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VECTOR VALUED FOURIER ANALYSIS ON HYPERGROUPS

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Abstract. The aim of this paper is to prove the vector-valued version of the classical Hausdorff-Young inequality for commutative hypergroups and compact hypergroups.

1. Introduction

Let K be a commutative hypergroup with dual object $\widehat{K}$. S. Degenfeld-Schonburg [2, 3] proved that the Fourier transform is a bounded linear transform from $L^p(K)$ into $L^{p'}(\widehat{K})$, where $1 \leqslant p \leqslant 2$ and p' is the conjugate exponent of p. This theorem is known as the Hausdorff-Young inequality. In the case of compact hypergroups, the above said inequality was verified by R.C. Vrem [12] in 1978 (see also [7]). This paper extends the results of Degenfeld-Schonburg and Vrem to vector-valued functions. In 1984, Milman [9] generalized the classical Hausdorff-Young inequality to Banach-valued functions. This leads to the theory of Fourier type of a Banach space with respect to a locally compact abelian group. Recently, J. García-Cuerva and J. Parcet [5], generalized Milmam's result to compact groups.

In Section 2 of this paper, we define the notion of Fourier type with respect to a commutative hypergroup and hence establish a vector-valued analogue of the Hausdorff-Young inequality for commutative hypergroups. As the dual object of a commutative hypergroup is, in general, not a hypergroup, the notion of Fourier cotype is also introduced in Section 2. Note that, this notion doesn't make sense if the hypergroup is a locally compact abelian group. We also show that the two notions, Fourier type and Fourier cotype are dual to each other. In Section 4 of this paper, we prove the vector-valued analogue of the Hausdorff-Young inequality for compact hypergroups. As a result, the notions of Fourier type and Fourier cotype with respect to a compact hypergroup are introduced. We also show that the notion of Fourier type and Fourier cotype introduced in Section 2 and Section 4 are one and the same if K is a compact commutative hypergroup.

It is well-known that operator spaces and completely bounded maps play a major role in non-commutative harmonic analysis. Even in this paper, in the case of compact non-commutative hypergroups, one has to define norms for vector-valued matrices. So

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it is very natural to consider the vector spaces to be operator spaces rather than just Banach spaces. If one notices, the authors of [5] have used only the operator spaces in order to develop the theory of Fourier type for compact groups. In a similar spirit, we have also used the theory of operator spaces in Section 4. In this regard, in Section 3 of this paper we give some basic definitions and notions of operator spaces that are needed in Section 4.

For any undefined notations or definitions on hypergroups, the reader is asked to refer [1] or [6].

2. Vector-valued Fourier transform on commutative hypergroup

We start with the definition of a hypergroup. In [6], Jewett refers to hypergroups as convos.

DEFINITION 2.1. [1, 6] A nonempty locally compact Hausdorff space K is said to be a *hypergroup* if there exists a binary operation $*$ on $M(K)$, the space of all complex valued bounded regular measures on K, satisfying the following conditions:

(i) $(M(K),*)$ is a complex associative algebra.

(ii) For every $x,y\in K$, $p_x * p_y$ is a probability measure with the compact support and the mapping $(x,y)\mapsto p_x*p_y$ is continuous from $K\times K$ to $M(K)$, where p_x is the point mass measure at x.

(iii) There exists a unique element $e\in K$ such that for all $x\in K$, $p_x*p_e=p_e*p_x=p_x$.

(iv) There exists a unique homeomorphism $x\mapsto \check{x}$ of K such that

  (a) $\check{\check{x}}=x$ for all $x\in K$,

  (b) if $\check{\mu}$ is defined by $\int_K f(x)\,d\check{\mu}(x)=\int_K f(\check{x})\,d\mu(x)$ for all $f\in C_c(K)$, then $(p_x*p_y)\check{}=p_{\check{y}}*p_{\check{x}}$ for all $x,y\in K$,

  (c) $e\in \operatorname{spt}(p_x*p_y)$ if and only if $y=\check{x}$.

(v) The mapping $(x,y)\mapsto \operatorname{spt}(p_x*p_y)$ is continuous from $K\times K$ to $\mathscr{C}(K)$, where $\mathscr{C}(K)$ denotes the space of all nonempty compact subsets of K equipped with the Michael topology (See [8]).

NOTE. *It follows from the definition that the bilinear map $(\mu,\nu)\mapsto\mu*\nu$, restricted to the space of non-negative measures, is weakly continuous.*

DEFINITION 2.2. A *left Haar measure* on a hypergroup K is a non zero regular Borel measure m such that $p_x*m=m$ for all $x\in K$. In this note, by a Haar measure we mean a left Haar measure.

It is well known that commutative and compact hypergroups admit a Haar measure. In fact, a Haar measure on a hypergroup (if exists) is unique upto a scalar multiple [6].

Throughout this section, K will stand for a commutative hypergroup and X will denote a Banach space. We now proceed by recalling the notions related to the Fourier analysis of a commutative hypergroup.

2.1. Fourier analysis on commutative hypergroups

Since K is commutative, it possesses a Haar measure, unique upto a scalar. We shall once for all fix dm as the Haar measure for K. Then, it is well known that $L^1(K)$ becomes a commutative Banach $*$-algebra. Denote the space of all bounded continuous complex valued functions on K by $C^b(K)$. Let

$$\Upsilon_b(K) = \{\chi \in C^b(K) : \chi(e) = 1 \text{ and } \chi(x * y) = \chi(x)\chi(y) \ \forall\, x, y \in K\}.$$

Equip $\Upsilon_b(K)$ with the compact-open topology. By [1, Theorem 2.2.2], the structure space of $L^1(K)$ can be identified with $\Upsilon_b(K)$. Let

$$\widehat{K} = \{\chi \in \Upsilon_b(K) : \chi(\check{x}) = \overline{\chi(x)} \ \forall\, x \in K\}.$$

Equip $\widehat{K}$ also with the compact-open topology. As mentioned in [1, Example 2.2.49] the set $\Upsilon_b(K)$ need not be equal to $\widehat{K}$. Further, note that $\widehat{K}$ need not possess a hypergroup structure. For example, see [6, Example 9.3C].

The Fourier transform of $f \in L^1(K)$ is defined by

$$\mathscr{F}_K(f)(\chi) = \widehat{f}(\chi) = \int_K f(x)\overline{\chi(x)}\, dm(x) \ \forall\, \chi \in \widehat{K}. \tag{1}$$

By [1, Theorem 2.2.4], the mapping $f \mapsto \mathscr{F}_K(f)$ is a norm-decreasing $*$-algebra homomorphism from $L^1(K)$ into $L^\infty(\widehat{K})$. Furthermore, $\mathscr{F}_K(f)$ vanishes at infinity. Also, as in the case of locally compact abelian groups, there exists a unique positive Borel measure π_K on $\widehat{K}$ such that

$$\int_K |f(x)|^2\, dx = \int_{\widehat{K}} |\mathscr{F}_K(f)(\chi)|^2\, d\pi_K(\chi) \ \forall\, f \in L^2(K) \cap L^1(K).$$

In fact, the Fourier transform extends to a unitary operator from $L^2(K)$ onto $L^2(\widehat{K})$. We would like to remark here that the support of π_K, denoted $\mathscr{S}$, need not be equal to $\widehat{K}$. See [1, Example 2.2.49].

2.2. Fourier type w.r.t. a commutative hypergroup

DEFINITION 2.3. Let $1 \leqslant p \leqslant 2$ and let p' be the conjugate exponent of p. We say that X has Fourier type p with respect to K if the operator $\mathscr{F}_K \otimes id_X : L^p(K) \otimes X \to L^{p'}(\mathscr{S}) \otimes X$ defined by $\mathscr{F}_K \otimes id_X(f \otimes a) = \mathscr{F}_K(f) \otimes a$, for all $f \in L^p(K), a \in X$, can be extended to a bounded linear operator $\mathscr{F}_{K,X}$ from $L^p(K,X)$ to $L^{p'}(\mathscr{S},X)$.

PROPOSITION 2.4. *Let K be a commutative hypergroup and let X be a Banach space. Then X has Fourier type 1 with respect to K.*

Proof. Let $f \in L^1(K,X)$. Then

$$\begin{aligned}\|\mathscr{F}_{K,X}(f)(\chi)\|_X &= \left\| \int_K f(x)\overline{\chi(x)}\, dm(x) \right\|_X \leqslant \int_K \|f(x)\|_X |\overline{\chi(x)}|\, dm(x) \\ &\leqslant \sup_{x \in K} |\overline{\chi(x)}| \|f\|_{L^1(K,X)} = \|f\|_{L^1(K,X)}.\end{aligned}$$

By taking supremum over all $\chi \in \mathscr{S}$, we get $\|\mathscr{F}_{K,X} f\| \leqslant \|f\|_{L^1(K,X)}$. Therefore, X has Fourier type 1. $\square$

PROPOSITION 2.5. *Let K be a commutative hypergroup and let $\mathscr{H}$ be a Hilbert space. Then $\mathscr{H}$ has Fourier type 2 with respect K.*

Proof. The proof of this proposition follows using the density of $L^2(K)\otimes\mathscr{H}$ in $L^2(K,\mathscr{H})$ and the Plancherel's theorem for scalar case. $\square$

The following corollary says that when p get closer to 2, we have stronger conditions on Fourier type.

COROLLARY 2.6. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant 2$ and let the Banach space X have Fourier type p_2 with respect to a commutative hypergroup K. Then X has Fourier type p_1 with respect to K.*

Proof. The proof follows from Proposition 2.4 and complex interpolation. $\square$

2.3. Fourier cotype w.r.t. a commutative hypergroup

For a locally compact abelian group G, the dual object $\widehat{G}$ is also a locally compact abelian group. Therefore, using the Pontrjagin duality, we can see that the inverse Fourier transform $\mathscr{F}_G^{-1}$ and the Fourier transform $\mathscr{F}_{\widehat{G}}$ are one and the same. In the vector valued setting, it is enough to study $\mathscr{F}_{\widehat{G},X}$ instead of $\mathscr{F}_{G,X}^{-1}$. For this reason, we do not have any other Fourier type of a Banach space X with respect to G. But, as observed earlier, the dual $\widehat{K}$ of a commutative hypegroup K is not a hypergroup. That is why, the study of $\mathscr{F}_K\otimes id_X$ is not enough for the study of the operator $\mathscr{F}_K^{-1}\otimes id_X$. For this reason, the need for a notion like Fourier cotype with respect to K appears, which does not exist for a locally compact abelian group.

We shall now begin to define the notion of a Fourier cotype of a Banach space with respect to K.

DEFINITION 2.7. Let $1 \leqslant p \leqslant 2$ and let p' be the conjugate exponent of p. The Banach space X has Fourier cotype p' with respect to the commutative hypergroup K if the operator $\mathscr{F}_K^{-1}\otimes id_X : L^p(\mathscr{S})\otimes X \to L^{p'}(K)\otimes X$ defined by $\mathscr{F}_K^{-1}\otimes id_X(f\otimes a) = \mathscr{F}_K^{-1}(f)\otimes a$, for all $f\in L^p(\mathscr{S})$, $a \in X$, can be extended to a bounded operator $\mathscr{F}_{K,X}^{-1}$ from $L^p(\mathscr{S},X)$ into $L^{p'}(K,X)$.

PROPOSITION 2.8. *Every Banach space X has Fourier cotype ∞ with respect to K.*

Proof. The proof of this is same as the proof of Proposition 2.4. $\square$

We also have the following proposition, whose proof is a routine check.

PROPOSITION 2.9. *Let X be a Hilbert space. Then X has Fourier cotype 2 with respect to K.*

Using the complex interpolation we have the following result as a consequence of Proposition 2.8.

COROLLARY 2.10. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant 2$ and let the Banach space X have Fourier type p_2' with respect to K. Then X has Fourier type p_1' with respect to K.*

Now, we show that Fourier type and Fourier cotype are dual notions. This is the *Vector-valued analogue of the Hausdorff-Young inequality* for commutative hypergroups.

THEOREM 2.11. *Let $1 < p < 2$ and p' be its conjugate exponent.*

(i) The Banach space X has Fourier type p with respect to K iff X^* has Fourier cotype p' with respect to K.

(ii) The Banach space X has Fourier cotype p' with respect to K iff X^* has Fourier type p with respect to K.

Proof. We shall prove only (i) as the proof of (ii) will follow similarly. Again, in order to prove (i), it is enough to show only one side as the proof of the other side follows similar lines. So, let $f \in L^p(\mathscr{S}, X)$. As $L^{p'}(K, X^*)$ and $L^p(K, X)^*$ are isometrically isomorphic to each other, for a given $\varepsilon > 0$, $\exists\, g \in L^p(K, X)$ of norm one such that

$$\begin{aligned}
\|\mathscr{F}^{-1}_{K,X^*}(f)\|_{L^{p'}(K,X^*)} &\leqslant (1+\varepsilon)\left|\int_K \mathscr{F}^{-1}_{K,X^*}(f)(x)g(x)\, dm(x)\right| \\
&= (1+\varepsilon)\left|\int_{\mathscr{S}} f(\chi)\widehat{g}(\chi)\, d\pi_K(\chi)\right| \\
&\leqslant (1+\varepsilon)\int_{\mathscr{S}} |f(\chi)\widehat{g}(\chi)|\, d\pi_K(\chi) \leqslant \|f\|_{L^{p'}(K,X^*)}\|\widehat{g}\|_{L^p(\widehat{K},X)}.
\end{aligned}$$

As, X has Fourier type p, the proof follows. □

3. Operator spaces

For dealing with noncommutative hypergroups in the next section, we need to make an extensive use of the theory of operator spaces. The books of Effros and Ruan [4] and Pisier [10] are basic references on this topic. The aim of this section is to recall some basic notations and definitions pertaining to operator spaces and the norms on the their corresponding matrix levels.

3.1. Operator spaces and completely bounded maps

An operator space is a closed subspace of $\mathscr{B}(\mathscr{H})$ where $\mathscr{B}(\mathscr{H})$ denotes the space of all bounded linear operators on the Hilbert space $\mathscr{H}$. It follows from GNS construction and the above definition that every C^*-algebra is an operator space. In particular, if $(\Omega,\mathscr{A},\mu)$ is a σ-finite measure space, then $L^\infty(\Omega,\mathscr{A},\mu)$ is an operator space. Further, closed subspaces, duals and preduals of operator spaces are also operator spaces. Therefore, $L^1(\Omega,\mathscr{A},\mu)$ is also an operator space.

Given an operator space $X \subset \mathscr{B}(\mathscr{H})$, we shall denote by $M_n(X)$ the space of all $n\times n$ matrices with entries from X and equipped with the norm arising from the natural embedding of $M_n(X)$ inside $\mathscr{B}\left(\overset{n}{\underset{i=1}{\oplus}}\mathscr{H}\right)$.

Let X and Y be any two operator spaces and let $\varphi : X \to Y$ be a linear transformation. For any $n \in \mathbb{N}$, the n^{th}-amplification of φ, denoted φ_n, is defined as a linear transformation $\varphi_n : M_n(X) \to M_n(Y)$ given by $\varphi_n([x_{ij}]) := [\varphi(x_{ij})]$. The linear transformation φ is said to be completely bounded if $\sup\{\|\varphi_n\| | n \in \mathbb{N}\} < \infty$. We shall denote by $\mathscr{CB}(X,Y)$ the space of all completely bounded linear mappings from X to Y equipped with the cb-norm, denoted $\|\cdot\|_{cb}$,

$$\|\varphi\|_{cb} := \sup\{\|\varphi_n\| | n \in \mathbb{N}\},\ \varphi \in \mathscr{CB}(X,Y).$$

We shall say that φ is a complete isometry if φ_n is an isometry $\forall\, n \in \mathbb{N}$. Similarly, we shall say that φ is a complete isomorphism if φ is an isomorphism such that both φ and φ^{-1} are completely bounded.

Let $\{X_0,X_1\}$ be a compatible couple of Banach spaces in the sense of complex interpolation. Suppose that X_0 and X_1 are also operator spaces. If X_θ, $0 < \theta < 1$, denotes the interpolation space $[X_0,X_1]_\theta$, then, by [10, Pg. 53], X_θ is also an operator space. Further, if $\{Y_0,Y_1\}$ is another compatible couple of Banach spaces which are also operator spaces and if $\varphi : X_0 + X_1 \to Y_0 + Y_1$ is such that $\|\varphi\|_{cb(X_o,Y_0)} \leqslant c_0$ and $\|\varphi\|_{cb(X_1,Y_1)} \leqslant c_1$, then for $0 < \theta < 1$, we have $\|\varphi\|_{cb(X_\theta,Y_\theta)} \leqslant c_0^{1-\theta} c_1^{\theta}$.

Given two operator spaces $X \subseteq \mathscr{B}(\mathscr{H})$ and $Y \subseteq \mathscr{B}(\mathscr{K})$, we define their minimal tensor product, denoted $X \otimes_{min} Y$, as the completion of their algebraic tensor product $X \otimes Y$ inside $\mathscr{B}(\mathscr{H} \otimes_2 \mathscr{K})$, where $\otimes_2$ denotes the Hilbert space tensor product. It is worth noting that, if X is an operator space, then $M_n \otimes_{min} X$ and $M_n(X)$ are completely isometric. This tensor product is the operator space analogue of the Banach space injective tensor product.

Similarly, just as we have projective tensor product for Banach spaces, there exists projective tensor product for operator spaces. If X and Y are operator spaces, then we shall denote by $X\widehat{\otimes}Y$ the operator space projective tensor product. For more details, see [4, 10]. It is worth noting that the dual of $X\widehat{\otimes}Y$ is completely isometrically isomorphic to $\mathscr{CB}(X,Y^*)$.

3.2. Schatten class operators

The space of p-Schatten class operators are the non-commutative analogues of the classical $\ell^p(n)$ spaces. The space of Schatten p-class operators, denoted S_n^p, is defined as the space M_n of $n \times n$ complex matrices equipped with the norm given by

$$\|A\|_{S_n^p} := \begin{cases} (tr(|A|^P))^{1/p}, & \text{if } 1 \leqslant p < \infty, \\ \sup\{\|Ax\|_{\ell^2(n)} : \|x\|_{\ell^2(n)} \leqslant 1\}, & \text{if } \quad p = \infty. \end{cases}$$

More generally, if X is an operator space, then, for $1 \leqslant p \leqslant \infty$, we define the operator spaces $S_n^p(X)$ as follows. If $p = \infty$, then $S_n^p(X)$ is defined as the operator space $M_n(X) (\cong M_n \otimes_{min} X)$. If $p = 1$, then $S_n^p(X)$ is defined as the operator space $S_n^1 \widehat{\otimes} X$. Here S_n^1 is given the operator space structure coming from the isometric identification $(S_n^1)^* \cong S_n^\infty$. If $1 < p < \infty$, then the operator space structure on $S_n^p(X)$ is defined by means of the complex interpolation, i.e.,

$$S_n^p(X) = [S_n^1(X), S_n^\infty(X)]_{1/p}.$$

The next theorem summarizes some of the properties of these spaces. This theorem will be used in this paper repeatedly.

THEOREM 3.1. [11, Chapter 1]

(i) *Let $1 \leqslant p \leqslant \infty$. If X and Y are any two operator spaces, then the cb-norm of a completely bounded map $\varphi : X \to Y$ is equal to* $\sup\limits_{n \geqslant 1} \|I_{M_n} \otimes \varphi\|_{\mathscr{B}\left(S_n^p(X), S_n^p(Y)\right)}$.

(ii) *Let $1 \leqslant p \leqslant \infty$ and let X be an operator space. Then the dual of $S_n^p(X)$ is completely isometrically isomorphic to $S_n^{p'}(X^*)$, where p' is the conjugate exponent of p.*

(iii) *Let $1 \leqslant p \leqslant q \leqslant \infty$ and let X be an operator space. Then the identity mapping from $S_n^p(X)$ onto $S_n^q(X)$ is a complete contraction.*

(iv) *Let $1 \leqslant p \leqslant \infty$ and let X be an operator space. Let $n \geqslant 1$. Then,*

$$S_n^p\left(L^p(K,X)\right) \cong L^p\left(K, S_n^p(X)\right)$$

completely isometrically, where K denotes a hypergroup.

4. Fourier type and cotype with respect to a compact hypergroup

In this section, we assume that the hypergroup K is compact. Also, throughout this section, X will denote an operator space. We begin this section with the definition of the Fourier transform on K.

4.1. Fourier analysis on compact hypergroups

It is well known that K possesses a unique Haar measure dm such that $\int_K dm(x) = 1$. Further, a continuous irreducible representation of K is always finite-dimensional. Let $\widehat{K}$ denote the set of all continuous irreducible representations upto unitary equivalence, called as the dual object. The dual object $\widehat{K}$ is given the discrete topology. For $f \in L^1(K)$ and $\pi \in \widehat{K}$, the Fourier coefficient of f at π is defined as

$$\hat{f}(\pi) = \int_K f(x)\bar{\pi}(x)dm(x),$$

where $\bar{\pi}$ denotes the conjugate representation of K.

DEFINITION 4.1. For $1 \leqslant p \leqslant \infty$, define the space $\mathscr{L}^p(\widehat{K},X)$ as follows: for $1 \leqslant p < \infty$

$$\mathscr{L}^p(\widehat{K},X) := \left\{ A \in \prod_{\pi\in\widehat{K}} M_{d_\pi}(X) : \|A\|_{\mathscr{L}^p(\widehat{K},X)} = \left(\sum_{\pi\in\widehat{K}} k_\pi \|A^\pi\|_{S^p_{d_\pi}(X)} \right)^{\frac{1}{p}} < \infty \right\},$$

and for $p = \infty$

$$\mathscr{L}^\infty(\widehat{K},X) := \left\{ A \in \prod_{\pi\in\widehat{K}} M_{d_\pi}(X) : \|A\|_{\mathscr{L}^\infty(\widehat{K},X)} = \sup_{\pi\in\widehat{K}} \|A^\pi\|_{S^\infty_{d_\pi}(X)} < \infty \right\}.$$

If $X = \mathbb{C}$, then we write $\mathscr{L}^p(\widehat{K})$ for $\mathscr{L}^p(\widehat{K},X)$. Further, note that, it is necessary to have X to be an operator space.

We now present some of the properties of $\mathscr{L}^p(\widehat{K},X)$ spaces.

THEOREM 4.2.

(i) *The map* $\Lambda : \mathscr{L}^\infty(\widehat{K},X) \to cb(\mathscr{L}^1(\widehat{K}),X)$ *given by* $\Lambda(A) = \sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} (A^\pi_{ij}(\cdot))$ *is a completely isometric isomorphism.*

(ii) *The identity map from* $\mathscr{L}^1(\widehat{K})\hat{\otimes}X$ *to* $\mathscr{L}^1(\widehat{K},X)$ *is a complete isometric.*

(iii) *The mapping* $A \mapsto \sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} (A^\pi_{ij}(\cdot))$ *from* $\mathscr{L}^1(\widehat{K},X^*)$ *onto* $C_0(\widehat{K},X)^*$ *is a completely isometric isomorphism.*

(iv) *The above mapping is also a complete isometry from* $\mathscr{L}^p(\widehat{K},X^*)$ *onto* $\mathscr{L}^{p'}(\widehat{K},X)^*$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

(v) *(Plancherel) The Fourier transform is a unitary map from* $L^2(K)$ *onto* $\mathscr{L}^2(\widehat{K},\mathbb{C})$.

For more on compact hypergroups, we refer to [13].

4.2. Fourier type w.r.t. a compact hypergroup

DEFINITION 4.3. The *vector-valued Fourier coefficient* of $f \in L^1(K,X)$ is defined by

$$\widehat{f}(\pi) := \int_K f(x)\,\overline{\pi}(x)\,dm(x), \;\; \pi \in \widehat{K}.$$

It is clear that $\widehat{f}(\pi) \in M_{d_\pi}(X)$. Further, we note that the above operator-valued integral $\widehat{f}(\pi)$ is interpreted in the weak sense. Also, as $\|\pi(x)\| \leqslant 1$, $\forall x \in K$, the above operator-valued Fourier transform is well-defined and hence, we can write the Fourier transform operator, denoted $\mathscr{F}_{K,X}$, as

$$\mathscr{F}_{K,X} : L^1(K,X) \to \prod_{\pi\in\widehat{K}} M_{d_\pi}(X)$$

given by

$$\mathscr{F}_{K,X}(f)(\pi) = \widehat{f}(\pi).$$

DEFINITION 4.4. Let $1 \leqslant p \leqslant 2$ and $p' = \frac{p}{p-1}$. An operator space X is said to be of *Fourier type* p with respect to the compact hypergroup K if the Fourier transform $\mathscr{F}_{K,X} : L^p(K) \otimes X \to \mathscr{L}^{p'}(\widehat{K}) \otimes X$ can be extended to a completely bounded operator $\Lambda^1_{K,X,p} : L^p(\widehat{K}, X) \to \mathscr{L}^{p'}(\widehat{K}, X)$.

In this case, the completely bounded norm of $\Lambda^1_{K,X,p}$ is denoted by $\mathscr{C}^1_p(X,K)$.

The following two results are simple consequence of Property 1 of Theorem 3.1.

LEMMA 4.5. *Let X be an operator space and let F be a closed subspace of X then we have $\mathscr{C}^1_p(F,K) \leqslant \mathscr{C}^1_p(X,K)$ for any $1 \leqslant p \leqslant 2$.*

COROLLARY 4.6. *For $1 \leqslant p \leqslant 2$, we have $\mathscr{C}^1_p(X,K) \geqslant 1$.*

LEMMA 4.7. *For $f \in L^1(K,X)$, we have $\|\widehat{f}\|_{\mathscr{L}^\infty(\widehat{K},X)} \leqslant \|f\|_{L^1(K,X)}$.*

Proof. Since X is an operator space, there exists a Hilbert space $\mathscr{H}$ such that $X \subset \mathscr{B}(\mathscr{H})$. For $h = (h_1, h_2, \ldots, h_{d_\pi}) \in \ell^2_{\mathscr{H}}(d_\pi)$ we have

$$\begin{aligned}
\|\widehat{f}(\pi)\|_{S^\infty_{d_\pi}(X)} = \|\widehat{f}(\pi)\|_{\mathscr{B}(\ell^2_{\mathscr{H}}(d_\pi))} &= \left\| \left(\int_K f(x) \otimes \bar{\pi}(x)\, dm(x) \right) h \right\|_{\mathscr{B}(\ell^2_{\mathscr{H}}(d_\pi))} \\
&= \left\| \int_K f(x) \left(\sum_{j=1}^{d_\pi} \overline{\pi_{ij}(x)} h_j \right) dm(x) \right\|_{\mathscr{B}(\ell^2_{\mathscr{H}}(d_\pi))} \\
&\leqslant \sup_{\|h\|_{\ell^2_{\mathscr{H}}(d_\pi)} \leqslant 1} \left(\sum_{i=1}^{d_\pi} \left[\int_K \left\| f(x) \left(\sum_{j=1}^{d_\pi} \overline{\pi_{ij}(x)} h_j \right) \right\|_{\mathscr{H}} dm(x) \right]^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

By Minkowski inequality we get

$$\begin{aligned}
\|\widehat{f}(\pi)\|_{S^\infty_{d_\pi}(X)} &\leqslant \sup_{\|h\|_{\ell^2_{\mathscr{H}}(d_\pi)} \leqslant 1} \int_K \|f(x)\|_X \left(\sum_{i=1}^{d_\pi} \left\| \sum_{j=1}^{d_\pi} \overline{\pi_{ij}(x)} h_j \right\|^2_{\mathscr{H}} \right)^{\frac{1}{2}} dm(x) \\
&= \sup_{\|h\|_{\ell^2_{\mathscr{H}}(d_\pi)} \leqslant 1} \left(\sum_{i=1}^{d_\pi} \left\| \sum_{j=1}^{d_\pi} \overline{\pi_{ij}(x)} h_j \right\|^2_{\mathscr{H}} \right)^{\frac{1}{2}} \int_K \|f(x)\|_X \, dm(x).
\end{aligned}$$

Since $\|\pi(x)\| \leqslant 1$ for all $\pi \in \widehat{K}, x \in K$, it follows that

$$\sup_{\|h\|_{\ell^2_{\mathscr{H}}(d_\pi)} \leqslant 1} \left(\sum_{i=1}^{d_\pi} \left\| \sum_{j=1}^{d_\pi} \overline{\pi_{ij}(x)} h_j \right\|^2_{\mathscr{H}} \right)^{\frac{1}{2}} \leqslant 1.$$

Hence, we have

$$\|\widehat{f}\|_{\mathscr{L}^\infty(\widehat{K},X)} = \sup_{\pi\in\widehat{K}} \|\widehat{f}(\pi)\|_{S^\infty_{d_\pi}(X)} \leqslant \int_K \|f(x)\|_X \, dm(x) = \|f\|_{L^1(K,X)}. \quad \square$$

THEOREM 4.8. *Let $1 \leqslant p \leqslant 2$ and let X be an operator space of Fourier type p with respect to K. Then $\Lambda^1_{K,X,p}(f) = \mathscr{F}_{K,X}(f)$ for all $f \in L^p(K,X)$.*

Proof. Suppose $\{f_n\}_n$ is sequence $L^p(K)\otimes X$ such that $f_n \to f \in L^1(K,X)$ in norm. Then

$$\begin{aligned}\|\Lambda^1_{K,X,p}(f) - \mathscr{F}_{K,X}(f)\|_{\mathscr{L}^\infty(\widehat{K},X)} &\leqslant \|\Lambda^1_{K,X,p}(f-f_n)\|_{\mathscr{L}^\infty(\widehat{K},X)} + \|\mathscr{F}_{K,X}(f_n-f)\|_{\mathscr{L}^\infty(\widehat{K},X)}\\ &\leqslant \|\Lambda^1_{K,X,p}(f-f_n)\|_{\mathscr{L}^{p'}(\widehat{K},X)} + \|\mathscr{F}_{K,X}(f_n-f)\|_{\mathscr{L}^\infty(\widehat{K},X)}.\end{aligned}$$

Since X is of Fourier type p, $\Lambda^1_{K,X,p}$ is completely bounded. Therefore, by using Lemma 4.7, we have

$$\|\Lambda^1_{K,X,p}(f) - \mathscr{F}_{K,X}(f)\|_{\mathscr{L}^\infty(\widehat{K},X)} \leqslant \mathscr{C}^1_p(X,K)\|f-f_n\|_{L^p(K,X)} + \|f_n - f\|_{L^1(K,X)}.$$

As K is compact, we have

$$\|\Lambda^1_{K,X,p}(f) - \mathscr{F}_{K,X}(f)\|_{\mathscr{L}^\infty(\widehat{K},X)} \leqslant (\mathscr{C}^1_p(X,K)+1)\|f-f_n\|_{L^p(K,X)}.$$

Now, as $n\to\infty$ we get $\|\Lambda^1_{K,X,p}(f) - \mathscr{F}_{K,X}(f)\|_{\mathscr{L}^\infty(\widehat{K},X)} \to 0$.

Hence $\Lambda^1_{K,X,p}(f) = \mathscr{F}_{K,X}(f)$ for all $f \in L^p(K,X)$. $\square$

THEOREM 4.9. *Every operator space has Fourier type 1 with respect to compact hypergroup, i.e., $\mathscr{C}^1_1(X,K) = 1$ for every pair (X,K).*

Proof. Since $L^1(K,X)$ can be written as $L^1(K)\widehat{\otimes}X$ with max structure on $L^1(K)$, by using the Hausdorff-Young inequality for compact hypergroups, we get

$$\mathscr{C}^1_1(X,K) = \sup_{\|f\|_{L^1(K)}\leqslant 1} \|\widehat{f}\otimes(\cdot)\|_{cb(X,\mathscr{L}^\infty_X(\widehat{K}))} \leqslant \sup_{\|f\|_{L^1(K)}\leqslant 1} \|\widehat{f}\|_{\mathscr{L}^\infty(\widehat{K})} \leqslant 1.$$

Now, the proof follows from Corollary 4.6. $\square$

COROLLARY 4.10. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant 2$ and let K be a compact hypergroup. If the operator space X has the Fourier type p_2 with respect to K then X has Fourier type p_1 with respect to K. Moreover, we have $\mathscr{C}^1_{p_1}(X,K) \leqslant \mathscr{C}^1_{p_2}(X,K)^{\frac{p_2'}{p_1'}}$, where p_1' and p_2' are the conjugate exponent of p_1 and p_2 respectively.*

Proof. The proof of this corollary follows from Theorem 4.9 and complex interpolation. $\square$

LEMMA 4.11. *Let X be an operator space. Then we have*

$$\mathscr{F}_{K,X}(L^1(K,X)) \subset C_0(\widehat{K},X).$$

Proof. By Theorem 4.9, every operator space has Fourier type 1. Also, $\mathrm{Trig}(K)\otimes X$ is uniformly dense in $C(K)\otimes X$ and hence $L^1(K,X)$-norm dense in $L^1(K,X)$. By Lemma 4.7, the map $f \mapsto \widehat{f}$ from $L^1(K,X)$ to $\mathscr{L}^\infty(\widehat{K},X)$ is continuous and hence each $\widehat{f}$ will belong to closure of $C_c(\widehat{K},X)$, namely, $C_0(\widehat{K},X)$. $\square$

4.3. Fourier cotype with respect to a compact hypergroup

The inverse Fourier transform $\mathscr{F}_K^{-1}$ from $\mathscr{L}^2(\widehat{K})$ to $L^2(K)$ is defined as the inverse of Fourier operator $\mathscr{F}_K$ from $L^2(K)$ onto $\mathscr{L}^2(\widehat{K})$. For $1 \leqslant p \leqslant 2$, we have that $\mathscr{L}^p(\widehat{K}) \subset \mathscr{L}^2(\widehat{K})$. Therefore, the inverse Fourier transform $\mathscr{F}_K^{-1}f$, for any $f \in \mathscr{L}^p(\widehat{K})$, is well defined. Obviously, $\mathscr{F}_K^{-1}$ takes $\mathscr{L}^p(\widehat{K})$ into $L^2(K)$. Now, the Hausdorff- Young inequality for compact hypergroups assures that $\mathscr{F}_K^{-1}$ maps $\mathscr{L}^p(\widehat{K})$ into $L^{p'}(K)$, where p' is exponent conjugate of p.

Let $1 \leqslant p \leqslant 2$. For any $A\otimes e \in \mathscr{L}^p(\widehat{K})\otimes X$, define the vector-valued inverse Fourier transform $\mathscr{F}_{K,X}^{-1}(A\otimes e)$ of $A\otimes e$ as $\mathscr{F}_K^{-1}(A)\otimes e$. It can be easily seen that $\mathscr{F}_{K,X}^{-1}(\mathscr{L}^p(\widehat{K})\otimes X) \subset L^{p'}(K)\otimes X$. This serves as a motivation for following definition.

DEFINITION 4.12. Let $1 \leqslant p \leqslant 2$ and $p' = \frac{p}{p-1}$. An operator space is said to have the *Fourier cotype* p' with respect to the compact hypergroup K if the inverse Fourier transform $\mathscr{F}_{K,X}^{-1} : \mathscr{L}^p(\widehat{K})\otimes X \to L^{p'}(K)\otimes X$ can be extended to a completely bounded operator $\Lambda^2_{K,X,p'} : \mathscr{L}^p(\widehat{K},X) \to L^{p'}(\widehat{K},X)$.

In this case, the completely bounded norm of $\Lambda^2_{K,X,p'}$ is denoted by $\mathscr{C}^2_{p'}(X,K)$.

The following two results are simple consequence of Property 1 of Theorem 3.1.

LEMMA 4.13. *Let F be a closed subspace of X. Then, we have $\mathscr{C}^2_{p'}(F,K) \leqslant \mathscr{C}^2_{p'}(X,K)$, where p' is conjugate exponent of p for $1 \leqslant p \leqslant 2$.*

COROLLARY 4.14. *Let $1 \leqslant p \leqslant 2$ and $p' = \frac{p}{p-1}$. Then, we have $\mathscr{C}^2_p(X,K) \geqslant 1$.*

We have the explicit formula for the inverse Fourier transform of $\mathscr{F}_K^{-1}$ on $\mathscr{L}^2(\widehat{K})$ and hence on $\mathscr{L}^p(\widehat{K})$ for $1 \leqslant p \leqslant 2$. It is clear that this formula is also true if we take tensor product with a operator space. Having all this in mind, we define the inverse Fourier operator $\mathscr{F}_{K,X}^{-1}$ on $\mathscr{L}^p(\widehat{K})\otimes X$ by

$$\mathscr{F}_{K,X}^{-1}(A) = \sum_{\pi\in\hat{K}} k_\pi \sum_{i,j=1}^{d_\pi} A^\pi_{ij}\pi_{ij}(\cdot).$$

The following lemma says that $\Lambda^2_{K,X,p'}$ preserves the given formula for $\mathscr{F}_{K,X}^{-1}$ and $\Lambda^2_{K,X,p'}$ agrees with inverse of the vector valued Fourier transform on $\mathscr{L}^p(\widehat{K},X)$.

LEMMA 4.15. *Let X be an operator space having Fourier cotype p' with respect to K. Then:*

(i) for $A \in \mathscr{L}^p(\widehat{K},X)$, we have

$$\Lambda^2_{K,X,p'}(A) = \mathscr{F}^{-1}_{K,X}(A) = \sum_{\pi\in\hat{K}} k_\pi \sum_{i,j=1}^{d_\pi} A^\pi_{ij}\pi_{ij}(\cdot);$$

(ii) for $A \in \mathscr{L}^p(\widehat{K},X)$, we have $\mathscr{F}^{-1}_{K,X} \circ \Lambda^2_{K,X,p'}(A) = A$.

Proof.

(i) Since $A \in \mathscr{L}^p(\widehat{K},X)$, it has a countable support, say, $\{\pi_i\}_{=1}^{\infty} \subset \widehat{K}$. Now, define $A_n = \{A_n^\pi\}_{\pi\in\widehat{K}} \in \mathscr{L}^p(\widehat{K}) \otimes X$ by

$$A_n^\pi = \begin{cases} A^\pi, & \text{if } \pi = \pi_i \text{ for } 1 \leqslant i \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

For our convenience, we write $\mathscr{F}^{-1}_{K,X}(A_n) = f_n$ and $\mathscr{F}^{-1}_{K,X}(A) = f$. Then

$$\begin{aligned}\|\Lambda^2_{K,X,p'}(A) - \mathscr{F}^{-1}_{K,X}(A)\|_{L^{p'}(K,X)} &= \|\Lambda^2_{K,X,p'}(A) - f\|_{L^{p'}(K,X)} \\ &\leqslant \|\Lambda^2_{K,X,p'}(A - A_n)\|_{L^{p'}(K,X)} + \|f - f_n\|_{L^{p'}(K,X)}.\end{aligned}$$

As X has p' Fourier cotype, we get

$$\|\Lambda^2_{K,X,p'}(A) - \mathscr{F}^{-1}_{K,X}(A)\| \leqslant C^2_{p'}(X,K)\|A - A_n\|_{\mathscr{L}^p(\widehat{K},X)} + \|f - f_n\|_{L^{p'}(K,X)}.$$

Note that the first term on right hand side is arbitraryly small for large values of n. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, without loss of generality, we can assume that $\|f_{n_1} - f_{n_2}\| \leqslant 2^{-m}$ for all $n_1, n_2 \geqslant m$.

Therefore, $\|f - f_n\|_{L^{p'}(K,X)} \leqslant \sum_{k=n+1}^{\infty} \|f_k - f_{k-1}\|_{L^{p'}(K,X)} < \sum_{k=n+1}^{\infty} \frac{1}{2^k}$. So, as $n \to \infty$ we get that $\|\Lambda^2_{K,X,p'}(A) - \mathscr{F}^{-1}_{K,X}(A)\| \to 0$.

Hence $\Lambda^2_{K,X,p'}(A) = \mathscr{F}^{-1}_{K,X}(A)$ for all $A \in \mathscr{L}^p(\widehat{K},X)$.

(ii) We shall continue to follow the notations used in the proof of part (i). For a fixed $\pi \in \widehat{K}$, take n_π to be the smallest positive integer n_π such that $\pi \neq \pi_k$ for $k \geqslant n_\pi$. Then $\widehat{f}(\pi) - A^\pi = (\widehat{f} - \widehat{f_n})(\pi)$ for all $n \geqslant n_\pi$. Therefore, it is enough to estimate the entries of the matrix $(\widehat{f} - \widehat{f_n})(\pi)$. In fact, we have

$$\begin{aligned}\|((\widehat{f} - \widehat{f_n})(\pi))_{ij}\|_X &\leqslant \int_K \|(f - f_n)(x)\|_X |\pi_{ji}(x)|\, dm(x) \\ &\leqslant \|f - f_n\|_{L^{p'}(K,X)} \|\pi_{ji}\|_{L^p(K)} \leqslant \mathscr{C}^2_{p'}(X,K)\|A - A_n\|_{\mathscr{L}^p(\widehat{K},X)},\end{aligned}$$

which is arbitrary small as for large values of n. $\square$

THEOREM 4.16. *For any pair (X,K), we have $\mathscr{C}^2_\infty(X,K)=1$.*

Proof. Since $\mathscr{L}^1(\widehat{K})\otimes X$ is dense in $\mathscr{L}^1(\widehat{K},X)$, by using a density argument and Property (i) of Theorem 3.1, we need to show that for all $n\in\mathbb{N}$ and for any family of matrices $(A_{ij})\in M_n(\mathscr{L}^1(\widehat{K})\otimes X)$ we have

$$\|\sum_{\pi\in\hat{K}}k_\pi\sum_{i,j=1}^{d_\pi}A^\pi_{ij}\pi_{ij}(x)\|_{S^\infty_n(X)}\leqslant\|(A_{ij})\|_{S^\infty_n(\mathscr{L}^1(\widehat{K},X))}\ a.e.x\in K.$$

But this can be seen easily by considering a vector $A\in\mathscr{L}^1(\widehat{K},X)$ as an element of $cb(\mathscr{L}^\infty(\widehat{K}),X)$ via the mapping $B\mapsto\sum_{\pi\in\widehat{K}}k_\pi\sum_{i,j=1}^{d_\pi}A^\pi_{ij}B^\pi_{ij}$ from $\mathscr{L}^\infty(\widehat{K})$ to X.

Now, define $B_x\in\mathscr{L}^\infty(\widehat{K})$ by $B^\pi_x=\pi(x)$. Then, we have

$$\begin{aligned}\left\|\sum_{\pi\in\widehat{K}}k_\pi\sum_{i,j=1}^{d_\pi}(A^\pi_{ij}\pi(x))\right\|_{S^\infty_n(X)}&=\left\|\left(\sum_{\pi\in\widehat{K}}k_\pi\sum_{i,j=1}^{d_\pi}(A^\pi_{ij}\cdot)(B_x)\right)\right\|_{S^\infty_n(X)}\\&\leqslant\left\|\left(\sum_{\pi\in\widehat{K}}k_\pi\sum_{i,j=1}^{d_\pi}(A^\pi_{ij}\cdot)\right)\right\|_{cb(\mathscr{L}^\infty(\widehat{K}),S^\infty_n(X))}\\&\leqslant\left\|(A_{ij})\right\|_{S^\infty_n(\mathscr{L}^1(\widehat{K})\widehat{\otimes}X)}.\end{aligned}$$

Here the last inequality follows from the complete contraction given by $\mathscr{L}^1(\widehat{K})\widehat{\otimes}X\to\mathscr{L}^1(\widehat{K})\otimes_{\min}X\to cb(\mathscr{L}^\infty(\widehat{K}),X)$. Finally, the desired inequality follows from Theorem 4.2. Therefore, $\mathscr{C}^2_\infty(X,K)\leqslant 1$. The other side of inequality follows from Corollary 4.14 above. $\square$

The proof of the following corollary follows from Theorem 4.16 and interpolation.

COROLLARY 4.17. *Let $1\leqslant p_1\leqslant p_2\leqslant 2$ and let K be a compact hypergroup. If the operator space X has the Fourier cotype p_2' with respect to K then X has Fourier cotype p_1' with respect to K. Moreover, we have $\mathscr{C}^1_{p_1'}(X,K)\leqslant\mathscr{C}^1_{p_2'}(X,K)^{\frac{p_2'}{p_1'}}$, where p_1' and p_2' are the conjugate exponent of p_1 and p_2 respectively.*

Here is the main result of this paper. Also, this is the compact hypergroup analogue of Theorem 2.11.

THEOREM 4.18. *Let X be an operator space, $1\leqslant p\leqslant 2$ and let p' be the conjegate exponent of p. Then X^* has*

(i) Fourier cotype p' with resptect to K iff X has Fourier type p with respect to K;

(ii) Fourier type p' with respect to K iff X has Fourier cotype p with respect to K.

Proof. We shall prove only (i) as the proof of (ii) will follow similarly. The case $p=1$ follows from Theorem 4.9 and Theorem 4.16. Thus, we shall assume that $1 < p \leqslant 2$.

We shall first prove that $\mathscr{C}_p^1(X,K) \geqslant \mathscr{C}_{p'}^1(X^*,K)$. By Theorem 3.1 and the denseness of the algebraic tensor product, it is enough to show that

$$\left\| \sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} (A_{ij}^\pi \pi(\cdot)) \right\|_{S_n^{p'}(L_{X^*}^{p'}(K))} \leqslant C_p^1(X,K) \|(A_{ij})\|_{S_n^{p'}(\mathscr{L}_{X^*}^p(\widehat{K}))},$$

for any $A_{ij} \in \mathscr{L}^p(\widehat{K}) \otimes X^*,\ 1 \leqslant i,j \leqslant n$ and for all $n \geqslant 1$. Since, $S_n^{p'}(L_{X^*}^{p'}(K))$ and $L^{p'}_{S_n^p(X)^*}(K)$ are completely isometric, for a given $\varepsilon > 0$, $\exists\ f^\varepsilon \in L^p_{S_n^p(X)}(K)$ of norm 1 such that

$$\begin{aligned}&\left\| \sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} (A_{ij}^\pi \pi(\cdot)) \right\|_{S_n^{p'}(L_{X^*}^{p'}(K))} \\ &\leqslant (1+\varepsilon) \left| \int_K tr\left[\left(\sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} (A_{ij}^\pi \pi(x)) \right) (f_{ij}^\varepsilon(x)) \right] \, dm(x) \right|.\end{aligned}$$

Here, $f_{ij}^\varepsilon \in L^p(K) \otimes X$, the entries of f^ε. Note that $A_{ij} \in \mathscr{L}^p(\widehat{K})$. Therefore, it follows that

$$\begin{aligned}&\left\| \sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} (A_{ij}^\pi \pi(\cdot)) \right\|_{S_n^{p'}(L_{X^*}^{p'}(K))} \\ &\leqslant (1+\varepsilon) \left| \sum_{i,j=1}^{n} \sum_{\pi\in\widehat{K}} k_\pi \int_K \left\langle \sum_{i,j=1}^{d_\pi} (A_{ij}^\pi \pi(x)), f_{ij}^\varepsilon(x) \right\rangle \, dm(x) \right| \\ &= (1+\varepsilon) \left| \sum_{i,j=1}^{n} \sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} \langle (A_{ij}^\pi), \widehat{\check{f}_{ij}^\varepsilon}(\pi) \rangle \right| = (1+\varepsilon) \left| tr\left[(A_{ij}) (\widehat{\check{f}_{ij}^\varepsilon}) \right] \right| \\ &\leqslant (1+\varepsilon) \|(A_{ij})\|_{S_n^{p'}(L_{X^*}^p(K))} \|\widehat{(\check{f}_{ij}^\varepsilon)}\|_{S_n^p(L_{X^*}^{p'}(K))} \\ &\leqslant (1+\varepsilon) C_p^1(X,K) \|(A_{ij})\|_{S_n^{p'}(L_{X^*}^p(K))} \|f^\varepsilon\|_{S_n^p(L_X^p(K))} = (1+\varepsilon) C_p^1(X,K) \|(A_{ij})\|_{S_n^{p'}(L_{X^*}^p(K))}.\end{aligned}$$

Thus, it follows that $C_{p'}^2(X^*,K) \leqslant C_p^1(X,K)$.

We shall now prove the other way inequality. As mentioned in the beginning of the proof of the previous inequality, it is enough to show that

$$\|(\widehat{f_{ij}})\|_{s_n^p(\mathscr{L}_X^{p'}(\widehat{K}))} \leqslant C_{p'}^2(X^*,K) \|(f_{ij})\|_{S_n^{p'}(L_X^p(K))}$$

for any $f_{ij} \in L^p(K) \otimes X,\ 1 \leqslant i,j \leqslant n$ and $n \geqslant 1$. Again, as $S_n^{p'}(\mathscr{L}_X^{p'}(\widehat{K}))$ is completely isometrically isomorphic to $\mathscr{L}^{p'}_{S_n^p(X)}(\widehat{K})$, for a given $\varepsilon > 0$, $\exists\ A^\varepsilon \in \mathscr{L}^p_{S_n^p(X^*)(\widehat{K})}$ of norm one such that

$$\|(\widehat{f_{ij}})\|_{S_n^{p'}(\mathscr{L}_X^{p'}(\widehat{K}))} \leqslant (1+\varepsilon)\left|\sum_{\pi\in\widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi}\left[(A_{ij}^{\varepsilon,\pi})(\widehat{f_{ij}}(\pi))\right]\right|.$$

Now the remaining proof follows similar lines as in the previous inequality. Hence the proof. □

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