Operators and Matrices
Volume 13, Number 4 (2019), 1035–1054 doi:10.7153/oam-2019-13-72

CONSTANT NORMS AND NUMERICAL RADII OF MATRIX POWERS

HWA-LONG GAU*, KUO-ZHONG WANG AND PEI YUAN WU

(*Communicated by H. Radjavi*)

Abstract. For an n-by-n complex matrix A, we consider conditions on A for which the operator norms $\|A^k\|$ (resp., numerical radii $w(A^k)$), $k \geqslant 1$, of powers of A are constant. Among other results, we show that the existence of a unit vector x in $\mathbb{C}^n$ satisfying $|\langle A^k x, x\rangle| = w(A^k) = w(A)$ for $1 \leqslant k \leqslant 4$ is equivalent to the unitary similarity of A to a direct sum $\lambda B \oplus C$, where $|\lambda| = 1$, B is idempotent, and C satisfies $w(C^k) \leqslant w(B)$ for $1 \leqslant k \leqslant 4$. This is no longer the case for the norm: there is a 3-by-3 matrix A with $\|A^k x\| = \|A^k\| = \sqrt{2}$ for some unit vector x and for all $k \geqslant 1$, but without any nontrivial direct summand. Nor is it true for constant numerical radii without a common attaining vector. If A is invertible, then the constancy of $\|A^k\|$ (resp., $w(A^k)$) for $k = \pm 1, \pm 2, \ldots$ is equivalent to A being unitary. This is not true for invertible operators on an infinite-dimensional Hilbert space.

1. Introduction

For a bounded linear operator A on a complex Hilbert space H, its *operator norm*, *numerical range* and *numerical radius* are

$$\|A\| = \sup\{\|Ax\| : x \in H, \|x\| = 1\},$$

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\},$$

and

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\},$$

respectively, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and its associated vector norm in H, respectively. In this paper, we are concerned with the problem when the powers of A have constant norms or constant numerical radii. One type of operators for which this occurs is that of *idempotent* ones, namely, those A's satisfying $A^2 = A$. Thus it seems natural to ask whether the constancy condition for norms or numerical radii would imply A having a modulus-one multiple of an idempotent operator as a direct summand. Though this is indeed the case when H has dimension 2 (Proposition 3.2), it fails for H with larger dimensions (Proposition 3.4). Note that if an operator A on a finite-dimensional space H has an idempotent summand, then its powers have a common attaining vector for their norms and numerical radii. The latter means that there is a unit vector x in H for which $\|A^k x\| = \|A^k\| = \|A\|$ (resp., $|\langle A^k x, x\rangle| = w(A^k) = w(A)$)

Mathematics subject classification (2010): 15A60, 47A12, 47A30.
Keywords and phrases: Operator norm, numerical radius, idempotent matrix, irreducible matrix.
* Corresponding author.

for all $k \geqslant 1$. We show in Theorem 4.2 that the existence of a common attaining vector for the numerical radii for powers up to 4 would guarantee the idempotent summand. That "4" is the smallest such number is shown by the example in Proposition 4.4. Unfortunately, this is not true for the norm: the 3-by-3 matrix A_1 in Proposition 3.4 is a counterexample. Finally, we turn to invertible operators in Section 5. We show in Theorem 5.1 that an invertible operator A on a finite-dimensional space is such that $\|A^k\|$ (resp., $w(A^k)$), $k = \pm 1, \pm 2, \ldots$, is constant if and only if it is unitary. This is not the case for A acting on an infinite-dimensional space: we show in Theorem 5.3 that for any $\varepsilon > 0$, there is an invertible A such that $\|A^k\| = 1 + \varepsilon$ (resp., $w(A^k) = 1 + \varepsilon$) for all $k = \pm 1, \pm 2, \ldots$.

In Section 2 below, we consider the general question of which sequence of nonnegative numbers can be the norms (resp., numerical radii) of powers of an operator. Here only some preliminary results are obtained in preparation for later discussions on constant norms and constant numerical radii. The constancy problems are addressed in Section 3. Section 4 considers these problems under the extra condition that powers of the operator have a common attaining vector for their norms or numerical radii. The case of invertible operators is taken up in Section 5.

Recall that the *spectrum* and *spectral radius* of an operator A are $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ not invertible}\}$ and $\rho(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. A is *irreducible* if it is not unitarily similar to an operator of the form $B \oplus C$; otherwise, it is *reducible*. To show the irreducibility of A, we usually verify that the only projection P ($P = P^* = P^2$) commuting with A is either I or 0. A is *power-bounded* if $\sup\{\|A^k\| : k \geqslant 1\} < \infty$. We use $\mathrm{Re}\,A$ and $\mathrm{Im}\,A$ to denote the *real part* $(A + A^*)/2$ and *imaginary part* $(A - A^*)/(2i)$ of A, respectively. An operator on an n-dimensional space will be identified as an n-by-n matrix. The n-by-n Jordan block $J_n(\lambda)$ with eigenvalue λ is the matrix

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

We use $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, I_n and 0_n to denote the n-by-n diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, the n-by-n identity matrix and zero matrix, respectively, and $\mathbb{D}$ to denote the open unit disc $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. The *transpose* of a matrix A is denoted by A^T. For a subset M of H, we use $\vee M$ to denote the closed subspace of H spanned by vectors in M.

Our references for general properties of operators and finite matrices are, respectively, [4] and [6]; our reference for numerical ranges and numerical radii is [5, Chapter 1].

We end this section with a list of properties concerning the numerical radius from [5, Chapter 1] which are to be used frequently in later discussions.

PROPOSITION 1.1. *For an operator A on H, the following properties hold:*

(a) *if $A = A_1 \oplus A_2$, then $w(A) = \max\{w(A_1), w(A_2)\}$;*

(b) $w(A) \leqslant \|A\| \leqslant 2w(A)$*;*

(c) *if* λ *is an eigenvalue of* A *and is in the boundary of* $W(A)$*, then* A *is unitarily similar to* $[\lambda] \oplus B$*;*

(d) $w(A) = \sup\{\|\mathrm{Re}\,(e^{i\theta}A)\| : \theta \in \mathbb{R}\}$*;*

(e) *if* $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *with* a *and* c *real, then* $w(A) = \|\mathrm{Re}A\|$*;*

(f) *if* $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* a, c *and* 0 *are collinear, then* $w(A) = \left(|a+c| + \sqrt{|a-c|^2 + |b|^2}\right)/2$*;*

(g) $w(A^k) \leqslant w(A)^k$ *for* $k \geqslant 1$ *(power inequality).*

2. Norms and numerical radii of powers

We start with a known result, due to Wallen, from [4, Problem 92].

PROPOSITION 2.1. *Let* $p_k \geqslant 0$ *for* $k \geqslant 1$*. Then* $p_k = \|A^k\|$ *for some operator* A *and for all* $k \geqslant 1$ *if and only if the* p_k*'s satisfy* $p_{j+k} \leqslant p_j p_k$ *for* $j, k \geqslant 1$*.*

If $p_k > 0$ for all $k \geqslant 1$, then the operator $A = [a_{ij}]_{i,j=1}^{\infty}$ on $\ell^2(\mathbb{N})$ given by

$$a_{ij} = \begin{cases} p_i/p_{i-1}, & \text{if } j-i=1 \text{ and } i \geqslant 1 \ (p_0 \equiv 1), \\ 0, & \text{otherwise,} \end{cases}$$

satisfies $\|A^k\| = p_k$ for all k; otherwise, if k_0 is the smallest integer for which $p_{k_0} = 0$, then we would have $p_k = 0$ for all $k \geqslant k_0$ and hence $A = [a_{ij}]_{i,j=1}^{k_0} \oplus 0$, where

$$a_{ij} = \begin{cases} p_i/p_{i-1}, & \text{if } j-i=1 \text{ and } 1 \leqslant i \leqslant k_0 - 1, \\ 0, & \text{otherwise,} \end{cases}$$

will do.

It would be interesting to know whether a (modified) condition as above on the p_k's would guarantee the existence of an n-by-n matrix A satisfying $\|A^k\| = p_k$ for all k. The next proposition is a preliminary try.

PROPOSITION 2.2. *For* $p_1, p_2 \geqslant 0$*, the condition* $p_2 \leqslant p_1^2$ *is necessary and sufficient for the existence of a* 2*-by-*2 *matrix* A *with* $\|A^k\| = p_k$ *for* $k = 1, 2$*.*

Proof. To prove the sufficiency, we may assume that $p_1 > 0$ and let

$$A = \begin{bmatrix} p_2/p_1 & \sqrt{p_1^2 - (p_2/p_1)^2} \\ 0 & 0 \end{bmatrix}.$$

Then it is easily seen that $\|A^k\| = p_k$ for $k = 1, 2$. $\square$

Note that if $p_1, p_2 > 0$, $p_3 = 0$ and $p_2 \leqslant p_1^2$, then there is no 2-by-2 matrix A with $\|A^k\| = p_k$ for $1 \leqslant k \leqslant 3$. This is because if there is such an A, then $\|A^3\| = p_3 = 0$ implies that A is nilpotent, which yields $A^2 = 0$ or $p_2 = 0$, a contradiction. On the other hand, the arguments for the proof of Proposition 2.1 show that for $p_k > 0$, $1 \leqslant k \leqslant n-1$, and $p_n = 0$, the existence of an n-by-n matrix A satisfying $\|A^k\| = p_k$ for $1 \leqslant k \leqslant n$ is equivalent to the p_k's satisfying $p_{j+k} \leqslant p_j p_k$ for $j,k \geqslant 1$ with $j+k \leqslant n-1$. As pointed out by the referee, when $p_1, \dots, p_n > 0$, this characterization may turn out to be true. This we have difficult in proving at this stage.

We next turn to the numerical radius. Since $w(A^{j+k}) \leqslant w(A^j)w(A^k)$, $j,k \geqslant 1$, do not hold in general (by the example in [1]), the best replacement for $\|A^{j+k}\| \leqslant \|A^j\|\|A^k\|$, $j,k \geqslant 1$, is the power inequalities (cf. Proposition 1.1 (g)). Because of the difficulty in computing the numerical radius even for a general 2-by-2 matrix, whether such inequalities would be enough to characterize $w(A^k)$'s is quite open. There is one case when the numerical radius of a 2-by-2 matrix is easy to compute, namely, when $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is such that a, c and 0 are collinear (cf. Proposition 1.1 (f)). Using this, we may obtain the following analogue of Proposition 2.2.

PROPOSITION 2.3. *Let $w_1, w_2 \geqslant 0$. Then $w_2 \leqslant w_1^2$ is necessary and sufficient for the existence of a 2-by-2 matrix A with $w(A^k) = w_k$ for $k = 1, 2$.*

Proof. For the sufficiency, we may assume that $w_1 > 0$ and let

$$A = \begin{bmatrix} w_2/w_1 & 2\sqrt{w_1^2 - w_2} \\ 0 & 0 \end{bmatrix}.$$

From Proposition 1.1 (f), we can easily show that $w(A^k) = w_k$ for $k = 1, 2$. □

As before, if $w_1, w_2 > 0$, $w_3 = 0$, and $w_2 \leqslant w_1^2$, then there is no 2-by-2 matrix A satisfying $w(A^k) = w_k$ for $1 \leqslant k \leqslant 3$. However, the next proposition says that there are 3-by-3 matrices satisfying these conditions.

PROPOSITION 2.4. *For $w_3 = 0$, the conditions $w_1, w_2 \geqslant 0$ and $w_2 \leqslant w_1^2$ are necessary and sufficient for the existence of a 3-by-3 matrix A with $w(A^k) = w_k$ for $1 \leqslant k \leqslant 3$.*

Proof. To prove the sufficiency, we may assume that $w_1 > 0$ and let

$$A = \begin{bmatrix} 0 & a & \\ & 0 & b \\ & & 0 \end{bmatrix}, \quad a, b \geqslant 0.$$

Then $w(A) = \|\mathrm{Re}A\| = \sqrt{a^2 + b^2}/2$ (since $W(A)$ is a circular disc centered at 0) and

$$w(A^2) = w\left(\begin{bmatrix} 0 & 0 & ab \\ & 0 & 0 \\ & & 0 \end{bmatrix} \right) = \frac{1}{2}ab.$$

If $b = \left(2w_1^2 + 2(w_1^4 - w_2^2)^{1/2}\right)^{1/2} > 0$ (since $w_2 \leqslant w_1^2$) and $a = 2w_2/b$, then $w(A^k) = w_k$ for $1 \leqslant k \leqslant 3$. $\square$

Note that if $w_1 = w_2 = 1$, $w_3 = 1/2$, and $w_4 = 0$, then, through some tedious computations, we can show that no 4-by-4 matrix A of the form

$$\begin{bmatrix} 0 & a & & \\ & 0 & b & \\ & & 0 & c \\ & & & 0 \end{bmatrix}$$

satisfies $w(A^k) = w_k$ for $1 \leqslant k \leqslant 4$. However, this does not rule out the possibility that other 4-by-4 nilpotent A will satisfy these conditions.

We conclude this section with the following relevant information on the norms and numerical radii of matrix powers.

PROPOSITION 2.5. *For an n-by-n matrix A, the following conditions are equivalent:*

(a) $\|A^k\| = \|A\|^k$ (*resp.,* $w(A^k) = w(A)^k$) *for some k larger than or equal to the degree of the minimal polynomial of A;*

(b) $\|A^k\| = \|A\|^k$ (*resp.,* $w(A^k) = w(A)^k$) *for all $k \geqslant 1$;*

(c) $\|A\| = \rho(A)$ (*resp.,* $w(A) = \rho(A)$)*;*

(d) *A is unitarily similar to a matrix of the form $[a] \oplus B$, where $|a| = \|A\|$ and $\|B\| \leqslant |a|$ (resp., $|a| = w(A)$ and $w(B) \leqslant |a|$).*

This is proved in [8, Theorem 2.1] (resp., [3, Theorems 2 and 3]).

COROLLARY 2.6. *For an n-by-n matrix A, the following are equivalent:*

(a) $\|A^n\| = \|A\| = 1$ (*resp.,* $w(A^n) = w(A) = 1$)*;*

(b) $\|A^k\| = 1$ (*resp.,* $w(A^k) = 1$) *for all $k \geqslant 1$;*

(c) *A is unitarily similar to $[a] \oplus B$ with $\|B\| \leqslant |a| = 1$ (resp., $w(B) \leqslant |a| = 1$).*

3. Constant norms and constant numerical radii

We start by observing that if A is such that $\|A^k\| = c$ (resp., $w(A^k) = c'$) for all $k \geqslant 1$, then either $c = 0$ or $c \geqslant 1$ (resp., $c' = 0$ or $c' \geqslant 1$). This is even true for power-bounded A and is an easy consequence of Gelfand's formula $\rho(A) = \lim_k \|A^k\|^{1/k}$ (resp., $\rho(A) = \lim_k w(A^k)^{1/k}$) (cf. [4, Problem 88]).

The next lemma concerning properties of idempotent operators follows easily from results in [12].

LEMMA 3.1. *If A is an idempotent operator on H, then:*

(a) $A^k = A$ *for all* $k \geqslant 1$*;*

(b) A *is unitarily similar to an operator of the form* $0 \oplus I \oplus \begin{bmatrix} 0 & B \\ 0 & I \end{bmatrix}$ *on* $H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$*, where* B *on* H_3 *is such that* $\langle Bx, x \rangle > 0$ *for any nonzero vector* x *in* H_3 *and is unique up to unitary similarity;*

(c) $\|A\| = \sqrt{1+\|B\|^2}$*;*

(d) $w(A) = (1+\sqrt{1+\|B\|^2})/2$*;*

(e) A *is reducible when* $3 \leqslant \dim H < \infty$.

For a 2-by-2 matrix, the constancy of the norms or numerical radii of its powers does indeed imply that it has a multiple of an idempotent as a direct summand.

PROPOSITION 3.2. *Let* A *be a* 2*-by-*2 *matrix.*

(a) *The following conditions are equivalent:*

- (i) $\|A^k\| = 1$ *for all* $k \geqslant 1$*;*
- (ii) $w(A^k) = 1$ *for all* $k \geqslant 1$*;*
- (iii) A *is unitarily similar to* $\lambda \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, *where* $|\lambda| = 1$ *and* $|a| \leqslant 1$.

(b) *The following conditions are equivalent:*

- (i) $\|A^k\| = c > 1$ *for all* $k \geqslant 1$*;*
- (ii) $w(A^k) = c' > 1$ *for all* $k \geqslant 1$*;*
- (iii) A *is unitarily similar to* λB*, where* $|\lambda| = 1$ *and* B *is a non-Hermitian idempotent.*

Moreover, c *and* c' *are related by* $c' = (1+c)/2$. *Under* (i) (*resp.,* (ii))*,* B *may be taken as* $\begin{bmatrix} 1 & \sqrt{c^2-1} \\ 0 & 0 \end{bmatrix}$ (*resp.,* $\begin{bmatrix} 1 & 2\sqrt{c'(c'-1)} \\ 0 & 0 \end{bmatrix}$).

The following lemma is needed for the proof of (a) (ii)$\Rightarrow$(iii) and also needed later in Section 4. It is from [2, Lemma 1.5].

LEMMA 3.3. *Let* A *be a nonzero* 2*-by-*2 *matrix. Then* $A = w(A)\begin{bmatrix} 1 & x \\ y & z \end{bmatrix}$ *if and only if* $|x| \leqslant 1$*,* $y = -\overline{x}$*, and* $z = (1-|x|^2)u - |x|^2$ *for some* u *with* $|u| \leqslant 1$.

Proof of Proposition 3.2.

(a) The implications (iii)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii) are trivial. We now prove (i)$\Rightarrow$(iii). By Gelfand's formula, we have $\rho(A) = 1$ and hence A is unitarily similar to $\lambda \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$ with $|\lambda| = 1$, $a \geqslant 0$ and $|b| \leqslant 1$. Since $1 = \|A\| \geqslant \sqrt{1+a^2}$, we obtain $a = 0$ and thus A is unitarily similar to $\lambda \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ as required. On the other hand, under (ii), $w(A) = w\left(\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}\right) = 1$. The implication (ii)$\Rightarrow$(iii) follows immediately from Lemma 3.3.

Note that the equivalence (i)$\Leftrightarrow$(iii) (resp., (ii)$\Leftrightarrow$(iii)) also follows directly from Corollary 2.6.

(b) We need only prove (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii), and may assume that $A = \lambda B$, where $|\lambda| = 1$, $B = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$, $a > 0$, and $|b| \leqslant 1$. To prove (i)$\Rightarrow$(iii), we consider four cases separately:

(I) $b = 1$. In this case, we have $B^k = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}$ and hence $c = \|A^k\| = \|B^k\| \geqslant \sqrt{1+k^2a^2}$ for all $k \geqslant 1$. This implies that $a = 0$, which contradicts our assumption.

(II) $|b| = 1$ and $b \neq 1$. Let k_j, $j \geqslant 1$, be positive integers such that $\lim_j b^{k_j} = 1$. Then

$$\lim_j B^{k_j} = \lim_j \begin{bmatrix} 1 & a(1-b^{k_j})/(1-b) \\ 0 & b^{k_j} \end{bmatrix} = I_2 \tag{1}$$

and hence $c = \lim_j \|B^{k_j}\| = 1$. Again, this contradicts (i).

(III) $0 < b < 1$. Since

$$\lim_k B^k = \lim_k \begin{bmatrix} 1 & a(1-b^k)/(1-b) \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} 1 & a/(1-b) \\ 0 & 0 \end{bmatrix},$$

we obtain

$$c = \lim_k \|B^k\| = \left\| \begin{bmatrix} 1 & a/(1-b) \\ 0 & 0 \end{bmatrix} \right\| = \left(1 + \frac{a^2}{(1-b)^2}\right)^{1/2}. \tag{2}$$

On the other hand, it can be computed that

$$c = \|B\| = \left\| \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \right\| = \left(\frac{1}{2}\left(a^2+b^2+1+(a^4+b^4+1+2(a^2b^2+a^2-b^2))^{1/2}\right)\right)^{1/2}. \tag{3}$$

The equality of (2) and (3) yields that

$$a^2\left((1-b)^4 + (a^2+b^2)(1-b)^2\right) = a^2\left(a^2 + (1-b)^2\right).$$

As $a > 0$, we deduce from above that

$$a^2 = \frac{(1-b)^4 + (b^2-1)(1-b)^2}{1-(1-b)^2} = \frac{2(1-b)^2(b-1)}{2-b} < 0,$$

which is absurd.

(IV) $0 < |b| < 1$ and b is not in $(0,1)$. Let $k_0 \geqslant 1$ be such that $\operatorname{Re} b^{k_0} < 0$. Then, as in (2),

$$\left(1 + \frac{a^2|1-b^{k_0}|^2}{|1-b|^2}\right)^{1/2} \leqslant \|B^{k_0}\| = c = \left(1 + \frac{a^2}{|1-b|^2}\right)^{1/2},$$

from which we obtain $a|1-b^{k_0}| \leqslant a$. As $a>0$, we have $\operatorname{Re}(1-b^{k_0}) \leqslant |1-b^{k_0}| \leqslant 1$ or $\operatorname{Re} b^{k_0} \geqslant 0$, which contradicts our choice of k_0.

As each of the above four cases leads to a contradiction, we arrive at the remaining option of $b=0$. Thus $A=\lambda B$ with $|\lambda|=1$ and $B=\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$ idempotent. As $c=\|A\|=\| \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \|=\sqrt{1+a^2}$, we have $a=\sqrt{c^2-1}$ as asserted.

We next prove (ii)$\Rightarrow$(iii). As before, we consider four cases separately:

(I) $b=1$. In this case, we have

$$c'=w(A^k)=w(B^k)=w(\begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix})=1+\frac{1}{2}ka, \ \ k \geqslant 1.$$

This yields $a=0$, a contradiction.

(II) $|b|=1$ and $b \neq 1$. Let k_j, $j \geqslant 1$, be as before. Then $\lim_j B^{k_j}=I_2$ as in (1) and hence $c'=\lim_j w(B^{k_j})=1$, again contradicting (ii).

For the remaining two cases, let $b=|b|e^{2\pi\theta i}$, where θ is real.

(III) $0<|b|<1$ and $\theta=m/n$ with $n \geqslant 1$ and m integers. Then

$$\lim_k B^k=\lim_k \begin{bmatrix} 1 & a(1-b^k)/(1-b) \\ 0 & b^k \end{bmatrix}=\begin{bmatrix} 1 & a/(1-b) \\ 0 & 0 \end{bmatrix}.$$

Hence

$$c'=\lim_k w(B^k)=w(\begin{bmatrix} 1 & a/(1-b) \\ 0 & 0 \end{bmatrix})=\frac{1}{2}(1+(1+|\frac{a}{1-b}|^2)^{1/2}), \tag{4}$$

by Lemma 3.1 (d). Since $b^n=|b|^n e^{2\pi mi}=|b|^n>0$, we also have

$$c'=w(B^n)=w(\begin{bmatrix} 1 & a(1-b^n)/(1-b) \\ 0 & b^n \end{bmatrix})=\frac{1}{2}((1+b^n)+(1-b^n)(1+|\frac{a}{1-b}|^2)^{1/2}) \tag{5}$$

by Proposition 1.1 (f). The equality of (4) and (5) yields that $b^n=b^n(1+|a/(1-b)|^2)^{1/2}$. Since $b \neq 0$, we obtain $a=0$, which contradicts our assumption.

(IV) $0<|b|<1$ and θ irrational. In this case, the set $\{e^{2\pi n\theta i} : n \geqslant 1\}$ is dense in $\partial\mathbb{D}$. Let k_0 be such that $\operatorname{Re} b^{k_0}<0$ and b^{k_0} is very close to $\operatorname{Re} b^{k_0}$. Let $C=\begin{bmatrix} 1 & |a/(1-b)| \\ 0 & 0 \end{bmatrix}$ and let $x=[x_1 \ x_2]^T$ be a unit vector in $\mathbb{C}^2$ with $x_1, x_2>0$ such that $\langle Cx, x\rangle=w(C)$ (cf. [7, Proposition 3.3]). Letting

$$D=\begin{bmatrix} 1 & |a(1-b^{k_0})/(1-b)| \\ 0 & b^{k_0} \end{bmatrix},$$

we claim that $\operatorname{Re}\langle Dx, x\rangle>w(C)$. Indeed, this is the same as

$$x_1^2+|\frac{a(1-b^{k_0})}{1-b}|x_1x_2+(\operatorname{Re} b^{k_0})x_2^2>x_1^2+|\frac{a}{1-b}|x_1x_2$$

or

$$d\frac{|1-b^{k_0}|-1}{\operatorname{Re} b^{k_0}}+1<0, \quad \text{where } d=\Big|\frac{a}{1-b}\Big|\frac{x_1}{x_2}.$$

Since $a \neq 0$ as before, we deduce from

$$x_1^2+\Big|\frac{a}{1-b}\Big|x_1x_2=w(C)=\frac{1}{2}\big(1+(1+\Big|\frac{a}{1-b}\Big|^2)^{1/2}\big)>1 \ \text{ and } x_2^2=1-x_1^2$$

that $|a/(1-b)|x_1x_2>x_2^2$, which yields $d>1$. As

$$\frac{|1-b^{k_0}|-1}{\operatorname{Re} b^{k_0}} \approx \frac{|1-\operatorname{Re} b^{k_0}|-1}{\operatorname{Re} b^{k_0}}=\frac{-\operatorname{Re} b^{k_0}}{\operatorname{Re} b^{k_0}}=-1,$$

we do have

$$d\frac{|1-b^{k_0}|-1}{\operatorname{Re} b^{k_0}}+1 \approx -d+1<0,$$

which proves our claim. Thus

$$c'=w(A^{k_0})=w(D)\geqslant |\langle Dx,x\rangle| \geqslant \operatorname{Re}\langle Dx,x\rangle > w(C),$$

which contradicts the fact that

$$c'=\lim_k w(A^k)=\lim_k w(B^k)=w(C).$$

As before, $b=0$ is the only option for us and hence $A=\lambda B$ with $|\lambda|=1$ and $B=\begin{bmatrix}1 & a\\ 0 & 0\end{bmatrix}$. Finally, from $c'=w(A)=w(B)=(1+\sqrt{1+a^2})/2$, we have $a=2\sqrt{c'(c'-1)}$ and c and c' are related by $c'=(1+c)/2$. $\square$

After the positive results in the preceding proposition, it seems conceivable that an n-by-n matrix A with the $\|A^k\|$'s (resp., $w(A^k)$'s) constant should have a summand of the form λB, where $|\lambda|=1$ and B is idempotent. This turns out to be not the case as the following examples show.

PROPOSITION 3.4. *If*

$$A_1=\begin{bmatrix}\sqrt{2}/2 & -1/2 & 0\\ 1 & \sqrt{2}/2 & 0\\ \sqrt{2}/2 & -1/2 & 0\end{bmatrix} \quad (\textit{resp.}, \ A_2=\begin{bmatrix}1 & 1 & 0\\ 0 & -1 & \sqrt{3-\sqrt{6}}\\ 0 & 0 & 0\end{bmatrix}),$$

then $\|A_1^k\|=\sqrt{2}$ (*resp.,* $w(A_2^k)=\sqrt{6}/2$) *for all* $k\geqslant 1$, *and* A_1 (*resp.,* A_2) *is irreducible.*

Proof. For A_1, we have $A_1^*A_1=\operatorname{diag}(2,1,0)$ and, by induction,

$$A_1^{k^*}A_1^k=A_1^*(A_1^{k-1^*}A_1^{k-1})A_1=A_1^*\big(\operatorname{diag}(2,1,0)\big)A_1=\operatorname{diag}(2,1,0), \quad k\geqslant 2.$$

Hence $\|A_1^k\| = \sqrt{2}$ for all $k \geqslant 1$.

As for A_2, note that

$$A_2^2 = \begin{bmatrix} 1 & 0 & \sqrt{3-\sqrt{6}} \\ 0 & 1 & -\sqrt{3-\sqrt{6}} \\ 0 & 0 & 0 \end{bmatrix}$$

is idempotent. Hence $A_2^{2k-1} = A_2$ and $A_2^{2k} = A_2^2$ for $k \geqslant 1$. We now show that $w(A_2) = w(A_2^2) = \sqrt{6}/2$. To prove $w(A_2) = \sqrt{6}/2$, we make use of Proposition 1.1 (d). For any real θ, consider the matrices $U_\theta = \operatorname{diag}(e^{2i\theta}, e^{i\theta}, 1)$ and

$$B_\theta \equiv U_\theta^*(e^{i\theta}A_2)U_\theta = \begin{bmatrix} e^{i\theta} & 1 & 0 \\ 0 & -e^{i\theta} & \sqrt{3-\sqrt{6}} \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\operatorname{Re} B_\theta = \begin{bmatrix} \cos\theta & 1/2 & 0 \\ 1/2 & -\cos\theta & \sqrt{3-\sqrt{6}}/2 \\ 0 & \sqrt{3-\sqrt{6}}/2 & 0 \end{bmatrix}$$

has characteristic polynomial

$$p_\theta(x) \equiv \det(xI_3 - \operatorname{Re} B_\theta) = x^3 - (1 - \frac{\sqrt{6}}{4} + \cos^2\theta)x + \frac{3-\sqrt{6}}{4}\cos\theta.$$

Since, for each fixed θ, the derivative

$$p'_\theta(x) = 3x^2 - (1 - \frac{\sqrt{6}}{4} + \cos^2\theta) \geqslant 3x^2 - 2 + \frac{\sqrt{6}}{4} > 0 \text{ for } x \geqslant \frac{\sqrt{6}}{2},$$

$p_\theta(x)$ is strictly increasing on $[\sqrt{6}/2, \infty)$. This together with

$$\begin{aligned} p_\theta(\frac{\sqrt{6}}{2}) &= \frac{3}{4}\sqrt{6} - (1 - \frac{\sqrt{6}}{4} + \cos^2\theta)\frac{\sqrt{6}}{2} + \frac{3-\sqrt{6}}{4}\cos\theta \\ &\geqslant \frac{3}{4}\sqrt{6} - (1 - \frac{\sqrt{6}}{4} + 1)\frac{\sqrt{6}}{2} + \frac{3-\sqrt{6}}{4}(-1) = 0 = p_\pi(\frac{\sqrt{6}}{2}) \end{aligned}$$

yields that $p_\theta(x) > 0$ on $(\sqrt{6}/2, \infty)$ for all θ in $(-\pi, \pi)$. Thus all eigenvalues of $\operatorname{Re} B_\theta$ are less than or equal to $\sqrt{6}/2$ for any θ and $\sqrt{6}/2$ is an eigenvalue of $\operatorname{Re} B_\pi$. This says that $\|\operatorname{Re} B_\theta\| \leqslant \sqrt{6}/2$ for all θ and $\|\operatorname{Re} B_\pi\| = \sqrt{6}/2$. Applying Proposition 1.1 (d) then yields that $w(A_2) = \sup_\theta \|\operatorname{Re} B_\theta\| = \sqrt{6}/2$. Moreover, from Lemma 3.1 (d), we obtain $w(A_2^2) = \big(1 + \sqrt{1 + 2(3-\sqrt{6})}\big)/2 = \sqrt{6}/2$. Hence $w(A_2^k) = \sqrt{6}/2$ for all $k \geqslant 1$.

The irreducibility of A_1 (resp., A_2) follows by showing via simple computations that the only projections commuting with A_1 (resp., A_2) are 0_3 and I_3, which we omit. $\square$

The preceding proposition shows that A_1 and A_2 there have no (proper) summand of the form λB with $|\lambda|=1$ and B idempotent. Nor are they themselves idempotent either by simple computations or by Lemma 3.1 (e).

In view of the fact that A_1^8 and A_2^2 are both idempotent, the referee suggested that the following may be true: If the n-by-n matrix A is such that $\|A^k\|$'s (resp., $w(A^k)$'s) are constant, then some power of A has a modulus-one multiple of an idempotent as a direct summand. We leave it has as a conjecture.

4. Common attaining vector

As seen in last section, the condition of constant norms or constant numerical radii of matrix powers A^k, $k \geqslant 1$, does not guarantee the existence of a multiple of an idempotent summand for A. In the present section, we consider a strengthening of the constancy condition. Since an idempotent A is equal to all its powers, there is a unit vector x such that $\|A^k x\| = \|A^k\| = \|A\|$ (resp., $|\langle A^k x, x\rangle| = w(A^k) = w(A)$) for all $k \geqslant 1$. It is thus natural to add such a common attaining vector condition to the constancy to check whether this would lead to the expected conclusion. This is indeed the case for the numerical radius as to be proved in Theorem 4.2 below, but, unfortunately, not for the norm in general: the 3-by-3 A_1 in Proposition 3.4 is such that $\|A_1^k x\| = \|A_1^k\| = \sqrt{2}$ for $x = [1\ 0\ 0]^T$ and all $k \geqslant 1$. However, for 2-by-2 matrices, the next proposition shows that our expectation is fulfilled.

PROPOSITION 4.1. *For a 2-by-2 matrix A, the existence of a unit vector x in $\mathbb{C}^2$ such that $\|A^k x\| = \|A^k\| = \|A\|$ for $k = 1,2$ (resp., $1 \leqslant k \leqslant 3$) is equivalent to the unitary similarity of A to $\lambda \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, ηB, or $\begin{bmatrix} 0 & \xi \\ \|A\| & 0 \end{bmatrix}$ (resp., $\lambda \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ or ηB), where $|\lambda| = |\eta| = |\xi| = 1$, $|a| \leqslant 1$, and B is idempotent.*

Proof. We need only prove the necessity. Assuming that $A \neq 0_2$, let y be a unit vector orthogonal to x, and let A be represented as $C = [a_{ij}]_{i,j=1}^2$ with respect to the orthonormal basis $\{x, y\}$ of $\mathbb{C}^2$. Since $\|Ax\| = \|A\|$, we have

$$|a_{11}|^2 + |a_{21}|^2 = \|A\|^2. \tag{6}$$

Moreover, the equalities $|\langle A^*Ax, x\rangle| = \|Ax\|^2 = \|A\|^2 = \|A^*A\|\|x\|^2$ yield that $A^*Ax = \alpha x$ for some scalar α. Hence $\langle Ax, Ay\rangle = \langle A^*Ax, y\rangle = \alpha\langle x, y\rangle = 0$, which, in terms of the entries of C, is the orthogonality of $[a_{11}\ a_{21}]^T$ and $[a_{12}\ a_{22}]^T$. It follows that $[a_{12}\ a_{22}]^T = \beta[-\overline{a}_{21}\ \overline{a}_{11}]^T$ for some β. Thus

$$C = \begin{bmatrix} a_{11} & -\beta\overline{a}_{21} \\ a_{21} & \beta\overline{a}_{11} \end{bmatrix}$$

and $C^*C = \operatorname{diag}(\|A\|^2, |\beta|^2\|A\|^2)$ from (6). On the other hand, since $\|A^2x\| = \|A^2\| = \|A\|$, we can repeat the above arguments to obtain $C^{2*}C^2 = \operatorname{diag}(\|A^2\|^2, *) =$

$\operatorname{diag}(\|A\|^2, *)$. It follows that

$$\begin{bmatrix} \|A\|^2 & 0 \\ 0 & * \end{bmatrix} = C^*(C^*C)C = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} \\ -\overline{\beta}a_{21} & \overline{\beta}a_{11} \end{bmatrix} \begin{bmatrix} \|A\|^2 & 0 \\ 0 & |\beta|^2\|A\|^2 \end{bmatrix} \begin{bmatrix} a_{11} & -\beta\overline{a}_{21} \\ a_{21} & \beta\overline{a}_{11} \end{bmatrix}$$
$$= \begin{bmatrix} (|a_{11}|^2 + |\beta|^2|a_{21}|^2)\|A\|^2 & 0 \\ \overline{\beta}a_{11}a_{21}\|A\|^2(-1+|\beta|^2) & * \end{bmatrix}.$$

The equalities of the (1, 1)- and (2, 1)-entries yield

$$(|a_{11}|^2 + |\beta|^2|a_{21}|^2)\|A\|^2 = \|A\|^2 \quad \text{and} \quad \overline{\beta}a_{11}a_{21}\|A\|^2(-1+|\beta|^2) = 0,$$

respectively. The former gives

$$|a_{11}|^2 + |\beta|^2|a_{21}|^2 = 1 \tag{7}$$

and the latter $a_{11} = 0$, $a_{21} = 0$, $\beta = 0$ or $|\beta| = 1$. We treat these four cases separately:

(I) $a_{11} = 0$. Then $C = \begin{bmatrix} 0 & -\beta\overline{a}_{21} \\ a_{21} & 0 \end{bmatrix}$, $|a_{21}| = \|A\|$ by (6), and $|\beta a_{21}| = 1$ by (7). Thus B is unitarily similar to $\begin{bmatrix} 0 & \xi \\ \|A\| & 0 \end{bmatrix}$, with $|\xi| = 1$.

(II) $a_{21} = 0$. Then $C = \operatorname{diag}(a_{11}, \beta\overline{a}_{11})$ with $|a_{11}| = \|A\| = 1$ by (6) and (7). Thus A is unitarily similar to $C = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & \beta\overline{a}_{11}/a_{11} \end{bmatrix}$ with $|a_{11}| = 1$ and $|\beta\overline{a}_{11}/a_{11}| \leqslant \|C\| = \|A\| = 1$.

(III) $\beta = 0$. Then $|a_{11}| = 1$ by (7), and A is unitarily similar to

$$C = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 0 \end{bmatrix} \equiv a_{11}B,$$

with B idempotent.

(IV) $|\beta| = 1$. Then C is unitary by (7) and the orthogonality of its two columns, and hence is unitarily similar to $\lambda\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, with $|\lambda| = |a| = 1$.

Finally, assume that $A = \begin{bmatrix} 0 & \xi \\ \|A\| & 0 \end{bmatrix}$, with $|\xi| = 1$, satisfies $\|A^3\| = \|A\|$. Then

$$\|A\| = \|A^3\| = \left\| \begin{bmatrix} 0 & \xi^2\|A\| \\ \xi\|A\|^2 & 0 \end{bmatrix} \right\| = \max\{\|A\|, \|A\|^2\},$$

which yields $\|A\| \geqslant \|A\|^2$. On the other hand, we also have

$$\|A\|^2 \geqslant \|A^2\| = \left\| \begin{bmatrix} \xi\|A\| & 0 \\ 0 & \xi\|A\| \end{bmatrix} \right\| = \|A\|.$$

Thus $\|A\|^2 = \|A\|$ or $\|A\| = 1$. Hence $A = \begin{bmatrix} 0 & \xi \\ 1 & 0 \end{bmatrix}$ is unitary and is unitarily similar to $\lambda\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ with $|\lambda| = |a| = 1$. $\square$

We now turn to the case for constant numerical radii. The next theorem is our main result in this section.

THEOREM 4.2. *For an n-by-n matrix A, the existence of a unit vector x in $\mathbb{C}^n$ such that $|\langle A^k x,x\rangle| = w(A^k) = w(A)$ for $1 \leqslant k \leqslant 4$ is equivalent to the unitary similarity of A to $\lambda B \oplus C$, where $|\lambda| = 1$, B is idempotent, and $w(C^k) \leqslant w(B)$ for $1 \leqslant k \leqslant 4$.*

For its proof, we need the following lemma, which is an easy consequence of Lemma 3.3.

LEMMA 4.3. *Let A be an n-by-n matrix, x be a unit vector in $\mathbb{C}^n$ such that $|\langle Ax,x\rangle| = w(A)$, and y be a unit vector orthogonal to x. Then:*

(a) $|\langle Ax,y\rangle| \leqslant w(A)$*;*

(b) $|\langle Ax,y\rangle| = w(A)$ *if and only if*

$$\begin{bmatrix} \langle Ax,x\rangle & \langle Ay,x\rangle \\ \langle Ax,y\rangle & \langle Ay,y\rangle \end{bmatrix} = w(A)\begin{bmatrix} 1 & \lambda \\ -\overline{\lambda} & -1 \end{bmatrix}$$

for some λ, $|\lambda| = 1$.

Note that the 2-by-2 matrix in (b) above is unitarily similar to $\begin{bmatrix} 0 & 2w(A) \\ 0 & 0 \end{bmatrix}$.

Proof of Theorem 4.2. We need only prove the necessity. For this, we may assume that $A \neq 0_n$ and $\langle Ax,x\rangle = w(A)$. Expand the vector x to an orthonormal basis $\{x,x_2,\ldots,x_n\}$ of $\mathbb{C}^n$ in such a way that $\{x,x_2\}$ forms a basis of a two-dimensional subspace containing $\vee\{x,Ax\}$. Let A be represented as $[a_{ij}]_{i,j=1}^n$ with respect to this basis. We have $a_{i1} = 0$ for $3 \leqslant i \leqslant n$. Since $a_{11} = w(A)$, we may apply Lemma 3.3 to the submatrices $\begin{bmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix}$, $2 \leqslant j \leqslant n$, to obtain $a_{21} = -\overline{a}_{12}$ and $a_{1j} = 0$ for $3 \leqslant j \leqslant n$. Let $C = [a_{ij}]_{i,j=2}^n$. If $a_{12} = 0$, then $A = [a_{11}] \oplus C$ and $A^2 = [a_{11}^2] \oplus C^2$. Note that $a_{11} = w(A) = w(A^2) = a_{11}^2$, where the last equality follows from

$$w(C^2) \leqslant w(C)^2 \leqslant w(A)^2 = a_{11}^2$$

via the power inequality (Proposition 1.1 (g)). Hence $a_{11} = 1$ and $A = [1] \oplus C$ with $w(C^k) \leqslant w(C)^k \leqslant 1$ for all k, $1 \leqslant k \leqslant 4$, as required.

In the following, we assume that $a_{12} \neq 0$. Let A^k be denoted by $[a_{ij}^{(k)}]_{i,j=1}^n$, $2 \leqslant k \leqslant 4$. We claim that $a_{11}^{(k)} = w(A^k)$ for each k. Indeed, for the $a_{ij}^{(2)}$'s, we have

$$a_{11}^{(2)} = a_{11}^2 - |a_{12}|^2, \;\; a_{12}^{(2)} = a_{12}(a_{11}+a_{22}), \;\; a_{21}^{(2)} = -\overline{a}_{12}(a_{11}+a_{22}),$$

and

$$a_{1k}^{(2)} = a_{12}a_{2k}, \;\; a_{k1}^{(2)} = -\overline{a}_{12}a_{k2} \quad \text{for } k \geqslant 3.$$

In particular, $a_{11}^{(2)} \geqslant 0$ by Lemma 4.3 (a), and hence, by $|\langle A^2x,x\rangle| = w(A^2)$, is equal to $w(A^2)$. By Lemma 4.3 (a) again, we have

$$a_{11} = w(A) = w(A^2) = a_{11}^{(2)} \geqslant |a_{12}^{(2)}| = |a_{12}||a_{11}+a_{22}|.$$

Together with $a_{11} \geqslant |a_{12}|$, this yields $a_{11}^2 \geqslant |a_{12}|^2|a_{11}+a_{22}|$. On the other hand, from Lemma 3.3, we infer that

$$a_{12}(a_{11}+a_{22}) = a_{12}^{(2)} = -\overline{a}_{21}^{(2)} = a_{12}(\overline{a}_{11}+\overline{a}_{22}).$$

Since $a_{12} \neq 0$, this implies that $a_{11}+a_{22}$ is real. Hence

$$\begin{aligned} a_{11}^{(3)} &= a_{11}a_{11}^{(2)} + a_{12}a_{21}^{(2)} = a_{11}^2 + a_{12}\big(-\overline{a}_{12}(a_{11}+a_{22})\big) = a_{11}^2 - |a_{12}|^2(a_{11}+a_{22}) \\ &\geqslant a_{11}^2 - |a_{12}|^2|a_{11}+a_{22}| \geqslant 0. \end{aligned}$$

Therefore, we also have $a_{11}^{(3)} = w(A^3)$ via our assumption $|\langle A^3x, x\rangle| = w(A^3)$. From

$$a_{11}^2 - |a_{12}|^2(a_{11}+a_{22}) = a_{11}^{(3)} = w(A^3) = w(A^2) = a_{11}^{(2)} = a_{11}^2 - |a_{12}|^2,$$

we obtain $a_{11}+a_{22} = 1$. From

$$a_{11} = w(A) = w(A^2) = a_{11}^{(2)} = a_{11}^2 - |a_{12}|^2, \tag{8}$$

we solve a_{11} as $(1 \pm \sqrt{1+4|a_{12}|^2})/2$. Then $a_{22} = (1 \mp \sqrt{1+4|a_{12}|^2})/2$ and, therefore, $a_{22}^2 - |a_{12}|^2 = a_{22}$. Note that, from $a_{12}a_{2k} = a_{1k}^{(2)} = -\overline{a}_{k1}^{(2)} = a_{12}\overline{a}_{k2}$, $k \geqslant 3$, we infer that $a_{2k} = \overline{a}_{k2}$ for $k \geqslant 3$. Thus

$$\begin{aligned} a_{22}^{(2)} &= -|a_{12}|^2 + a_{22}^2 + \sum_{k=3}^{n} a_{2k}a_{k2} = a_{22} + \sum_{k=3}^{n} |a_{2k}|^2, \\ a_{12}^{(3)} &= a_{11}a_{12}^{(2)} + a_{12}a_{22}^{(2)} = a_{11}a_{12}(a_{11}+a_{22}) + a_{12}a_{22}^{(2)} = a_{12}(a_{11}+a_{22}^{(2)}) \\ &= a_{12}(a_{11}+a_{22}+\sum_{k=3}^{n}|a_{2k}|^2) = a_{12}(1+\sum_{k=3}^{n}|a_{2k}|^2), \end{aligned} \tag{9}$$

and

$$a_{11}^{(4)} = a_{11}^{(3)}a_{11} + a_{12}^{(3)}a_{21} = a_{11}^2 + a_{12}a_{21}(1+\sum_{k=3}^{n}|a_{2k}|^2) = a_{11}^2 - |a_{12}|^2(1+\sum_{k=3}^{n}|a_{2k}|^2). \tag{10}$$

Since

$$a_{11} = a_{11}^{(3)} \geqslant |a_{12}^{(3)}| = |a_{12}|(1+\sum_{k=3}^{n}|a_{2k}|^2)$$

by (9), we have

$$a_{11}^2 \geqslant |a_{12}|^2(1+\sum_{k=3}^{n}|a_{2k}|^2)^2 \geqslant |a_{12}|^2(1+\sum_{k=3}^{n}|a_{2k}|^2).$$

Applying this to (10) yields $a_{11}^{(4)} \geqslant 0$ and thus $a_{11}^{(4)} = w(A^4)$. Therefore, by (8) and (10), we have

$$a_{11}^2 - |a_{12}|^2 = a_{11} = w(A) = w(A^4) = a_{11}^{(4)} = a_{11}^2 - |a_{12}|^2(1+\sum_{k=3}^{n}|a_{2k}|^2),$$

which then implies that $\sum_{k=3}^{n}|a_{2k}|^2=0$ or $a_{2k}=a_{k2}=0$ for all $k\geqslant 3$. This shows that $A=B\oplus C$, where $B=\begin{bmatrix} a_{11} & a_{12}\\ -\bar{a}_{12} & a_{22}\end{bmatrix}$ is unitarily similar to $\begin{bmatrix} 1 & \sqrt{2(|a_{12}|^2-a_{11}a_{22})}\\ 0 & 0\end{bmatrix}$ (since $a_{11}+a_{22}=1$, $a_{11}a_{22}+|a_{12}|^2=a_{11}(1-a_{11})+|a_{12}|^2=0$ by (8), and $a_{11}^2+a_{22}^2+2|a_{12}|^2=(a_{11}+a_{22})^2-2a_{11}a_{22}+2|a_{12}|^2=1+2(|a_{12}|^2-a_{11}a_{22})$) and hence is idempotent, and C satisfies $w(C^k)\leqslant w(A^k)=w(A)=w(B)$ for $1\leqslant k\leqslant 4$. $\square$

In the preceding theorem, the condition $|\langle A^kx,x\rangle|=w(A^k)=w(A)$ for k up to 4 cannot be further reduced to "3" in general as the next example shows.

PROPOSITION 4.4. *If*

$$A=\begin{bmatrix} 2 & \sqrt{2} & 0\\ -\sqrt{2} & -1 & \varepsilon\\ 0 & \varepsilon & 0\end{bmatrix},$$

where $\varepsilon=1/100$, *and* $x=[1\ 0\ 0]^T$, *then* $\langle A^kx,x\rangle=w(A^k)=w(A)$ *for* $1\leqslant k\leqslant 3$, *and* A *is irreducible.*

Proof. It can be shown by tedious computations, which we omit, that $\mathrm{Re}\,(e^{i\theta}A^k)\leqslant 2I_3$ for any real θ and any k, $1\leqslant k\leqslant 3$. Moreover, since

$$\mathrm{Re}A=\begin{bmatrix} 2 & 0 & 0\\ 0 & -1 & \varepsilon\\ 0 & \varepsilon & 0\end{bmatrix},\ \mathrm{Re}\,(A^2)=\begin{bmatrix} 2 & 0 & 0\\ 0 & -1+\varepsilon^2 & -\varepsilon\\ 0 & -\varepsilon & \varepsilon^2\end{bmatrix},\ \text{and}$$

$$\mathrm{Re}\,(A^3)=\begin{bmatrix} 2 & 0 & 0\\ 0 & -1-2\varepsilon^2 & -\varepsilon(1-\varepsilon^2)\\ 0 & -\varepsilon(1-\varepsilon^2) & -\varepsilon^2\end{bmatrix},$$

we have $\|\mathrm{Re}\,(A^k)\|=2$ for $1\leqslant k\leqslant 3$. Combining these together yields $w(A^k)=2$ for $1\leqslant k\leqslant 3$ (cf. Proposition 1.1 (d)). The irreducibility of A can be verified as before. $\square$

Note that for a 3-by-3 matrix, its irreducibility also implies that it is not unitarily similar to a multiple of an idempotent matrix (cf. Lemma 3.1 (e)).

For 2-by-2 matrices, the critical number can indeed be reduced to "3" as shown in the following proposition. Its proof is modeled after that of Theorem 4.2.

PROPOSITION 4.5. *For a* 2*-by-*2 *matrix* A, *the existence of a unit vector* x *in* $\mathbb{C}^2$ *such that* $|\langle A^kx,x\rangle|=w(A^k)=w(A)$ *for* $1\leqslant k\leqslant 3$ *is equivalent to the unitary similarity of* A *to* $\lambda\begin{bmatrix} 1 & 0\\ 0 & a\end{bmatrix}$ *or to* ηB, *where* $|\lambda|=|\eta|=1$, $|a|\leqslant 1$, *and* B *is idempotent.*

Proof. To prove the necessity, we may assume, as in the proof of Theorem 4.2, that $A=\begin{bmatrix} a_{11} & a_{12}\\ -\bar{a}_{12} & a_{22}\end{bmatrix}$ with $a_{11}=w(A)$. If $a_{12}=0$, then $a_{11}=1$ and $|a_{22}|\leqslant 1$ as before. On the other hand, if $a_{12} \neq 0$, then $a_{11}+a_{22}=1$ and $a_{11}a_{22}+|a_{12}|^2=0$. Thus A is unitarily similar to $\begin{bmatrix} 1 & \sqrt{2(|a_{12}|^2-a_{11}a_{22})} \\ 0 & 0 \end{bmatrix}$, which is easily seen to be idempotent. □

That, for 2-by-2 matrices, the number "3" cannot be further reduced to "2" is seen by the example below.

PROPOSITION 4.6. *If* $A = \begin{bmatrix} a & \sqrt{a(a-1)} \\ -\sqrt{a(a-1)} & 0 \end{bmatrix}$, *where* $a=\sqrt{5}/2$, *and* $x=[1\ 0]^T$, *then* $\langle A^k x, x\rangle = w(A^k) = a$ *for* $k=1,2$, *and* A *is irreducible and is not a multiple of an idempotent matrix.*

Proof. It can be computed that the eigenvalues of A and

$$A^2 = \begin{bmatrix} a & a\sqrt{a(a-1)} \\ -a\sqrt{a(a-1)} & -a(a-1) \end{bmatrix}$$

are $(a \pm \sqrt{-3a^2+4a})/2$ and $\big(-a(a-2) \pm \sqrt{a^3(-3a+4)}\big)/2$, respectively. For $a = \sqrt{5}/2$, these are all real numbers. Hence, by Proposition 1.1 (e),

$$w(A) = \|\mathrm{Re}A\| = \left\| \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\| = a = \frac{\sqrt{5}}{2}$$

and

$$w(A^2) = \|\mathrm{Re}\,(A^2)\| = \left\| \begin{bmatrix} a & 0 \\ 0 & -a(a-1) \end{bmatrix} \right\| = a = \frac{\sqrt{5}}{2}.$$

The irreducibility of A can be easily verified as before. □

5. Invertible matrices and operators

In this final section, we consider invertible matrices or operators A for which the norms (resp., numerical radii) of powers of both A and A^{-1} are constant. For invertible matrices, all is well as demonstrated in the following theorem.

THEOREM 5.1. *For an invertible* n*-by-*n *matrix* A, *the following conditions are equivalent:*

(a) $\|A^k\| = c$ *for all* $k=\pm 1, \pm 2, \ldots$;

(a′) $w(A^k) = c'$ *for all* $k=\pm 1, \pm 2, \ldots$;

(b) $\|A^k\| = c$ *and* $\|A^{-k}\| = d$ *for all* $k \geqslant 1$;

(b′) $w(A^k) = c'$ *and* $w(A^{-k}) = d'$ *for all* $k \geqslant 1$;

(c) $\|A^k\| = c$ *and* $\|A^{-k}\| \leqslant d$ *for all* $k \geqslant 1$;

(c′) *$w(A^k) = c'$ and $w(A^{-k}) \leqslant d'$ for all $k \geqslant 1$;*

(d) *A is unitary.*

For the proof, we need the next lemma, which is based on a classical result of Kronecker's.

LEMMA 5.2. *Let A be an n-by-n matrix.*

(a) *If A is power-bounded, then there is a sequence of positive integers k_j, $j \geqslant 1$, and an idempotent matrix B such that $\lim_j A^{k_j} = B$.*

(b) *If A is invertible and both A and A^{-1} are power-bounded, then there are positive integers k_j, $j \geqslant 1$, such that $\lim_j A^{k_j} = \lim_j A^{-k_j} = I_n$.*

Proof.

(a) Let $A = X^{-1}JX$, where X is invertible and $J = \left(\sum_{i=1}^{m} \oplus J_{n_i}(\lambda_i)\right) \oplus C$, where $|\lambda_i| = 1$ for all i and $\rho(C) < 1$, is the Jordan canonical form of A. If $n_i > 1$ for some i, then

$$J_{n_i}(\lambda_i)^k = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & & * \\ & \lambda_i^k & \ddots & \\ & & \ddots & k\lambda_i^{k-1} \\ & & & \lambda_i^k \end{bmatrix}$$

and hence $\|J_{n_i}(\lambda_i)^k\| \geqslant k|\lambda_i^{k-1}| = k$ for any $k \geqslant 1$. This leads to

$$\|A^k\| \geqslant \frac{\|J^k\|}{\|X\|\|X^{-1}\|} \geqslant \frac{\|J_{n_i}(\lambda_i)^k\|}{\|X\|\|X^{-1}\|} \geqslant \frac{k}{\|X\|\|X^{-1}\|},$$

for all k, which contradicts the power-boundedness of A. Hence $n_i = 1$ for all i, $1 \leqslant i \leqslant m$, and $J = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \oplus C$. Moreover, Kronecker's theorem says that there are positive integers k_j, $j \geqslant 1$, such that $\lim_j \lambda_i^{k_j} = 1$ for all i (cf. [14, Lemma 2.2]). Since $\rho(C) < 1$, C is similar to a matrix D with $\lim_k D^k = 0_{n-m}$ (cf. [4, Problem 153]). Thus we also have $\lim_k C^k = 0_{n-m}$. Combining these together yields that $\lim_j J^{k_j} = I_m \oplus 0_{n-m}$. Thus A^{k_j} converges to the idempotent $B \equiv X^{-1}(I_m \oplus 0_{n-m})X$ as j approaches infinity.

(b) If A is power-bounded, then Gelfand's formula $\rho(A) = \lim_k \|A^k\|^{1/k}$ implies that $\rho(A) \leqslant 1$. Similarly, the power-boundedness of A^{-1} yields that $\rho(A^{-1}) \leqslant 1$. Thus all the eigenvalues of A have moduli equal to 1. Then the arguments in (a) show that the Jordan form J of A is a diagonal unitary matrix. Hence Kronecker's theorem yields the existence of positive integers k_j, $j \geqslant 1$, such that J^{k_j} together with A^{k_j} converges to I_n. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Since the implications (a)⇒(b)⇒(c), (a′)⇒(b′)⇒(c′), and (d)⇒(a), (a′) are trivial, we need only prove (c)⇒(d) and (c′)⇒(d).

Note that, under (c), both A and A^{-1} are power-bounded. Thus Lemma 5.2 (b) says that $\lim_j A^{k_j} = I_n$ for some positive integers k_j, $j \geqslant 1$. Hence $c = \lim_j \|A^{k_j}\| = \|I_n\| = 1$. In particular, we have $\|A\| = 1$. As noted in the proof of Lemma 5.2 (b), all eigenvalues of A have moduli equal to 1. If A is unitarily similar to the upper-triangular $A' = [a_{ij}]_{i,j=1}^n$, then $|a_{ii}| = 1$ for all i. Moreover,

$$1 = \|A\| \geqslant \Big(\sum_{j=i}^{n} |a_{ij}|^2\Big)^{1/2} \geqslant |a_{ii}| = 1,$$

for all i yields that $a_{ij} = 0$ for all $i < j$. Hence A' is diagonal unitary and A is unitary.

Now we assume that (c′) holds. Since $\|A^k\| \leqslant 2w(A^k) = 2c'$ and $\|A^{-k}\| \leqslant 2w(A^{-k}) \leqslant 2d'$, for all $k \geqslant 1$ (cf. Proposition 1.1 (b)), A and A^{-1} are both power-bounded with eigenvalues all contained in $\partial\mathbb{D}$. As before, apply Lemma 5.2 (b) to obtain that $c' = 1$. Let λ_1 be an eigenvalue of A. Since $|\lambda_1| = 1$ and $w(A) = 1$, λ_1 is in $\partial W(A)$. Hence Proposition 1.1 (c) implies that A is unitarily similar to a matrix of the form $[\lambda_1] \oplus A_1$. We may repeat the above arguments to A_1 to obtain the unitary similarity of A_1 to $[\lambda_2] \oplus A_2$, where λ_2 is another eigenvalue of A. By induction, A is unitarily similar to a diagonal unitary matrix and hence is itself unitary. □

Finally, we turn to the case when A acts on an infinite-dimensional space H. Some of the tools we used in proving Theorem 5.1 and Lemma 5.2 remain true. For example, instead of relying on the Jordan form of a matrix, we have available a result of Sz.-Nagy's [11, Theorem 1]: *An invertible operator A with A and A^{-1} both power-bounded is similar to a unitary operator*. Using Kronecker's theorem, Wermer obtained in [15, Theorem 6] that *if A is a diagonal unitary operator*, *then there are positive integers k_j, $j \geqslant 1$, such that A^{k_j} converges to I in the strong operator topology*. Another result relevant to our present discussions is the one by Stampfli: *For an invertible operator A, the following conditions are equivalent*: (a) $\|A\|, \|A^{-1}\| \leqslant 1$, (b) $\|A\|, w(A^{-1}) \leqslant 1$, (c) $w(A), \|A^{-1}\| \leqslant 1$, (d) $w(A), w(A^{-1}) \leqslant 1$, *and* (e) *$A$ is unitary* (cf. [9, Corollaries 1 and 2 of Theorem 1] or [10, Corollary 4 of Theorem 2]). Unfortunately, despite all such supporting evidences, Theorem 5.1 per se is not true for an infinite-dimensional A. This is shown in our final theorem.

THEOREM 5.3. *For any $\varepsilon > 0$, there is an invertible operator A on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ such that $\|A^k\| = \|A^{-k}\| = 1+\varepsilon$ (resp., $w(A^k) = w(A^{-k}) = 1+\varepsilon$) for all $k \geqslant 1$. In particular, A is not unitary.*

Proof. We first consider the constant norm case. Let $a = 1/(1+\varepsilon)$ and let $A' = [a_{ij}]_{i,j=-\infty}^{\infty}$ on $\ell^2(\mathbb{Z})$ be such that

$$a_{ij} = \begin{cases} a, & \text{if } (i,j) = (0,1), \\ 1, & \text{if } j = i+1 \text{ and } (i,j) \neq (0,1), \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Then A' is invertible and A'^k, denoted by $[a_{ij}^{(k)}]_{i,j=-\infty}^{\infty}$ for $k \geqslant 1$, is given by

$$a_{ij}^{(k)} = \begin{cases} a, & \text{if } (i,j) = (0,k), (-1,k-1), \ldots, \big(-(k-1),1\big), \\ 1, & \text{if } j = i+k \text{ and } i \neq 0, -1, \ldots, -(k-1), \\ 0, & \text{otherwise.} \end{cases}$$

As $0 < a < 1$ and 1 appears as entries of A'^k, it is easily seen that $\|A'^k\| = 1$ for all $k \geqslant 1$. On the other hand, A'^{-1*} has the same matrix form as A' except that its (0,1)-entry is $1/a$ instead of a. Thus, as above, we easily obtain that $\|A'^{-k}\| = \|A'^{-k*}\| = 1/a$ for all k. Letting $A = A' \oplus A'^{-1}$ on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, we have $\|A^k\| = \|A^{-k}\| = 1/a = 1+\varepsilon$ for all $k \geqslant 1$.

For the constant numerical radii, we let $a = 1+\varepsilon - \sqrt{\varepsilon(\varepsilon+2)}$. Then $0 < a < 1$. Let A' be as in (11). Note that A'^k, $k \geqslant 1$, is unitarily similar to the direct sum of k copies of A'. Hence $w(A'^k) = w(A') = 1$ and $w(A'^{-k}) = w(A'^{-1}) = (a^2+1)/(2a) = 1+\varepsilon$ for $k \geqslant 1$ (cf. [13, Theorem 4.9]). As before, $A \equiv A' \oplus A'^{-1}$ is the operator on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ satisfying $w(A^k) = w(A^{-k}) = 1+\varepsilon$ for all $k \geqslant 1$. $\square$

A final remark: in line with Sz.-Nagy's result, the operator A in the preceding theorem is similar to the unitary operator $W \oplus W$, where $W = [w_{ij}]_{i,j=-\infty}^{\infty}$ is the simple bilateral shift given by $w_{ij} = 1$ if $i = j+1$, and $w_{ij} = 0$ otherwise. This is seen by $(XY)^{-1}A'(XY) = W$ and $X^{-1}A'^{-1}X = W$, where X is the diagonal operator $\operatorname{diag}(\ldots,1,1,\underline{1},1/a,1/a,\ldots)$ on $\ell^2(\mathbb{Z})$ with its (0, 0)-entry underlined, and $Y = [y_{ij}]_{i,j=-\infty}^{\infty}$ is such that $y_{ij} = 1$ if $i+j = -1$, and $y_{ij} = 0$ otherwise.

Acknowledgement. We thank Chi-Kwong Li for many stimulating inputs to our work for this paper. The three authors were all supported by Ministry of Science and Technology of Republic of China under projects MOST 108-2115-M-008-006 (H.-L. Gau), MOST 108-2115-M-009-001 (K.-Z. Wang), and MOST 108-2115-M-009-014 (P. Y. Wu), respectively.

We also thank the referee for pointing out two related conjectures, which we have incorporated into the text, together with a number of errors and typos.

REFERENCES

[1] V. CHKLIAR, *Numerical radii of simple powers*, Linear Algebra Appl., **265** (1997), 119–121.

[2] M. A. DRITSCHEL AND H. J. WOERDEMAN, *Model theory and linear extreme points in the numerical radius unit ball*, Mem. Amer. Math. Soc., **129** (1997).

[3] M. GOLDBERG, E. TADMORE AND G. ZWAS, *The numerical radius and spectral matrices*, Linear Multilinear Algebra, **2** (1975), 317–326.

[4] P. R. HALMOS, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.

[5] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[6] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013.

[7] C.-K. LI, B.-S. TAM AND P. Y. WU, *The numerical range of a nonnegative matrix*, Linear Algebra Appl., **350** (2002), 1–23.

[8] V. PTÁK, *Lyapunov equations and Gram matrices*, Linear Algebra Appl., **49** (1983), 33–55.

[9] J. G. STAMPFLI, *Minimal range theorems for operators with thin spectra*, Pacific J. Math., **23** (1967), 601–612.
[10] J. G. STAMPFLI, *A local spectral theory for operators*, J. Func. Anal., **4** (1969), 1–10.
[11] B. SZ.-NAGY, *On uniformly bounded linear transformations in Hilbert space*, Acta Sci. Math. (Szeged), **11** (1947), 152–157.
[12] S.-H. TSO AND P. Y. WU, *Matricial ranges of quadratic operators*, Rocky Mountain J. Math., **29** (1999), 1139–1152.
[13] K.-Z. WANG AND P. Y. WU, *Numerical ranges of weighted shifts*, J. Math. Anal. Appl., **381** (2011), 897–909.
[14] K.-Z. WANG, P. Y. WU AND H.-L. GAU, *Crawford numbers of powers of a matrix*, Linear Algebra Appl., **433** (2010), 2243–2254.
[15] J. WERMER, *On invariant subspaces of normal operators*, Proc. Amer. Math. Soc., **3** (1952), 270–277.

(Received August 7, 2018)

Hwa-Long Gau
Department of Mathematics
National Central University
Chungli 32001, Taiwan
e-mail: hlgau@math.ncu.edu.tw

Kuo-Zhong Wang
Department of Applied Mathematics
National Chiao Tung University
Hsinchu 30010, Taiwan
e-mail: kzwang@math.nctu.edu.tw

Pei Yuan Wu
Department of Applied Mathematics
National Chiao Tung University
Hsinchu 30010, Taiwan
e-mail: pywu@math.nctu.edu.tw