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ON THE MAXIMAL NUMERICAL RANGE OF A HYPONORMAL OPERATOR

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Dedicated to Professor M. S. Moslehian

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Abstract. Let A be a bounded linear operator acting on a complex Hilbert space. Let $\sigma(A)$ and $W_0(A)$ denote the spectrum and the maximal numerical range of A, respectively. In [10], it was shown that if A is a subnormal operator, then

$$W_0(A) = co(\{\lambda \in \sigma(A) : |\lambda| = \|A\|\}),$$

where $co(.)$ stands for the convex hull of the corresponding set. We extend this result to hyponormal operators. We give a geometric interpretation of the obtained result and deduce a necessary and sufficient condition to have $0 \in W_0(A)$ for a hyponormal operator A. Some properties of normaloid operators are also given.

1. Introduction

First, let us set some notations and recall some results from the literature.
Let L be a subset of the complex plane $\mathbb{C}$. As usual, the symbols $\overline{L}$, ∂L and $co(L)$ stand for the closure, the boundary and the convex hull of L, respectively. Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$. For $A \in \mathscr{B}(\mathscr{H})$, the numerical range of A is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Ax, x\rangle$ associated with the operator. More precisely,

$$W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H},\ \|x\| = 1\}.$$

Thus the numerical range of an operator, like the spectrum, is a subset of the complex plane. It is a celebrated result due to Toeplitz[13] and Hausdorff [8] that $W(A)$ is a bounded convex set in the complex plane, for more detail, see [6]. It is closed if $dim(\mathscr{H}) < \infty$, but it is not always closed if $dim(\mathscr{H}) = \infty$.
For $A \in \mathscr{B}(\mathscr{H})$, let $\sigma(A)$, $r(A)$ and $w(A)$ denote the spectrum, the spectral radius and the numerical radius of A, respectively. Recall that they are given by

$$\sigma(A) = \{\lambda \in \mathbb{C} :\ A - \lambda I \text{ is not invertible}\},$$

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$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} \text{ and } w(A) = \sup\{|z| : z \in W(A)\}.$$

It is well known that $\sigma(A)$ is a compact set and $co(\sigma(A)) \subseteq \overline{W(A)}$. For more material about the specral radius, the numerical radius and other information on the basic theory of algebraic numerical range, we mention here the books [1, 2, 5, 6] as standard sources of references.

It is a basic fact that $w(.)$ is a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the C^*-norm $\|.\|$. In fact, for any operator $A \in \mathscr{B}(\mathscr{H})$, the following inequalities are well known

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|.$$

An operator $A \in \mathscr{B}(\mathscr{H})$ is called normaloid if $w(A) = \|A\|$ or equivalently $r(A) = \|A\|$, see [5, Theorem 1.3-2]. Familiar examples of normaloid operators are hyponormal (normal and subnormal) operators, see [11, Theorem 1].

There is another set that is close to $W(A)$; that is the maximal numerical range $W_0(A)$ of A. It it was introduced by Stampfli [12] and defined by

$$W_0(A) = \{\lim_n \langle Ax_n, x_n\rangle : x_n \in \mathscr{H}, \ \|x_n\| = 1, \ \lim_n \|Ax_n\| = \|A\|\}.$$

It was shown in [12, Lemma 2] that $W_0(A)$ is nonempty, closed, convex, and contained in the closure of the numerical range; $W_0(A) \subseteq \overline{W(A)}$. When $\mathscr{H}$ is finite dimensional, $W_0(A)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Ax\| = \|A\|$). We also will denote by $\delta_{A,B}$ the generalized derivation induced by $A, B \in \mathscr{B}(\mathscr{H})$ and which is defined as follows

$$\delta_{A,B} : \mathscr{B}(\mathscr{H}) \longrightarrow \mathscr{B}(\mathscr{H}), \ X \longmapsto \delta_{A,B}(X) = (L_A - R_B)X,$$

where L_A, R_B are the left and the right multiplications defined on $\mathscr{B}(\mathscr{H})$ by $L_A(X) = AX$ and $R_B(X) = XB$, respectively. The generalized derivation was studied by many authors; see for instance [3, 12] and the references therein.

It is interesting to know that Stampfli [12] introduced the maximal numerical range (specially) for the purpose of calculating the norm of the generalized derivation. Indeed, he has given the following elegant formula, see [12, Theorem 8]. For any $A, B \in \mathscr{B}(\mathscr{H})$

$$\|\delta_{A,B}\| = \inf_{\lambda \in \mathbb{C}} (\|A - \lambda\| + \|B - \lambda\|).$$

However, the maximal numerical range recently became the interest of several researchers, see for instance [7, 9, 10]. We expect the present work to contribute to shed more light on the maximal numerical range.

Throughout this paper, for any operator $A \in \mathscr{B}(\mathscr{H})$ we denote by $\sigma_n(A)$ the subset of $\sigma(A)$ defined by

$$\sigma_n(A) = \{\lambda \in \sigma(A) : \ |\lambda| = \|A\|\}.$$

In Section 2, for any normaloid operator $A \in \mathscr{B}(\mathscr{H})$ and any $\lambda \in \sigma_n(A)$, we show the following:

(1) $\left\|A^k + \lambda^k\right\| = \|A\|^k + \left|\lambda^k\right| = 2\|A\|^k$ for $k = 1, 2, 3, \ldots$;

(2) for any nonzero natural number k, the operator $A^k+\lambda^k$ is normaloid;

(3) for any operator $B\in\mathscr{B}(\mathscr{H})$,

$$w^{'}(B)\,\|A\|^k \leqslant w(BA^k) \quad \text{for } k=1,2,3,\dots,$$

where $w^{'}(B)=\inf\{|z|:\ z\in W(B)\}$. And A is normaloid if the inequality

$$w^{'}(B)\,\|A\|^l \leqslant w(BA^l)$$

is satisfied for any operator $B\in\mathscr{B}(\mathscr{H})$ and some nonzero natural number l.

Motivated by results of Spitkovsky [10] and basing on the fact that any hyponormal operator T has a normal dilation N with $\sigma(N)\subseteq\sigma(T)$, see [4, Theorem 3.2], we aim in Section 3 to show that if A is a hyponormal operator, then

$$W_0(A)=co(\sigma_n(A)).$$

We give a geometric interpretation of the obtained result and deduce a necessary and sufficient condition to have $0\in W_0(A)$ for a hyponormal operator A.

From now on, $\mathscr{B}(\mathscr{H})$ denotes the algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$.

2. Some properties of normaloid operators

Let $A\in\mathscr{B}(\mathscr{H})$ be an arbitrary operator, then $\sigma(A)\subseteq\overline{W(A)}$ and $W_0(A)\subseteq\overline{W(A)}$. But we don't know whether the intersection $\sigma(A)\cap W_0(A)$ is empty or not. However, if A is a normaloid operator this intersection is always a nonempty set. Indeed, since A is normaloid, then $r(A)=\|A\|$ and, $\sigma(A)$ being a compact set, we can find a scalar $\lambda\in\sigma(A)$ such that $|\lambda|=\|A\|$. Then, $\sigma_n(A)$ is a nonempty subset of $\sigma(A)$ if A is normaloid. On the other hand, it is shown in [10, Lemma 1] that for any operator $A\in\mathscr{B}(\mathscr{H})$

$$W_0(A)\cap C_A=\sigma_n(A),$$

where $C_A=\{z:|z|=\|A\|\}$. Hence, since $W_0(A)$ is convex, $co(\sigma_n(A))$ is always a subset of $W_0(A)$ for any operator $A\in\mathscr{B}(\mathscr{H})$. However, if the operator A is not normaloid, the set $\sigma_n(A)$ is empty. Therefore, we will be interested in this section in the normaloidness case (i.e., $\sigma_n(A)\neq\emptyset$). Recall that if A is normaloid, then for any nonzero natural number k, the operator A^k is normaloid (see [5, Theorem 6.2-1]) and we also have

$$co(\sigma_n(A^k))\subseteq W_0(A^k). \tag{2.1}$$

Recall also that, by the spectral mapping theorem, if $\lambda\in\sigma_n(A)$, then $\lambda^k\in\sigma_n(A^k)$ for any nonzero natural number k.

THEOREM 2.1. *Let $A\in\mathscr{B}(\mathscr{H})$ be a normaloid operator. Then, for any $\lambda\in\sigma_n(A)$ we have*

$$\left\|A^k+\lambda^k\right\|=\|A\|^k+\left|\lambda^k\right|=2\,\|A\|^k \quad (k=1,2,\dots). \tag{2.2}$$

Proof. The result is evident if $A = 0$. Therefore, assume A is a nonzero operator. Let $\lambda \in \sigma_n(A)$ and let k be any nonzero natural number. Since A is normaloid, by Equation (2.1), $\lambda^k \in W_0(A^k)$, so, $\frac{1}{|\lambda^k|}\lambda^k = \frac{1}{\|A^k\|}\lambda^k \in W_0(\frac{1}{\|A^k\|}A^k)$. We have $W_0(\frac{1}{|\lambda^k|}\lambda^k) = \{\frac{1}{|\lambda^k|}\lambda^k\}$, then $W_0(\frac{1}{\|A^k\|}A^k) \cap W_0(\frac{1}{|\lambda^k|}\lambda^k) \neq \emptyset$. From [12, Theorem 7], $\left\|\delta_{A^k,-\lambda^k}\right\| = \|A^k\| + |\lambda^k|$. Since A is normaloid, then $\|A^k\| = \|A\|^k$, see [5, Theorem 6.2-1]. On the other hand $\left\|\delta_{A^k,-\lambda^k}\right\| = \|L_{A^k+\lambda^k}\| = \|A^k + \lambda^k\|$, we conclude that $\|A^k + \lambda^k\| = 2\|A\|^k$. □

REMARK 2.2. Let $A \in \mathscr{B}(\mathscr{H})$. We always have

$$\max_{\lambda\in\sigma(A)} \|A+\lambda\| \leqslant \max_{\lambda\in\sigma(A)} (\|A\| + |\lambda|) \leqslant 2\|A\|.$$

If A is normaloid, for any $\lambda \in \sigma_n(A)$ we have from equation (2.2) $\|A+\lambda\| = 2\|A\|$, then we obtain

$$\max_{\lambda\in\sigma(A)} \|A+\lambda\| = 2\|A\|. \tag{2.3}$$

This leads us to ask if identity (2.3) holds, is A normaloid? The following corollary answers this question and then gives another characterization of normaloid operators in $\mathscr{B}(\mathscr{H})$.

COROLLARY 2.3. *Let $A \in \mathscr{B}(\mathscr{H})$. Then, A is normaloid if and only if*

$$\max_{\lambda\in\sigma(A)} \|A+\lambda\| = 2\|A\|.$$

Proof. The necessity follows from Remark 2.2 so that we only need to prove the sufficiency. Note first that, by an argument of compactness, there exists $\mu \in \sigma(A)$ such that

$$\max_{\lambda\in\sigma(A)} (\|A\| + |\lambda|) = \|A\| + |\mu|.$$

If A is not normaloid, we have $|\mu| < \|A\|$ and we obtain

$$\max_{\lambda\in\sigma(A)} \|A+\lambda\| \leqslant \max_{\lambda\in\sigma(A)} (\|A\| + |\lambda|) = \|A\| + |\mu| < 2\|A\|.$$

This proves the sufficiency. □

THEOREM 2.4. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normaloid operator. Then, for any $\lambda \in \sigma_n(A)$ and any nonzero natural number k the operator $A^k + \lambda^k$ is normaloid.*

Proof. Let $A \in \mathscr{B}(\mathscr{H})$ be a normaloid operator. Let $\lambda \in \sigma_n(A)$ and let k be any nonzero natural number. We always have

$$w(A^k + \lambda^k) \leqslant \left\|A^k + \lambda^k\right\| \leqslant 2\|A\|^k.$$

By equation (2.1), $\lambda^k \in W_0(A^k)$ and therefore, $\lim_n \langle A^k x_n, x_n\rangle = \lambda^k$ for some sequence of unit vectors $x_n \in \mathscr{H}$. Then,

$$\lim_n \langle (A^k+\lambda^k)x_n, x_n\rangle = 2\lambda^k.$$

It follows that $2\lambda^k \in \overline{W(A^k+\lambda^k)}$ and hence, $2\|A\|^k = 2|\lambda|^k \leqslant w(A^k+\lambda^k)$. Consequently $w(A^k+\lambda^k) = \left\|A^k+\lambda^k\right\|$. That is just to say that the operator $A^k+\lambda^k$ is normaloid. □

The following theorem gives another characterization of normaloid operators in terms of inequality.

THEOREM 2.5. *Let $A \in \mathscr{B}(\mathscr{H})$. Then, the following are equivalent statements:*

i) A is normaloid;

ii) for any operator $B \in \mathscr{B}(\mathscr{H})$ and nonzero natural number k, we have

$$w^{'}(B)\,\|A\|^k \leqslant w(BA^k); \tag{2.4}$$

iii) there is a nonzero natural number l such that for any $B \in \mathscr{B}(\mathscr{H})$, we have

$$w^{'}(B)\,\|A\|^l \leqslant w(BA^l).$$

Proof. $i) \Rightarrow ii)$. Assume that A is normaloid and let k be a nonzero natural number. Since the operator A^k is normaloid, then there is λ with $\left|\lambda^k\right| = \left\|A^k\right\| = \|A\|^k$ and $\lambda^k \in \sigma_{app}(A^k)$. Let (x_n) be a sequence of unit vectors such that $\lim_n \left\|A^k x_n - \lambda^k x_n\right\| = 0$. For any $B \in \mathscr{H}$

$$\begin{aligned}\left|\langle BA^k x_n, x_n\rangle\right| &= \left|\langle B(\lambda^k I + (A^k - \lambda^k I))x_n, x_n\rangle\right| \geqslant |\lambda|^k\,|\langle Bx_n, x_n\rangle| - \left|\langle B(A^k x_n - \lambda^k x_n), x_n\rangle\right| \\ &\geqslant w^{'}(B)\,\|A\|^k - \|B\|\left\|A^k x_n - \lambda^k x_n\right\|.\end{aligned}$$

Inequality (2.4) follows.

$ii) \Rightarrow iii)$. It is obvious.

$iii) \Rightarrow i)$. Let l be a nonzero natural number l such that for any $B \in \mathscr{B}(\mathscr{H})$,

$$w^{'}(B)\,\|A\|^l \leqslant w(BA^l).$$

Take $B = I$, we get $\|A\|^l \leqslant w(A^l) \leqslant (w(A))^l \leqslant \|A\|^l$. It results that $(w(A))^l = \|A\|^l$, that is $w(A) = \|A\|$, and hence A is normaloid. □

3. Maximal numerical range of a hyponormal operator

Let $A \in \mathscr{B}(\mathscr{H})$ be an operator. It is shown in [10, Corollary 2] that the following equality

$$W_0(A) = co(\sigma_n(A)) \tag{3.1}$$

holds for subnormal (and then for normal) operators A. In this section, we extend this property to hyponormal operators by using the fact that every hyponormal operator A has a normal dilation N with $\sigma(N) \subseteq \sigma(A)$, see [4, Theorem 3.2]. First, let us recall the definition of the dilation of an operator. Let A and B be bounded linear operators on the complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. B is said to be a dilation of A (or A is dilated to B) if B is unitarily equivalent to a 2×2 operator matrix of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$. This is equivalent to requiring the existence of an isometry V from $\mathscr{H}$ to $\mathscr{K}$ such that $A = V^*BV$. For this end, we need the following auxiliary lemma.

LEMMA 3.1. *Let $A \in \mathscr{B}(\mathscr{H})$. If A has a normal dilation N on some complex Hilbert space $\mathscr{K}$ such that $\sigma(N) \subseteq \sigma(A)$, then*

$$W_0(A) \subseteq W_0(N).$$

Proof. Since N is normal and $\sigma(N) \subseteq \sigma(A)$, then $\|N\| = r(N) \leqslant r(A) \leqslant \|A\|$. Let $\lambda \in W_0(A)$, then there is a sequence of unit vectors $x_n \in \mathscr{H}$ such that

$$\lim_n \langle Ax_n, x_n \rangle = \lambda \quad \text{and} \quad \lim_n \|Ax_n\| = \|A\|.$$

Let V be an isometry from $\mathscr{H}$ to $\mathscr{K}$ such that $A = V^*NV$ and set $y_n = Vx_n$, so y_n is a unit vector in $\mathscr{K}$. Therefore, we have

$$\lim_n \langle Ny_n, y_n \rangle = \lim_n \langle NVx_n, Vx_n \rangle = \lim_n \langle V^*NVx_n, x_n \rangle = \lim_n \langle Ax_n, x_n \rangle = \lambda.$$

Moreover, since

$$\|Ax_n\| = \|V^*NVx_n\| \leqslant \|V^*\| \, \|NVx_n\| \leqslant \|V^*\| \, \|N\| \, \|Vx_n\| \leqslant \|N\| \leqslant \|A\|,$$

we conclude that $\lim_n \|NVx_n\| = \|N\|$; that is, $\lim_n \|Ny_n\| = \|N\|$. It results that $\lambda \in W_0(N)$ and consequently, $W_0(A) \subseteq W_0(N)$ as desired. □

REMARK 3.2. In fact, in the previous lemma, we have $\|A\| = \|N\|$. Indeed, since $A = V^*NV$, then $\|A\| \leqslant \|V^*\| \, \|N\| \, \|V\| = \|N\|$.

THEOREM 3.3. *Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator, then*

$$W_0(A) = co(\sigma_n(A)).$$

Proof. Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. According to [4, Theorem 3.2], A has a normal dilation N on some complex Hilbert space $\mathscr{K}$ with $\sigma(N) \subseteq \sigma(A)$. From Lemma 3.1 and using [10, Corollary 2]

$$W_0(A) \subseteq W_0(N) = co(\sigma_n(N)) \subseteq co(\sigma_n(A)) \quad \text{(because } \|N\| = \|A\|\text{)}.$$

Since $co(\sigma_n(A))$ is always a subset of $W_0(A)$, we derive that $W_0(A) = co(\sigma_n(A))$. This completes the proof. □

REMARK 3.4. The converse of the previous theorem does not hold in general. Indeed, let B the backward shift defined on the Hilbert space ℓ_2 by

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

It is known that $\sigma(B)$ is the closed unit disk $\overline{\mathbb{D}} = \{\lambda \in \mathbb{C} : \ |\lambda| \leqslant 1\}$ and $\|B\| = 1$. Then $\sigma_n(B) = C(0,1)$; the unit circle, and $W_0(B) \subseteq \overline{\mathbb{D}}$. Since $co(\sigma_n(B)) = \overline{\mathbb{D}}$, we get

$$W_0(B) = co(\sigma_n(B)).$$

However, by taking $x = (1,0,0,\ldots)$, we have

$$\langle (B^*B - BB^*)x, x\rangle = -1 < 0,$$

and so B is not a hyponormal operator.

We end this section by giving a geometric interpretation of Theorem 3.3 and we locate the position of the maximal numerical range $W_0(A)$ in the closed disk $\overline{D}(O, \|A\|)$ relatively to the center of mass of a hyponormal operator A. First, let us recall the definition and some properties of the center of mass of an operator A. In [12, Corollary of Theorem 2], it was shown that there exists a unique scalar c_A (called *center of mass* of A) satisfying the following (called *Pythagorean relation*)

$$\|A - c_A\|^2 + |\lambda|^2 \leqslant \|(A - c_A) + \lambda\|^2, \text{ for all } \lambda \in \mathbb{C} \tag{3.2}$$

and $0 \in W_0(A)$ if and only if $c_A = 0$. Taking $\lambda = c_A$ in inequality (3.2), we get

$$\|A - c_A\|^2 + |c_A|^2 \leqslant \|A\|^2. \tag{3.3}$$

We will denote by $w_0^{'}(A)$ the infinimum modulus of $W_0(A)$, that is,

$$w_0^{'}(A) = \inf\{|z| : \ z \in W_0(A)\}.$$

THEOREM 3.5. *Let $A \in \mathscr{B}(\mathscr{H})$ be any operator. Then, $w_0^{'}(A) \geqslant |c_A|$.*

Proof. By an argument of compactness, there exists $\alpha \in W_0(A)$ such that $|\alpha| = w_0^{'}(A)$. Hence, there is a sequence of unit vectors $x_n \in \mathscr{H}$ satisfying

$$\alpha = \lim_n \langle Ax_n, x_n\rangle \ \text{ and } \ \lim_n \|Ax_n\| = \|A\|.$$

Therefore, we have

$$\begin{aligned}\|A-c_A\|^2 &\geqslant \|(A-c_A)x_n\|^2 = \|Ax_n\|^2 + |c_A|^2 - 2Re(\overline{c_A}\langle Ax_n, x_n\rangle) \\ &\geqslant \|Ax_n\|^2 + |c_A|^2 - 2\,|c_A|\,|\langle Ax_n, x_n\rangle)|\,.\end{aligned}$$

It results that

$$\|A-c_A\|^2 \geqslant \|A\|^2 + |c_A|^2 - 2\,|c_A|\,w_0'(A) = \|A\|^2 - (w_0'(A))^2 + (w_0'(A) - |c_A|)^2.$$

Thus,

$$\|A-c_A\|^2 + (w_0'(A))^2 \geqslant \|A\|^2 + (w_0'(A) - |c_A|)^2.$$

We see that

$$\|A-c_A\|^2 + (w_0'(A))^2 \geqslant \|A\|^2$$

and from inequality (3.3), we get $w_0'(A) \geqslant |c_A|$. □

GEOMETRIC INTERPRETATION 3.6. Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. Assume that $c_A \neq 0$. From Theorem 3.5, $W_0(A)$ is outside of the open disk $D(O, |c_A|)$. Let $\alpha \in W_0(A)$ such that $|\alpha| = w_0'(A)$ (we may have $\alpha = c_A$), then we have two cases.
First case: $\alpha \in \sigma_n(A)$. It is clear that $W_0(A) = \sigma_n(A) = \{\alpha\}$. For example, A is a normal operator acting on the complex Hilbert space $\mathscr{H} = \mathbb{C}^2$ with $\sigma(A) = \{\alpha, \beta\}$ and $|\beta| < |\alpha|$ ($|\alpha| = \|A\|$).
Second case: $|\alpha| < \|A\|$. By Theorem 3.3 and the fact that $|\alpha| = d(0, W_0(A))$, there is $\lambda_1, \lambda_2 \in \sigma_n(A)$ with $\lambda_1 \neq \lambda_2$ such that α is the midpoint of $[\lambda_1, \lambda_2]$; the closed line segment connecting λ_1 with λ_2 ($\alpha = \dfrac{\lambda_1+\lambda_2}{2}$). Being convex, $W_0(A)$ must be contained in the gray area of $\overline{D}(O, \|A\|)$ (see Figure 1 below, for more details).

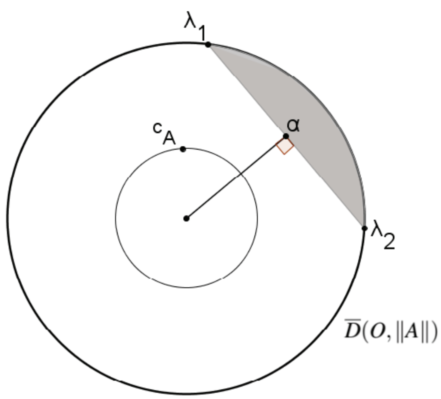

Figure 1: Geometric place of the maximal numerical range

Now, we examine the case where $c_A = 0$ (i.e., $0 \in W_0(A)$). According to Theorem 3.3, the set $\sigma_n(A)$, unlike the first case (see Figure 1 above), cannot be contained in a portion of the circle $C(0, \|A\|)$ smaller than a semicircle (equivalently, $W_0(A)$ cannot be contained in a portion of the disk $\overline{D}(O, \|A\|)$ smaller than a half-disk or $W_0(A) = [-\lambda, \lambda]$, where $\lambda \in \mathbb{C}$). However, if the operator A is not hyponormal, this result may fail. Indeed, let us give an example. Note that we obtain the desired result from the example in [10] by a simple and short method.

EXAMPLE 3.7. Let B be the operator on the complex Hilbert space $\mathscr{H} = \mathbb{C}^3$ represented by $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $\|B\| = 1$ and $\sigma(B) = \{0, 1\}$ (so, B is normaloid). Let (e_1, e_2, e_3) be the standard orthonormal basis of $\mathscr{H}$, then $Be_2 = e_1$, so $\|Be_2\| = 1$, hence e_2 is a maximal vector and therefore $0 = \langle Be_2, e_2 \rangle \in W_0(B)$. We see that $0 \notin co(\sigma_n(B)) = \{1\}$ (consequently, equality (3.1) does not hold for normaloid operators, in general). It is clear that B is not hyponormal and $c_B = 0$, however $\sigma_n(B)$ is just a singleton.

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