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ABSTRACT KOROVKIN THEORY FOR DOUBLE SEQUENCES VIA POWER SERIES METHOD IN MODULAR SPACES

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Abstract. In the present paper, we obtain an abstract version of the Korovkin type approximation theorems for double sequences of positive linear operators on modular spaces in the sense of power series method. We present an example that satisfies our theorem but not satisfies the classical one and also, we study an extension to non-positive operators.

1. Introduction

The classical version of Korovkin theorem deals with the approximation only using test functions that provides the approximation in whole space [10]. This theorem has been extended with the use of summability methods since they make convergent a non-convergent sequence. The well known summability methods are power series methods which includes Abel and Borel methods. The first usage of Korovkin theorem in modular spaces was by Bardaro and Mantellini [4]. Then, this theorem was studied by various authors on modular spaces in papers [6], [8] and [9]. In this paper, we study an abstract Korovkin theorems by means of power series method for double sequences on modular spaces. We present an example that satisfies our theorem but not satisfies the classical one and also, we study an extension to non-positive operators.

We begin the following notations and definitions of power series method for double sequences used in this paper.

Let (p_{mn}) be a double sequence of nonnegative numbers with $p_{00} > 0$ and such that the following power series

$$p(t,s) := \sum_{m,n=0}^{\infty} p_{mn} t^m s^n$$

has radius of convergence R with $R \in (0,\infty]$ and $t,s \in (0,R)$. If, for all $t,s \in (0,R)$,

$$\lim_{t,s\to R^-} \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n x_{mn} = A,$$

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then we say that the double sequence $x = (x_{mn})$ is convergent to A in the sense of power series method [7]. The power series method for double sequences is regular if and only if

$$\lim_{t,s\to R^-} \frac{\sum_{m=0}^{\infty} p_{m\upsilon} t^m}{p(t,s)} = 0 \text{ and } \lim_{t,s\to R^-} \frac{\sum_{n=0}^{\infty} p_{\mu n} s^n}{p(t,s)} = 0, \text{ for any } \mu, \upsilon, \tag{1}$$

hold [7]. Throughout the paper we assume that power series method is regular.

REMARK 1. Note that for $R = 1$, if $p_{mn} = 1$ and $p_{mn} = \frac{1}{(m+1)(n+1)}$, the power series methods are Abel summability method and logarithmic summability method, respectively. For $R = \infty$ and $p_{mn} = \frac{1}{m!n!}$, the power series method coincides with Borel summability method.

Now, we recall the concept of Pringsheim convergence for double sequences.

A double sequence $x = (x_{mn})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{mn} - L| < \varepsilon$ whenever $m,n > N$, where L is called the Pringsheim limit of x and denoted by $P - \lim_{m,n} x_{mn} = L$ (see [17]). We shall call such an x, briefly, "P-convergent". A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \leqslant M$, for all $(m,n) \in \mathbb{N}^2$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Now let us remind that some well known notations and properties of modular spaces.

Assume that Z be a locally compact Hausdorff topological space given with a uniform structure $\mathscr{U} \subset 2^{Z\times Z}$ that generates the topology of Z (see, [12]). Let $\mathscr{B}$ be the $\sigma-$algebra of all Borel subsets of Z and $\mu : \mathscr{B} \to \mathbb{R}$ is a positive $\sigma-$finite regular measure. Let $L^0(Z)$ be the space of all real valued $\mu-$measurable functions on Z with identification up to sets of measure μ zero, $C(Z)$ be the space of all continuous real valued functions on Z, $C_b(Z)$ be the space of all continuous real valued and bounded functions on Z and $C_c(Z)$ be the subspace of $C_b(Z)$ of all functions with compact support on Z. In this case, we say that a functional $\rho : L^0(Z) \to [0,\infty]$ is a modular on $L^0(Z)$ if it satisfies, (i) $\rho(f) = 0$ if and only if $f = 0$ $\mu-$almost everywhere on Z, (ii) $\rho(-f) = \rho(f)$, for every $f \in L^0(Z)$, (iii) $\rho(\alpha f + \beta g) \leqslant \rho(f) + \rho(g)$, for every $f,g \in L^0(Z)$ and for any $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$.

A modular ρ is $N-$quasi convex if there exists a constant $N \geqslant 1$ such that the inequality

$$\rho(\alpha f + \beta g) \leqslant N\alpha\rho(Nf) + N\beta\rho(Ng)$$

holds for every $f,g \in L^0(Z)$, $\alpha,\beta \geqslant 0$ with $\alpha + \beta = 1$. Note that if $N = 1$, then ρ is called convex. Furthermore, a modular ρ is $N-$quasi semiconvex if there exists a constant $N \geqslant 1$ such that $\rho(\alpha f) \leqslant N\alpha\rho(Nf)$ holds for every $f \in L^0(Z)$ and $\alpha \in (0,1]$.

The modular space $L^\rho(Z)$ generated by the modular ρ, is given by

$$L^\rho(Z) := \left\{ f \in L^0(Z) : \lim_{\lambda\to 0^+} \rho(\lambda f) = 0 \right\}$$

and $E^{\rho}(Z) := \{f \in L^{\rho}(Z) : \rho(\lambda f) < \infty, \text{ for all } \lambda > 0\}$ is the space of the finite elements of $L^{\rho}(Z)$. Also, we note that if ρ is $N-$quasi semiconvex, then the space

$$\left\{f \in L^{0}(Z) : \rho(\lambda f) < \infty, \text{ for some } \lambda > 0\right\}$$

coincides with $L^{\rho}(Z)$.

Now we recall the modular and strong convergence for double sequences.

DEFINITION 1. Let (f_{mn}) be a double function sequence whose terms belong to $L^{\rho}(Z)$. Then, (f_{mn}) is said to be *modularly convergent* to a function $f \in L^{\rho}(Z)$ iff

$$P-\lim_{m,n} \rho(\lambda_0 (f_{mn} - f)) = 0, \text{ for some } \lambda_0 > 0.$$

Also, (f_{mn}) is $F-$*norm convergent* (or, *strongly convergent*) to f iff

$$P-\lim_{m,n} \rho(\lambda (f_{mn} - f)) = 0, \text{ for every } \lambda > 0.$$

The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leqslant M\rho(f)$, for every $f \in L^{0}(Z)$, see [14].

Now we also give convergences in the sense of power series method in modular spaces:

DEFINITION 2. Let (f_{mn}) be a double function sequence whose terms belong to $L^{\rho}(Z)$. Then, (f_{mn}) is said to be *modularly convergent* to a function $f \in L^{\rho}(Z)$ in the sense of power series method iff

$$\lim_{t,s\to R^{-}} \rho\left(\lambda_0 \left(\frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n f_{mn} - f\right)\right) = 0, \text{ for some } \lambda_0 > 0.$$

Also, (f_{mn}) is $F-$*norm convergent* (or, *strongly convergent*) to f in the sense of power series method iff

$$\lim_{t,s\to R^{-}} \rho\left(\lambda \left(\frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n f_{mn} - f\right)\right) = 0, \text{ for every } \lambda > 0.$$

Recall the following notions that we use in this paper.

A modular ρ is said to be monotone if $\rho(f) \leqslant \rho(g)$, for $|f| \leqslant |g|$. A modular ρ is finite if $\chi_A \in L^{\rho}(Z)$ whenever $A \in \mathscr{B}$ with $\mu(A) < \infty$. A modular ρ is strongly finite if $\chi_A \in E^{\rho}(Z)$, for all $A \in \mathscr{B}$ such that $\mu(A) < \infty$ and a modular ρ is said to be absolutely continuous if there exists an $\alpha > 0$ such that: for every $f \in L^{0}(Z)$ with $\rho(f) < \infty$, the following conditions hold:

- for each $\varepsilon > 0$ there exists a set $A \in \mathscr{B}$ such that $\mu(A) < \infty$ and $\rho\left(\alpha f \chi_{Z\setminus A}\right) \leqslant \varepsilon$,
- for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho(\alpha f \chi_B) \leqslant \varepsilon$, for every $B \in \mathscr{B}$ with $\mu(B) < \delta$.

If a modular ρ is monotone and finite, then $C(Z) \subset L^{\rho}(Z)$. If ρ is monotone and strongly finite, then $C(Z) \subset E^{\rho}(Z)$. Also, if ρ is monotone, strongly finite and absolutely continuous, $\overline{C_c(Z)} = L^{\rho}(Z)$ with respect to the modular convergence in the ordinary sense (for details and properties see also [11, 13, 15]).

2. Korovkin type theorems via power series method

In this section we prove some Korovkin type theorems with respect to an abstract finite set of test functions $e_0, e_1, \ldots, e_k$ in the sense of power series method. First, we recall a Korovkin type theorem via modular convergence for double sequences:

Set $e_0(v) \equiv 1$, for all $v \in Z$, let e_r, $r = 1, 2, \ldots, k$ and a_r, $r = 0, 1, 2, \ldots, k$, be functions in $C_b(Z)$. Put

$$P_u(v) = \sum_{r=0}^{k} a_r(u) e_r(v), \; u, v \in Z, \tag{2}$$

and suppose that $P_u(v)$, $u, v \in Z$, satisfies the following properties:

$(P1)$ $P_u(u) = 0$, for all $u \in Z$;

$(P2)$ for every neighbourhood $U \in \mathscr{U}$ there is a positive real number η with $P_u(v) \geqslant \eta$ whenever $u, v \in Z$, $(u, v) \notin U$ (see [3]).

THEOREM 1. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that e_r and a_r, $r = 0, 1, 2, \ldots, k$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators from D into $L^0(Z)$ with $C_b(Z) \subset D \subset L^0(Z)$ such that the inequality*

$$P - \limsup_{m,n} \rho\left(\lambda\left(T_{mn} h\right)\right) \leqslant R\rho(\lambda h)$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. If $(T_{mn} e_r)$ is strongly convergent to e_r, $r = 0, 1, 2, \ldots, k$, in $L^{\rho}(Z)$ then $(T_{mn} f)$ is modularly convergent to f in $L^{\rho}(Z)$ and f is any function belonging to $D \cap L^{\rho}(Z)$ with $f - C_b(Z) \subset X_{\mathbb{T}}$.

Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators from D into $L^0(Z)$ with $C_b(Z) \subset D \subset L^0(Z)$. Let ρ be monotone and finite modular on $L^0(Z)$. Assume further that the double sequence $\mathbb{T}$, together with modular ρ, satisfies the following property:

there exists a subset $X_{\mathbb{T}} \subset D \cap L^{\rho}(Z)$ with $C_b(Z) \subset X_{\mathbb{T}}$ the inequality

$$\limsup_{t,s \to R^-} \rho\left(\lambda \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n T_{mn} h\right) \leqslant R\rho(\lambda h) \tag{3}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R.

Throughout the paper S_{ts} is defined by

$$S_{ts}f := S_{ts}(f;.) := \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n T_{mn}(f;.),$$

for each $t,s \in (0,R)$.

In order that to obtain our main theorem, we first give the following result.

THEOREM 2. *Let ρ be a monotone, strongly finite and $N-$quasi semiconvex modular. Suppose that e_r and a_r, $r = 0,1,2,...,k$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators from D into $L^0(Z)$. If*

$$\lim_{t,s\to R^-} \rho\left(\lambda_0\left(S_{ts}e_r - e_r\right)\right) = 0, \;\; \textit{for some } \lambda_0 > 0, \tag{4}$$

$r = 0,1,2,...,k$, in $L^\rho(Z)$, then for every $f \in C_c(Z)$

$$\lim_{t,s\to R^-} \rho\left(\gamma(S_{ts}f - f)\right) = 0, \;\; \textit{for some } \gamma > 0 \tag{5}$$

in $L^\rho(Z)$. If

$$\lim_{t,s\to R^-} \rho\left(\lambda\left(S_{ts}e_r - e_r\right)\right) = 0, \;\; \textit{for every } \lambda > 0,$$

$r = 0,1,2,...,k$, in $L^\rho(Z)$, then for every $f \in C_c(Z)$

$$\lim_{t,s\to R^-} \rho\left(\lambda\left(S_{ts}f - f\right)\right) = 0, \textit{for every } \lambda > 0$$

in $L^\rho(Z)$.

Proof. We first claim that, for every $f \in C_c(Z)$,

$$\lim_{t,s\to R^-} \rho\left(\gamma(S_{ts}f - f)\right) = 0, \;\; \text{for some } \gamma > 0. \tag{6}$$

To see this assume that $f \in C_c(Z)$. Then, since Z is endowed with the uniformity $\mathscr{U}$, f is uniformly continuous and bounded on Z. By the uniform continuity of f, choose $\varepsilon \in (0,1]$, there exists a set $U \in \mathscr{U}$ such that $|f(u) - f(v)| \leqslant \varepsilon$ whenever $u,v \in Z$, $(u,v) \in U$.

For all $u,v \in Z$ let $P_u(v)$ be as in (2), and $\eta > 0$ satisfy condition $(P2)$. Then for $u,v \in Z$, $(u,v) \notin U$, we have $|f(u) - f(v)| \leqslant \frac{2M}{\eta} P_u(v)$ where $M := \sup_{v\in Z} |f(v)|$. Therefore, in any case we get $|f(u) - f(v)| \leqslant \varepsilon + \frac{2M}{\eta} P_u(v)$, for all $u,v \in Z$, namely,

$$-\varepsilon - \frac{2M}{\eta} P_u(v) \leqslant f(u) - f(v) \leqslant \varepsilon + \frac{2M}{\eta} P_u(v). \tag{7}$$

Since T_{mn} is linear and positive, by applying T_{mn} to (7), for every $m,n \in \mathbb{N}$, we have

$$-\varepsilon S_{ts}(e_0;u) - \frac{2M}{\eta} S_{ts}(P_u;u) \leqslant f(u) S_{ts}(e_0;u) - S_{ts}(f;u) \leqslant \varepsilon S_{ts}(e_0;u) + \frac{2M}{\eta} S_{ts}(P_u;u).$$

Hence

$$\begin{aligned}\left|S_{ts}(f;u)-f(u)\right| &\leqslant \left|S_{ts}(f;u)-f(u)S_{ts}(e_0;u)\right|+\left|f(u)\right|\left|S_{ts}(e_0;u)-e_0(u)\right| \\ &\leqslant \varepsilon S_{ts}(e_0;u)+\frac{2M}{\eta}S_{ts}(P_u;u)+M\left|S_{ts}(e_0;u)-e_0(u)\right| \\ &\leqslant \varepsilon+(\varepsilon+M)\left|S_{ts}(e_0;u)-e_0(u)\right|+\frac{2M}{\eta}\sum_{r=0}^{k}a_r(u)\left|S_{ts}(e_r;u)-e_r(u)\right|.\end{aligned}$$

Let $\gamma>0$. Now for each $r=0,1,2,...,k$ and $u\in Z$, choose $M_0>0$. The last inequality gives that

$$\gamma\left|S_{ts}(f;u)-f(u)\right| \leqslant \gamma\varepsilon+K\gamma\sum_{r=0}^{k}\left|S_{ts}(e_r;u)-e_r(u)\right|,$$

where $K:=\varepsilon+M+\frac{2M}{\eta}M_0$. Now, applying the modular ρ to both sides of the above inequality, since ρ is monotone, we get

$$\rho\left(\gamma\left(S_{ts}f-f\right)\right)\leqslant\rho\left(\gamma\varepsilon+K\gamma\sum_{r=0}^{k}\left|S_{ts}e_r-e_r\right|\right).$$

Thus, we can see that

$$\rho\left(\gamma\left(S_{ts}f-f\right)\right)\leqslant\rho\left((k+2)\gamma\varepsilon\right)+\sum_{r=0}^{k}\rho\left((k+2)K\gamma\left(S_{ts}e_r-e_r\right)\right).$$

Let $\lambda_0>0$ be as in the hypothesis (4), such $\lambda_0>0$, by hypothesis, does exist. Let $\gamma>0$ be with $(k+2)K\gamma\leqslant\lambda_0$ and since ρ is $N-$quasi semiconvex and strongly finite, we have

$$\rho\left(\gamma\left(S_{ts}f-f\right)\right)\leqslant N\varepsilon\rho\left((k+2)\gamma N\right)+\sum_{r=0}^{k}\rho\left(\lambda_0\left(S_{ts}e_r-e_r\right)\right) \tag{8}$$

without loss of generality, where $\varepsilon\in(0,1]$. By taking limit superior as $t,s\to R^-$ in the both sides and by using the hypothesis (4), we get

$$\lim_{t,s\to R^-}\rho\left(\gamma\left(S_{ts}f-f\right)\right)=0,$$

which proves our claim (6).

The last part of theorem can be proved similarly to the first one. □

Now, let us give our main theorem of this paper.

THEOREM 3. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that e_r and a_r, $r=0,1,2,...,k$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators satisfying (3). If*

$$\lim_{t,s\to R^-}\rho\left(\lambda\left(S_{ts}e_r-e_r\right)\right)=0,\ \textit{for every}\ \lambda>0,$$

$r=0,1,2,...,k$ in $L^{\rho}(Z)$, then for every $f\in D\cap L^{\rho}(Z)$ with $f-C_b(Z)\subset X_{\mathbb{T}}$,

$$\lim_{t,s\to R^-}\rho\left(\lambda_0\left(S_{ts}f-f\right)\right)=0,\ \text{for some } \lambda_0>0$$

in $L^{\rho}(Z)$ and D, $X_{\mathbb{T}}$ are as before.

Proof. Let $f\in D\cap L^{\rho}(Z)$ with $f-C_b(Z)\subset X_{\mathbb{T}}$. It is known from [5, 13] that there exists a sequence $(g_{kj})\subset C_c(Z)$ such that $\rho(3\lambda_0^* f)<\infty$ and $P-\lim\limits_{k,j}\rho\left(3\lambda_0^*\left(g_{kj}-f\right)\right)=0$, for some $\lambda_0^*>0$. This means that, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ with

$$\rho\left(3\lambda_0^*\left(g_{kj}-f\right)\right)<\varepsilon,\ \text{for every } k,j\geqslant k_0. \tag{9}$$

For all $m,n\in\mathbb{N}$, by linearity and positivity of the operators T_{mn}, we have

$$\begin{aligned}\lambda_0^*\left|S_{ts}(f;u)-f(u)\right|\leqslant{}&\lambda_0^*\left|S_{ts}\left(f-g_{k_0k_0};u\right)\right|+\lambda_0^*\left|S_{ts}\left(g_{k_0k_0};u\right)-g_{k_0k_0}(u)\right|\\&+\lambda_0^*\left|g_{k_0k_0}(u)-f(u)\right|\end{aligned}$$

holds for every $u\in Z$. Now, applying modular ρ in the last inequality and using the monotonicity of ρ, we get

$$\begin{aligned}\rho\left(\lambda_0^*\left(S_{ts}f-f\right)\right)\leqslant{}&\rho\left(3\lambda_0^*\left(S_{ts}\left(f-g_{k_0k_0}\right)\right)\right)+\rho\left(3\lambda_0^*\left(S_{ts}g_{k_0k_0}-g_{k_0k_0}\right)\right)\\&+\rho\left(3\lambda_0^*\left(g_{k_0k_0}-f\right)\right).\end{aligned}\tag{10}$$

Then using the (9) in (10), we have

$$\rho\left(\lambda_0^*\left(S_{ts}f-f\right)\right)\leqslant\varepsilon+\rho\left(3\lambda_0^*\left(S_{ts}\left(f-g_{k_0k_0}\right)\right)\right)+\rho\left(3\lambda_0^*\left(S_{ts}g_{k_0k_0}-g_{k_0k_0}\right)\right).$$

By property (3) and also using the facts that $g_{k_0k_0}\in C_c(Z)$ and $f-g_{k_0k_0}\in X_{\mathbb{T}}$, we obtain

$$\begin{aligned}\limsup_{t,s\to R^-}\rho\left(\lambda_0^*\left(S_{ts}f-f\right)\right)&\leqslant\varepsilon+R\rho\left(3\lambda_0^*\left(f-g_{k_0k_0}\right)\right)+\limsup_{t,s\to R^-}\rho\left(3\lambda_0^*\left(S_{ts}g_{k_0k_0}-g_{k_0k_0}\right)\right)\\&\leqslant\varepsilon(1+R)+\limsup_{t,s\to R^-}\rho\left(3\lambda_0^*\left(S_{ts}g_{k_0k_0}-g_{k_0k_0}\right)\right)\end{aligned}$$

also, resulting from previous theorem,

$$0=\lim_{t,s\to R^-}\rho\left(3\lambda_0^*\left(S_{ts}g_{k_0k_0}-g_{k_0k_0}\right)\right)=\limsup_{t,s\to R^-}\rho\left(3\lambda_0^*\left(S_{ts}g_{k_0k_0}-g_{k_0k_0}\right)\right),$$

which gives

$$0\leqslant\limsup_{t,s\to R^-}\rho\left(\lambda_0^*\left(S_{ts}f-f\right)\right)\leqslant\varepsilon(1+R).$$

From arbitrariness of $\varepsilon>0$, it follows that $\limsup\limits_{t,s\to R^-}\rho\left(\lambda_0^*\left(S_{ts}f-f\right)\right)=0$. Furthermore,

$$\lim_{t,s\to R^-}\rho\left(\lambda_0^*\left(S_{ts}f-f\right)\right)=0,$$

and this completes the proof. □

REMARK 2. Note that, in Theorem 3, in general it is not possible to obtain strong convergence in the sense of power series method unless the modular ρ satisfies the Δ_2-condition.

Now, we give an application showing that in general, our results are stronger than classical ones.

EXAMPLE 1. Let us consider $Z=[0,1]^2\subset\mathbb{R}^2$ and let $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function with φ is convex, $\varphi(0)=0$, $\varphi(x)>0$, *for any* $x>0$, *and* $\lim_{x\to\infty}\varphi(x)=\infty$. Then, the functional ρ^{φ} defined by

$$\rho^{\varphi}(f):=\int_0^1\int_0^1\varphi(|f(x,y)|)\,dxdy,\quad \text{for } f\in L^0(Z),$$

is a convex modular on $L^0(Z)$ and

$$L^{\varphi}(Z):=\left\{f\in L^0(Z):\rho^{\varphi}(\lambda f)<+\infty,\quad \text{for some } \lambda>0\right\}$$

is the Orlics space generated by φ.

For every $(x,y)\in Z$, let $e_0(x,y)=a_3(x,y)=1$, $e_1(x,y)=x$, $e_2(x,y)=y$, $e_3(x,y)=a_0(x,y)=x^2+y^2$, $a_1(x,y)=-2x$, $a_2(x,y)=-2y$. For every $m,n\in\mathbb{N}$, $u_1,u_2\in[0,1]$, let $K_{mn}(u_1,u_2)=(m+1)(n+1)u_1^m u_2^n$ and for $f\in C(Z)$ and $x,y\in[0,1]$ set

$$M_{mn}(f;x,y)=\int_0^1\int_0^1 K_{mn}(u_1,u_2)f(u_1x,u_2y)\,du_1du_2.$$

Then we get

$$\int_0^1\int_0^1 K_{mn}(u_1,u_2)\,du_1du_2=(m+1)\left(\int_0^1 u_1^m du_1\right)(n+1)\left(\int_0^1 u_2^n du_2\right)=1,$$

and hence, $M_{mn}(e_0;x,y)=e_0(x,y)=1$. Also, we know from [2] that

$$|M_{mn}(e_1;x,y)-e_1(x,y)|\leqslant\frac{1}{m+2},\quad |M_{mn}(e_2;x,y)-e_2(x,y)|\leqslant\frac{1}{n+2},$$
$$\left|M_{mn}\left(e_1^2;x,y\right)-e_1^2(x,y)\right|\leqslant\frac{2}{m+3},\quad \left|M_{mn}\left(e_2^2;x,y\right)-e_2^2(x,y)\right|\leqslant\frac{2}{n+3},$$

and for each $m,n\geqslant 2$, $f\in L^{\varphi}(Z)$ we get $\rho^{\varphi}(M_{mn}f)\leqslant 32\rho^{\varphi}(f)$. Moreover, (M_{mn}) satisfies the condition (14) in [16] with $X_{\mathbb{M}}=L^{\varphi}(Z)$ and $(M_{mn}f)$ is modulary convergent to $f\in L^{\varphi}(Z)$. Using the operators $\mathbb{M}=(M_{mn})$, we define the double sequence of positive linear operators $\mathbb{T}=(T_{mn})$ on $L^{\varphi}(Z)$ as follows:

$$T_{mn}(f;x,y)=(1+s_{mn})M_{mn}(f;x,y),\ \text{for } f\in L^{\varphi}(Z),$$

$x,y\in[0,1]$ and $m,n\in\mathbb{N}$, where $s_{mn}=1$, m,n are squares and 0 otherwise. Also let $R=1$, $p(t,s)=\frac{1}{(1-t)(1-s)}$ and for $m,n\geqslant 0$, $p_{mn}=1$. As it is well known, in this case the power series method coincides with Abel method. Then, (s_{mn}) is convergent to 0 in the sense of power series method. If $\varphi(x)=x^p$, for $1\leqslant p<\infty$, $x\geqslant 0$ then $L^{\varphi}(Z)=L_p(Z)$ and we have for any function $f\in L^{\varphi}(Z)$, $\rho^{\varphi}(f)=\|f\|_p^p$.

Then, for every $L_1(Z)$, $\lambda>0$ that

$$\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}h)=\left\|\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}h\right\|_p^p$$
$$\leqslant 2^p\left\|\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nM_{mn}h\right\|_p^p\leqslant 2^p\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^n\|\lambda M_{mn}h\|_p^p$$
$$\leqslant 2^{p+5}\rho^{\varphi}(\lambda h).$$

Now, observe that $T_{mn}(e_0;x,y)-e_0(x,y)=s_{mn}$, hence, we can see, for any $\lambda>0$, that

$$\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_0-e_0)=\left\|\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_0-e_0\right\|_p^p$$
$$=\left\|\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^ns_{mn}\right\|_p^p,$$

then, since (s_{mn}) is convergent to 0 in the sense of power series method, we get

$$\lim_{t,s\to R^-}\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_0-e_0)=0.$$

Also, we have

$$\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_1-e_1)=\left\|\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_1-e_1\right\|_p^p$$
$$\leqslant 2^p\left\|\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nM_{mn}e_1-e_1\right\|_p^p\leqslant 2^p\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^n\|\lambda(M_{mn}e_1-e_1)\|_p^p,$$

from above inequality, since $(M_{mn}f)$ is modulary convergent to $f\in L^{\varphi}(Z)$, we have

$$\lim_{t,s\to R^-}\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_1-e_1)=0.$$

Similarly, we get

$$\lim_{t,s\to R^-}\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_2-e_2)=0,$$
$$\lim_{t,s\to R^-}\rho^{\varphi}(\lambda\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^ms^nT_{mn}e_3-e_3)=0.$$

So, our new operator $\mathbb{T} = (T_{mn})$ satisfies all conditions of Theorem 3 and therefore we obtain

$$\lim_{t,s\to R^-} \rho^{\varphi}(\lambda_0 \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n T_{mn} f - f) = 0,$$

for some $\lambda_0 > 0$, for any $f \in L_p(Z)$. However, $(T_{mn} e_0)$ is not modularly convergent, thus (T_{mn}) does not fulfil the Theorem 1.

3. An extension to non-positive operators

In [1, 2, 3], they relax the positivity condition of linear operators in the Korovkin theorems. Following this approach, we give some positive answers also for modular convergence in the sense of power series method and prove a Korovkin type approximation theorem.

Let I be a bounded interval of $\mathbb{R}$, $C^2(I)$ (resp. $C_b^2(I)$) be the space of all functions defined on I, (resp. bounded and) continuous together with their first and second derivatives, $C_+ := \{f \in C_b^2(I) : f \geqslant 0\}$, $C_+^2 := \{f \in C_b^2(I) : f'' \geqslant 0\}$.

Let e_r, $r = 1,2,...,k$ and a_r, $r = 0,1,2,...,k$, be functions in $C_b^2(I)$, $P_u(v)$, $u,v \in I$, be as in (7), and suppose that $P_u(v)$ satisfies the properties $(P1)$, $(P2)$ and

$(P3)$ there is a positive real constant S_0 such that $P_u''(v) \geqslant S_0$, for all $u,v \in I$ (here the second derivative is intended with respect to v).

Now we prove the following Korovkin type approximation theorem for not necessarily positive linear operators.

THEOREM 4. *Let ρ and σ_r be as in Theorem 2 and e_r, a_r, $r = 0,1,2,...,k$ and $P_u(v)$, $u,v \in I$, satisfies the properties $(P1)$, $(P2)$ and $(P3)$. Assume that $\mathbb{T} = (T_{mn})$ is a double sequence of linear operators and $T_{mn}(C_+ \cap C_+^2) \subset C_+$, for all $m,n \in \mathbb{N}$. If $T_{mn} e_r$ is modularly convergent to e_r in the sense of power series method in $L^\rho(I)$, for each $r = 0,1,2,...,k$, then $T_{mn} f$ is modularly convergent to f in the sense of power series method in $L^\rho(I)$, for every $f \in C_b^2(I)$.*

If $T_{mn} e_r$ is strongly convergent to e_r in the sense of power series method, $r = 0,1,2,...,k$, in $L^\rho(I)$, then $T_{mn} f$ is strongly convergent to f in the sense of power series method in $L^\rho(I)$, for every $f \in C_b^2(I)$.

Furthermore, if ρ is absolutely continuous, $\mathbb{T}$ satisfies the property (3) and $T_{mn} e_r$ is strongly convergent to e_r in the sense of power series method, $r = 0,1,2,...,k$, in $L^\rho(I)$, then $T_{mn} f$ is modularly convergent to f in the sense of power series method in $L^\rho(I)$, for every $f \in D \cap L^\rho(I)$ with $f - C_b(I) \subset X_{\mathbb{T}}$.

Proof. Let $f \in C_b^2(I)$. Since f is uniformly continuous and bounded on I, given $\varepsilon > 0$ with $0 < \varepsilon \leqslant 1$, there exists a $\delta > 0$ such that $|f(u) - f(v)| \leqslant \varepsilon$, for all $u,v \in I$, $|u - v| \leqslant \delta$. Let $P_u(v)$, $u,v \in I$, be as in (2) and let $\eta > 0$ be associated with δ, satisfying $(P2)$. As in Theorem 2, for every $\beta \geqslant 1$ and $u,v \in I$, we have

$$-\varepsilon - \frac{2M\beta}{\eta} P_u(v) \leqslant f(u) - f(v) \leqslant \varepsilon + \frac{2M\beta}{\eta} P_u(v), \tag{11}$$

where $M=\sup\limits_{v\in I}\left|f(v)\right|$. From (11) it follows that

$$h_{1,\beta}(v):=\varepsilon+\frac{2M\beta}{\eta}P_u(v)+f(v)-f(u)\geqslant 0, \tag{12}$$

$$h_{2,\beta}(v):=\varepsilon+\frac{2M\beta}{\eta}P_u(v)-f(v)+f(u)\geqslant 0. \tag{13}$$

Let H_0 satisfy $(P3)$. For each $v\in I$, we get

$$h''_{1,\beta}(v)\geqslant\frac{2M\beta H_0}{\eta}+f''(v),\ h''_{2,\beta}(v)\geqslant\frac{2M\beta H_0}{\eta}-f''(v).$$

Because of f'' is bounded on I, we can choose $\beta\geqslant 1$ in such a way that $h''_{1,\beta}(v)\geqslant 0$, $h''_{2,\beta}(v)\geqslant 0$, for each $v\in I$. Hence $h_{1,\beta},h_{2,\beta}\in C_+\cap C_+^2$ and then, by hypothesis

$$T_{mn}\left(h_{j,\beta};u\right)\geqslant 0,\ \text{for all } m,n\in\mathbb{N},\ u\in I \text{ and } j=1,2 \tag{14}$$

and hence

$$S_{ts}\left(h_{j,\beta};u\right)\geqslant 0,\ \text{for } t,s\in(0,R),\ u\in I \text{ and } j=1,2.$$

From (12)-(14) and the linearity of T_{mn}, we get

$$\varepsilon S_{ts}(e_0;u)+\frac{2M\beta}{\eta}S_{ts}(P_u;u)+S_{ts}(f;u)-f(u)S_{ts}(e_0;u)\geqslant 0,$$
$$\varepsilon S_{ts}(e_0;u)+\frac{2M\beta}{\eta}S_{ts}(P_u;u)-S_{ts}(f;u)+f(u)S_{ts}(e_0;u)\geqslant 0,$$

thus,

$$-\varepsilon S_{ts}(e_0;u)-\frac{2M\beta}{\eta}S_{ts}(P_u;u)\leqslant f(u)S_{ts}(e_0;u)-S_{ts}(f;u)$$
$$\leqslant\varepsilon S_{ts}(e_0;u)+\frac{2M\beta}{\eta}S_{ts}(P_u;u).$$

By arguing similarly as in the proof of Theorem 2, using the modular ρ and for $t,s\in(0,R)$, we have the assertion of the first part.

The other parts can be proved similarly as in the proofs of Theorem 2 and Theorem 3. □

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