{Z}Z^2$ is a linear combination of the columns in $\mathfrak{B}(2)\cup\{Z^3\}$. For reason of simplicity, we adopt the notation of the Relation (4.8), that is,

$$\overline{Z}Z^2 = P_{\overline{z}z^2}(Z,\overline{Z}) = \alpha Z^3 + R_{\overline{z}z^2}(Z,\overline{Z}), \tag{4.12}$$

where

$$\alpha = \frac{\gamma_{42}-b}{\gamma_{33}-a} \neq 0,$$

by using (4.11).

We conclude that, in the both cases *II-1* and *II-2*, we have extended the initial data $\gamma^{(5)}$ to $\gamma^{(6)}$ so that the associated moment matrix $M(3)$ has the following columns relation

$$\overline{Z}Z^2 = P_{\overline{z}z^2}(Z,\overline{Z}) = \alpha Z^3 + R_{\overline{z}z^2}(Z,\overline{Z}), \quad \text{with } \alpha \neq 0. \tag{4.13}$$

We also note that since $a \neq e$ we get,

$$|\alpha| = \left|\frac{\gamma_{42}-b}{\gamma_{33}-a}\right| = \frac{\sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}}{\gamma_{33}-a} \neq 1. \tag{4.14}$$

Step 2: $M(3)$ has a flat extension, and therefore has a representing measure. We will build moments $\{\gamma_{ij}\}_{i+j=7,8}$ for which the moment matrix $M(4)$ is a flat extension of $M(3)$.

The relation (4.13) yields that

$$\langle M(3)\overline{Z}Z^2,\overline{Z}^jZ^i\rangle = \langle M(3)P_{\overline{z}z^2}(Z,\overline{Z}),\overline{Z}^jZ^i\rangle, \quad \text{for all } i+j\leqslant 3.$$

By applying (2.4), one obtains

$$\Lambda_{\gamma^{(6)}}(\overline{z}^{i+1}z^{j+2}) = \Lambda_{\gamma^{(6)}}(\overline{z}^iz^jP_{\overline{z}z^2}), \qquad i+j\leqslant 3. \tag{4.15}$$

Since $|\alpha|\neq 1$, we derive that there exists a complex number $\gamma_{43}=\overline{\gamma_{43}}$ such that

$$\gamma_{43} = \Lambda(\overline{z}^3zP_{\overline{z}z^2}), \tag{4.16}$$

that is,

$$\gamma_{43} = \alpha\gamma_{34} + \sum_{i+j\leqslant 2}\alpha_{i,j}\gamma_{i+3,j+1}.$$

It follows, from (4.16) and (4.15), that $\overline{z}z^2 - P_{\overline{z}z^2}$ is a generating polynomial of $\gamma^{(6)}\cup\{\gamma_{34},\gamma_{43}\}$. Since $\begin{pmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33}\end{pmatrix} > 0$, then there exists a (unique) vector, denoted here

$$P_{z^4} = \beta z^3 + R_{z^4} = \beta z^3 + \sum_{\overline{z}^iz^j\in\mathfrak{B}(2)}\beta_{ij}\overline{z}^iz^j$$

the associated polynomial, such that

$$\begin{pmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33}\end{pmatrix}P_{z^4} = ((\gamma_{04},\gamma_{14},\gamma_{05},\gamma_{24},\gamma_{15},\gamma_{06})_{|\mathfrak{B}(2)},\gamma_{34})^T.$$

Therefore the sequence $\gamma^{(6)}\cup\{\gamma_{34},\gamma_{43}\}$ verifies that

$$\gamma_{i,j+4} = \Lambda(\overline{z}^iz^jP_{z^4}), \quad \text{for all } i+j\leqslant 2 \text{ and } (i,j)=(3,0), \tag{4.17}$$

and

$$\gamma_{i+4,j} = \Lambda(\overline{z}^iz^jP_{\overline{z}^4}), \quad \text{for all } i+j\leqslant 2 \text{ and } (i,j)=(0,3). \tag{4.18}$$

Thus $z^4 - P_{z^4}$ is a generating polynomial of $\gamma^{(6)}\cup\{\gamma_{34},\gamma_{43}\}$.

We will build $\gamma^{(8)}\equiv\{\gamma_{ij}\}_{i+j\leqslant 8}$, the extension of $\gamma^{(6)}\cup\{\gamma_{34},\gamma_{43}\}$, by using the generating polynomial P_{z^4} and the initial data $\{\gamma_{ij}\}_{i,j\leqslant 3}$, that is,

$$\gamma_{i,j+4} = \Lambda(\overline{z}^iz^jP_{z^4}) = \beta\gamma_{i,j+3} + \sum_{\overline{z}^lz^k\in\mathfrak{B}(2)}\beta_{lk}\gamma_{i+l,j+k}, \quad i+j\leqslant 4, \tag{4.19}$$

or, equivalently,

$$\gamma_{i+4,j} = \Lambda(\overline{z}^iz^jP_{\overline{z}^4}) = \overline{\beta}\gamma_{i+3,j} + \sum_{\overline{z}^lz^k\in\mathfrak{B}(2)}\overline{\beta_{lk}}\gamma_{i+k,j+l}, \quad i+j\leqslant 4. \tag{4.20}$$

Hence, Lemma 3.3 implies that $\overline{z}z^2 - P_{\overline{z}z^2}$ and $z^4 - P_{z^4}$ are two generating polynomials of $\gamma^{(8)}$. Therefore, in $M(4)$, the columns $Z^4,\overline{Z}Z^3,\overline{Z}^2Z^2,\overline{Z}^3Z,\overline{Z}^4$ are linear combinations of the columns $\{\overline{Z}^iZ^j\}_{i+j\leqslant 3}$ and thus $M(4)$ is a flat extension of $M(3)$. Indeed, it suffices to observe that $P_{z^4}, P_{\bar{z}^4}, P_{\bar{z}z^2}, P_{\bar{z}^2z} \in V \equiv Vect(Z^3, \overline{Z}^3, Z^2, \overline{Z}Z, \overline{Z}^2, \overline{Z}, Z, 1)$, threfore $zP_{\bar{z}z^2}, \bar{z}P_{\bar{z}^2z}, \bar{z}P_{\bar{z}z^2} \in V$. Also one has, for all $i+j \leqslant 4$,

$$\langle M(4)Z^4, \overline{Z}^i Z^j\rangle = \langle M(4)P_{z^4}, \overline{Z}^i Z^j\rangle;$$
$$\langle M(4)\overline{Z}^4, \overline{Z}^i Z^j\rangle = \langle M(4)P_{\bar{z}^4}, \overline{Z}^i Z^j\rangle;$$
$$\langle M(4)\overline{Z}Z^3, \overline{Z}^i Z^j\rangle = \langle M(4)zP_{\bar{z}z^2}, \overline{Z}^i Z^j\rangle;$$
$$\langle M(4)\overline{Z}^2Z^2, \overline{Z}^i Z^j\rangle = \langle M(4)\bar{z}P_{\bar{z}z^2}, Z^i Z^j\rangle;$$
$$\text{and } \langle M(4)\overline{Z}^3Z, \overline{Z}^i Z^j\rangle = \langle M(4)\bar{z}P_{\bar{z}^2z}, \overline{Z}^i Z^j\rangle.$$

This finishes the proof. □

5. Examples

We give in this section four examples illustrating the different solved cases.

5.1. The case $a = e$ and $b = f$

We consider the quintic sequence,

$$\begin{array}{llllll}
\gamma_{00} = 6 \\
\gamma_{01} = 1+i & \gamma_{10} = 1-i \\
\gamma_{20} = -2i & \gamma_{11} = 6 & \gamma_{02} = 2i \\
\gamma_{30} = -2-2i & \gamma_{21} = 2-2i & \gamma_{12} = 2+2i & \gamma_{03} = -2+2i \\
\gamma_{40} = 0 & \gamma_{31} = -4i & \gamma_{22} = 8 & \gamma_{13} = 4i & \gamma_{04} = 0 \\
\gamma_{50} = -4+4i & \gamma_{41} = -4-4i & \gamma_{32} = 4-4i & \gamma_{23} = 4+4i & \gamma_{14} = -4+4i & \gamma_{05} = -4-4i.
\end{array}$$

Then, we have

$$M(2) = \begin{pmatrix} 6 & 1+i & 1-i & 2i & 6 & -2i \\ 1-i & 6 & -2i & 2+2i & 2-2i & -2-2i \\ 1+i & 2i & 6 & -2+2i & 2+2i & 2-2i \\ -2i & 2-2i & -2-2i & 8 & -4i & 0 \\ 6 & 2+2i & 2-2i & 4i & 8 & -4i \\ 2i & -2+2i & 2+2i & 0 & 4i & 8 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -2+2i & 2+2i & 2-2i & -2-2i \\ 4i & 8 & -4i & 0 \\ 0 & 4i & 8 & -4i \\ 4+4i & 4-4i & -4-4i & -4+4i \\ -4+4i & 4+4i & 4-4i & -4-4i \\ -4-4i & -4+4i & 4+4i & 4-4i \end{pmatrix}.$$

The fact that $M(2)$ is positive definite implies,

$$W = (M(2))^{-1}B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \frac{3}{4}+\frac{3i}{4} & \frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & -\frac{3}{4}+\frac{3i}{4} \\ 0 & 0 & 0 & 0 \\ -\frac{3}{4}-\frac{3i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{4}+\frac{i}{4} & \frac{3}{4}-\frac{3i}{4} \end{pmatrix}$$

and

$$W^*M(2)W = \begin{pmatrix} 12 & -8i & -4 & 8i \\ 8i & 12 & -8i & -4 \\ -4 & 8i & 12 & -8i \\ -8i & -4 & 8i & 12 \end{pmatrix}.$$

Since $a = e = 12$ and $b = f = -8i$, according to the main theorem, our sequence provides a moment matrix for a 6 atoms measure. In fact, from W, we can see that $Z^3 + \frac{3(1+i)}{4}(\overline{Z}^2 - Z^2) - \overline{Z}$ and $Z^2\overline{Z} + \frac{(1-i)}{4}(\overline{Z}^2 - Z^2) - Z$ are two characteristic polynomials for this moment sequence. The common roots of the two polynomials are

$$\{\pm 1, \pm i, 0, 1+i\}.$$

Finally we can write that $\mu = \delta_1 + \delta_{-1} + \delta_i + \delta_{-1} + \delta_0 + \delta_{1+i}$.

5.2. The case $a < e$

We consider the quintic sequence,

$$\begin{array}{llllll}
\gamma_{00} = 7 & & & & & \\
\gamma_{01} = 0 & \gamma_{10} = 0 & & & & \\
\gamma_{20} = -4i & \gamma_{11} = 8 & \gamma_{02} = 4i & & & \\
\gamma_{30} = 0 & \gamma_{21} = 0 & \gamma_{12} = 0 & \gamma_{03} = 0 & & \\
\gamma_{40} = -4 & \gamma_{31} = -8i & \gamma_{22} = 12 & \gamma_{13} = 8i & \gamma_{04} = -4 & \\
\gamma_{50} = 0 & \gamma_{41} = 0 & \gamma_{32} = 0 & \gamma_{23} = 4+4i & \gamma_{14} = 0 & \gamma_{05} = 0.
\end{array}$$

Then, we have

$$M(2) = \begin{pmatrix} 7 & 0 & 0 & 4i & 8 & -4i \\ 0 & 8 & -4i & 0 & 0 & 0 \\ 0 & 4i & 8 & 0 & 0 & 0 \\ -4i & 0 & 0 & 12 & -8i & -4 \\ 8 & 0 & 0 & 8i & 12 & -8i \\ 4i & 0 & 0 & -4 & 8i & 12 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 8i & 12 & -8i & -4 \\ -4 & 8i & 12 & -8i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $M(2)$ is positive semi-definite implies, we can take W as follows:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & \frac{4}{3} & -\frac{i}{3} & 0 \\ 0 & \frac{i}{3} & \frac{4}{3} & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

therefore

$$W^*M(2)W = \begin{pmatrix} 8 & -12i & -8 & 4i \\ 12i & \frac{56}{3} & -\frac{44i}{3} & -8 \\ -8 & \frac{44i}{3} & \frac{56}{3} & -12i \\ -4i & -8 & 12i & 8 \end{pmatrix}.$$

Since $a = 8 < e = \frac{56}{3}$, according to the main theorem, our sequence is a moment sequence associated with a 7 atoms measure. In fact, examining (4.6) we can take $\gamma_{33} = 20$, then $\gamma_{42} = -16i$. Therefore $\gamma_{06} = -16i$, then $\bar{Z}Z^2 = \frac{i}{3}Z^3 + Z + \frac{i}{3}\bar{Z}$ is a characteristic polynomial and with $\gamma_{43} = 0$ we derive another characteristic polynomial given by $Z^4 = \frac{3i}{2}Z^2 + \bar{Z}Z - \frac{3i}{2}\bar{Z}^2$. The common roots of the two polynomials are

$$\{\pm 1, \pm i, 0, 1+i, -1-i\}.$$

Finally we get

$$\mu = \delta_0 + \delta_1 + \delta_{-1} + \delta_i + \delta_{-i} + \delta_{1+i} + \delta_{-1-i}.$$

5.3. The case $a > e$ and $a - e < 2|b - f|$

We consider the quintic sequence,

$\gamma_{00} = 7, \gamma_{01} = 1+5i, \gamma_{10} = 1-5i,$

$\gamma_{20} = -64-60i, \gamma_{11} = 230, \gamma_{02} = -64+60i,$

$\gamma_{30} = 277-161i, \gamma_{21} = -203+257i, \gamma_{12} = -203-257i, \gamma_{03} = 277+161i,$

$\gamma_{40} = 3722+4320i, \gamma_{31} = -4816-4200i, \gamma_{22} = 10778,$
$\gamma_{13} = -4816+4200i, \gamma_{04} = 3722-4320i,$

$\gamma_{50} = -59219-29695i, \gamma_{41} = 31021-11585i, \gamma_{32} = -16979+24353i,$
$\gamma_{23} = -16979-24353i, \gamma_{14} = 31021+11585i$ and $\gamma_{05} = -59219+29695i.$
Then $M(2) =$

$$\begin{pmatrix} 7 & 1+5i & 1-5i & -64-60i & 230 & -64+60i \\ 1-5i & 230 & -64+60i & -203+257i & -203-257i & 277+161i \\ 1+5i & -64-60i & 230 & 277-161i & -203+257i & -203-257i \\ -64+60i & -203-257i & 277+161i & 10778 & -4816+4200i & 3722-4320i \\ 230 & -203+257i & -203-257i & -4816-4200i & 10778 & -4816+4200i \\ -64-60i & 277-161i & -203+257i & 3722+4320i & -4816-4200i & 10778 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 277-161i & -203+257i & -203-257i & 277+161i \\ -4816-4200i & 10778 & -4816+4200i & 3722-4320i \\ 3722+4320i & -4816-4200i & 10778 & -4816+4200i \\ -16979+24353i & -16979-24353i & 31021+11585i & -59219+29695i \\ 31021-11585i & -16979+24353i & -16979-24353i & 31021+11585i \\ -59219-29695i & 31021-11585i & -16979+24353i & -16979-24353i \end{pmatrix}.$$

The fact that $M(2)$ is positive definite allows to calculate W and $W^*M(2)W$. Finally, using the same process in above, one gets the following expression of μ

$$\mu=\delta_2+\delta_{-6}+\delta_{4i}+\delta_{-7i}+\delta_{3+3i}+\delta_{5-3i}+\delta_{-3+8i}.$$

5.4. The remaining case $a=e$ and $b\neq f$

We consider the quintic sequence,

$$\begin{array}{lllll}
\gamma_{00}=7 & & & & \\
\gamma_{01}=9+9i & \gamma_{10}=9-9i & & & \\
\gamma_{20}=22i & \gamma_{11}=34 & \gamma_{02}=-22i & & \\
\gamma_{30}=-18+18i & \gamma_{21}=58+58i & \gamma_{12}=58-58i & \gamma_{03}=-18-18i & \\
\gamma_{40}=40 & \gamma_{31}=164i & \gamma_{22}=256 & \gamma_{13}=-164i & \gamma_{04}=40 \\
\gamma_{50}=304(1+i) & \gamma_{41}=128(i-1) & \gamma_{32}=480(i+1) & \gamma_{23}=480(1-i) & \gamma_{14}=-128(1+i) \\
\gamma_{05}=304(1-i). & & & &
\end{array}$$

Then, we obtain

$$M(2)=\begin{pmatrix}
7 & 9+9i & 9-9i & 22i & 34 & -22i \\
9-9i & 34 & -22i & 58+58i & 58-58i & -18-18i \\
9+9i & 22i & 34 & -18+18i & 58+58i & 58-58i \\
-22i & 58-58i & -18-18i & 256 & -164i & 40 \\
34 & 58+58i & 58-58i & 164i & 256 & -164i \\
22i & -18+18i & 58+58i & 40 & 164i & 256
\end{pmatrix}$$

and

$$B=\begin{pmatrix}
-18+18i & 58+58i & 58-58i & -18-18i \\
164i & 256 & -164i & 40 \\
40 & 164i & 256 & -164i \\
480+480i & 480-480i & -128-128i & 304-304i \\
-128+128i & 480+480i & 480-480i & -128-128i \\
304+304i & -128+128i & 480+480i & 480-480i
\end{pmatrix}.$$

Since $M(2)$ is not invertible. We can take W as follows,

$$W=\begin{pmatrix}
-\frac{1041}{319}+\frac{1041i}{319} & \frac{931}{319}+\frac{931i}{319} & -\frac{1041}{319}+\frac{1041i}{319} & \frac{3483}{319}+\frac{3483i}{319} \\
-\frac{1}{319}(2028i) & -\frac{2061}{319} & -\frac{1}{319}(752i) & -\frac{4613}{319} \\
-\frac{89}{319} & -\frac{1}{319}(1220i) & -\frac{89}{319} & -\frac{1}{319}(2496i) \\
\frac{39}{11}+\frac{39i}{11} & \frac{17}{11}-\frac{17i}{11} & \frac{17}{11}+\frac{17i}{11} & \frac{39}{11}-\frac{39i}{11} \\
\frac{24}{29}-\frac{24i}{29} & \frac{82}{29}+\frac{82i}{29} & \frac{82}{29}-\frac{82i}{29} & \frac{24}{29}+\frac{24i}{29} \\
0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$W^*M(2)W=\begin{pmatrix}
\frac{719500}{319} & -\frac{1}{319}(448504i) & \frac{152956}{319} & -\frac{1}{319}(622040i) \\
\frac{448504i}{319} & \frac{719500}{319} & -\frac{1}{319}(449800i) & \frac{152956}{319} \\
\frac{152956}{319} & \frac{449800i}{319} & \frac{719500}{319} & -\frac{1}{319}(448504i) \\
\frac{622040i}{319} & \frac{152956}{319} & \frac{448504i}{319} & \frac{719500}{319}
\end{pmatrix}.$$

In this case we get $a=e=\frac{719500}{319}$ and $b=-\frac{448504i}{319}\neq f=\frac{449800i}{319}$. This is the remaining case which is not covered by the main theorem. We notice that this sequence is a moment sequence associated with a 7 atoms representing measure:

$$\mu=\delta_1+\delta_i+\delta_{1+i}+\delta_{1+2i}+\delta_{1+3i}+\delta_{2+i}+\delta_{3+i}.$$

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