Operators
and
Matrices
Volume 13, Number 4 (2019), 1173–1190 doi:10.7153/oam-2019-13-78

A NOTE ON PRESERVERS OF PSEUDO SPECTRUM OF MATRIX PRODUCTS

MOHAMED BENDAOUD*, AYYOUB BENYOUNESS AND MUSTAPHA SARIH

(*Communicated by Y.-T. Poon*)

Abstract. Let $\mathscr{M}_2$ be the algebra of 2×2 complex matrices. For $\varepsilon>0$, complete descriptions are given of the maps of $\mathscr{M}_2$ leaving invariant the ε-pseudo spectrum of $A*B$, where $A*B$ stands either for the Jordan semi-triple product ABA or the skew product AB^* on matrices.

1. Introduction

Throughout this paper, $\mathscr{H}$ will denote a Hilbert space over the complex field $\mathbb{C}$ and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$ with identity operator I. For $T\in\mathscr{L}(\mathscr{H})$ we write T^* for its adjoint, $\sigma(T)$ for its spectrum, and $\|T\|$ the (spectral) norm of T. For $\varepsilon>0$, the $\varepsilon-$pseudo spectrum of T, $\sigma_\varepsilon(T)$, is defined by

$$\sigma_\varepsilon(T):=\bigcup_{E\in\mathscr{L}(\mathscr{H}),\,\|E\|<\varepsilon}\sigma(T+E),$$

and coincides with the set

$$\{z\in\mathbb{C}:\|(z-A)^{-1}\|>\varepsilon^{-1}\}$$

with the convention that $\|(z-A)^{-1}\|=\infty$ if $z\in\sigma(T)$. Unlike the spectrum, which is a purely algebraic concept, the $\varepsilon-$pseudo spectrum depends on the norm. The $\varepsilon-$pseudo spectral radius of T, $r_\varepsilon(T)$, is given by

$$r_\varepsilon(T):=\sup\{|z|:z\in\sigma_\varepsilon(T)\}.$$

Pseudo spectra is a useful tool for analyzing operators, furnishing a lot of information about the algebraic and geometric properties of operators and matrices. They play a very natural role in numerical computations, especially in those involving spectral perturbations. The monograph [20] gives an extensive account of the pseudo spectra, as well as investigations and applications in numerous fields.

Mathematics subject classification (2010): 47B49, 47A10, 47A25.
Keywords and phrases: Operator, pseudo spectrum, pseudo spectral radius, nonlinear preservers.
This work was supported by a grant from ARS-MIU, Morocco.
* Corresponding author.

General preserver problems with respect to various algebraic operations on $\mathscr{M}_n$, the algebra of $n \times n$ complex matrices, attracted a lot of attention of researchers in the fields; see for instance [7] where Chan et al. treated the preservers of zero product of matrices, Dobovišek et al. [14] concerned with the preservers of zero Jordan semi-triple product, Šemrl [19] studied non-linear commutativity preserving maps on matrices, and Hou et al. [16] determined the structure of mappings on $\mathscr{M}_n$ that preserve zero skew semi-triple product of matrices. On the subject focused on the structures of nonlinear transformations on $\mathscr{M}_n$ that respect the pseudo spectra of certain algebraic operations, we mention: [12] where the authors studied mappings on $\mathscr{M}_n$ that preserve the pseudo spectra of matrix Lie products, [2] concerned with the preservers of the pseudo spectra of the usual matrix products, and [3] general preserver problems that to do with preservers of pseudo spectra of matrix Jordan triple products are considered. It should be pointed out that the results of the all above cited papers hold for $n \geqslant 3$, and no details were given for $n = 2$. In this paper we consider the preservers of pseudo spectra of matrix products where $n = 2$, and, in particular, we complement [3, Theorem 4]. Contrary to what could be expected, the techniques used here do not allow us to include the special case of $n \geqslant 3$ as a consequence; which deserve their own independent study. In the next section, we study the pseudo spectra preservers of matrix Jordan semi-triple products. While the last section is devoted to the preservers of pseudo spectra of matrix skew products.

For other preserver problems on different types of products on matrices and operators, one may see [1, 4, 5, 6, 8, 9, 10, 11, 13, 15, 17, 18, 21] and their references.

2. Pseudo spectra preservers of matrix Jordan semi-triple products

We first fix some notation. The inner product on $\mathscr{H}$ will be denoted by $\langle .,. \rangle$. For $x, f \in \mathscr{H}$, as usual we denote by $x \otimes f$ the rank at most one operator on $\mathscr{H}$ given by $z \mapsto \langle z, f \rangle x$, and all at most rank one operators in $\mathscr{L}(\mathscr{H})$ can written into this form. For an operator $T \in \mathscr{L}(\mathscr{H})$ we will denote by T^{tr} the transpose of T relative to an arbitrary but fixed orthogonal basis of $\mathscr{H}$. For a subset σ of $\mathbb{C}$ we will denote by $\overline{\sigma}$ the complex conjugation set of σ, and for $\varepsilon > 0$ and $a \in \mathbb{C}$ we will denote by $D(a, \varepsilon)$ the open disc of $\mathbb{C}$ centered at a and of radius ε.

Before stating the main results of this section, we collect some lemmas needed in what follows. The first one summarizes some properties of the pseudo spectrum; see [20].

LEMMA 1. *For $\varepsilon > 0$ and $T \in \mathscr{L}(\mathscr{H})$, the following statements hold.*

(i) $\sigma(T) + D(0, \varepsilon) \subseteq \sigma_\varepsilon(T)$.

(ii) If T is normal, then $\sigma_\varepsilon(T) = \sigma(T) + D(0, \varepsilon)$.

(iii) $\sigma_\varepsilon(T^{tr}) = \sigma_\varepsilon(T)$, $\sigma_\varepsilon(T^*) = \overline{\sigma_\varepsilon(T)}$ *and* $\sigma_\varepsilon(UTU^*) = \sigma_\varepsilon(T)$ *for every unitary operator* $U \in \mathscr{L}(\mathscr{H})$.

The second lemma, quoted from [11], identifies the $\varepsilon-$pseudo spectra of some special operators.

LEMMA 2. *For $\varepsilon > 0$ and $T \in \mathscr{L}(\mathscr{H})$, the following statements hold.*

(i) Let $x, f \in \mathscr{H}$ be arbitrary. Then

$$r_{\varepsilon}(x \otimes f) = \frac{1}{2}\left(\sqrt{|\langle x, f\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\|x\|\|f\|} + |\langle x, f\rangle|\right).$$

Furthermore, $\langle x, f\rangle = 0$ if and only if $\sigma_{\varepsilon}(x \otimes f) = D(0, \sqrt{\varepsilon^2 + \|x\|\|f\|\varepsilon})$.

(ii) $T = aI$ for some scalar $a \in \mathbb{C}$ if and only if $\sigma_{\varepsilon}(T) = D(a, \varepsilon)$.

(iii) There exists a nontrivial projection P such that $T = aP$ for some nonzero scalar $a \in \mathbb{C}$ if and only if $\sigma_{\varepsilon}(aP) = D(0, \varepsilon) \cup D(a, \varepsilon)$.

(iv) T is self-adjoint if and only if $\sigma_{\varepsilon}(T) \subseteq \{z \in \mathbb{C} : |Im z| < \varepsilon\}$.

The next lemma is quoted from [2].

LEMMA 3. *Let $\varepsilon > 0$, $T \in \mathscr{L}(\mathscr{H})$ and $u, v \in \mathscr{H}$. If $r_{\varepsilon}(u \otimes f) = r_{\varepsilon}(v \otimes f)$ for every unit vector $f \in \mathscr{H}$, then u and v are linearly dependent.*

Let us review some more notation that we will need in the sequel. We will denote by $(e_i)_{1 \leqslant i \leqslant 2}$ the canonical basis of $\mathbb{C}^2$ and by $(E_{ij})_{1 \leqslant i,j \leqslant 2}$ the standard basis of $\mathscr{M}_2$, i.e., $E_{ij} = e_i \otimes e_j$ for all $1 \leqslant i, j \leqslant 2$.

We now have collected all the necessary ingredients and are therefore in a position to state and prove the main result of this section. The following theorem complements [3, Theorem 4] and characterizes nonlinear maps on $\mathscr{M}_2$ that preserve the pseudo spectrum of Jordan semi-triple product of matrices.

THEOREM 1. *Let $\varepsilon > 0$. A map $\Phi : \mathscr{M}_2 \to \mathscr{M}_2$ satisfies*

$$\sigma_{\varepsilon}(\Phi(A)\Phi(B)\Phi(A)) = \sigma_{\varepsilon}(ABA) \quad (A, B \in \mathscr{M}_2) \tag{1}$$

if and only if there exist a scalar $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$ and a unitary matrix $U \in \mathscr{M}_2$ such that Φ has the form

$$\Phi : A \mapsto \alpha U A U^* \ \text{ or } \ \Phi : A \mapsto \alpha U A^{tr} U^*,$$

where A^{tr} denotes the transpose of A.

Proof. Checking the ‘if’ part is straightforward, so we will only deal with the ‘only if’ part. So assume that Φ satisfices (1). We divide the proof of it into several steps.

STEP 1. For every unit vector $x \in \mathbb{C}^2$, there exist a scalar $\alpha_x \in \mathbb{C}$ with $\alpha_x^3 = 1$ and a unit vector $y_x \in \mathbb{C}^2$ such that

$$\Phi(x \otimes x) = \alpha_x y_x \otimes y_x. \tag{2}$$

Let $x \in \mathbb{C}^2$ be a unit vector. By the third statement of Lemma 2, we have

$$D(0,\varepsilon) \cup D(1,\varepsilon) = \sigma_\varepsilon((x \otimes x)^3) = \sigma_\varepsilon(\Phi(x \otimes x)^3),$$

implying that $\Phi(x \otimes x)^3$ is a non trivial orthogonal projection. Observe that $\Phi(x \otimes x)^3$ is not invertible, and so $\Phi(x \otimes x)^3$ is a rank one orthogonal projection and $\Phi(x \otimes x)^3 = z \otimes z$ for some unit vector $z \in \mathbb{C}^2$. Using the Shur decomposition, one can see that there exist $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathscr{M}_2$ and a unitary matrix $V \in \mathscr{M}_2$ such that $\Phi(x \otimes x) = VTV^*$. Thus,

$$\{a^3, c^3\} = \sigma(\Phi(x \otimes x))^3 = \sigma(\Phi(x \otimes x)^3) = \sigma(z \otimes z) = \{0, 1\}$$

and either ($a^3 = 1$ and $c = 0$) or ($a = 0$ and $c^3 = 1$). In the first case, we have

$$z \otimes z = \Phi(x \otimes x)^3 = V \begin{pmatrix} 1 & ba^2 \\ 0 & 0 \end{pmatrix} V^*;$$

which implies that $ba^2 = 0$ and $b = 0$. Similarly, in the case when $a = 0$ and $c^3 = 1$, we have $b = 0$ too, and consequently either

$$\Phi(x \otimes x) = aVE_{11}V^* = aV(e_1 \otimes e_1)V^* = aV(e_1) \otimes V(e_1),$$

or

$$\Phi(x \otimes x) = cVE_{22}V^* = aV(e_2 \otimes e_2)V^* = cV(e_2) \otimes V(e_2).$$

This together with the fact that the vectors $V(e_1)$ and $V(e_2)$ are unit vectors yields the desired conclusion in the step.

As the matrices E_{11} and E_{22} are unitary similar, the proof of the above step ensures that there is a unitary matrix $V \in \mathscr{M}_2$ and $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$ such $\Phi(E_{11}) = \alpha VE_{11}V^*$. Set

$$\Psi(A) = \frac{1}{\alpha} V^* \Phi(A) V$$

for every $A \in \mathscr{M}_2$, and note that, by Lemma 1, the map Ψ preserves the pseudo spectrum of Jordan semi-triple product of matrices and satisfies (2) and $\Psi(E_{11}) = E_{11}$.

STEP 2. $\Psi(I) = I$ and $\Psi(E_{22}) = E_{22}$.

Write $\Psi(I) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and note that

$$\begin{aligned} D(0,\varepsilon) \cup D(1,\varepsilon) &= \sigma_\varepsilon(E_{11} I E_{11}) = \sigma_\varepsilon(\Psi(E_{11})\Psi(I)\Psi(E_{11})) = \sigma_\varepsilon(E_{11}\Psi(I)E_{11}) \\ &= \sigma_\varepsilon(aE_{11}) = D(0,\varepsilon) \cup D(a,\varepsilon), \end{aligned}$$

which implies that $a = 1$. On the other hand, we have

$$D(0,\varepsilon) \cup D(1,\varepsilon) = \sigma_\varepsilon(IE_{11}I) = \sigma_\varepsilon(\Psi(I)E_{11}\Psi(I));$$

which proves that $\Psi(I)E_{11}\Psi(I) = \begin{pmatrix} 1 & b \\ c & bc \end{pmatrix}$ is a non trivial orthogonal projection. Consequently, $c = \bar{b}$ and $\begin{pmatrix} 1 & b \\ \bar{b} & |b|^2 \end{pmatrix}^2 = \begin{pmatrix} 1 & b \\ \bar{b} & |b|^2 \end{pmatrix}$; which shows that $b = c = 0$ and $\Psi(I) = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. Furthermore, by the second statement of Lemma 2 together with the fact that $\sigma_\varepsilon(\Psi(I)^3) = \sigma_\varepsilon(I^3) = D(1,\varepsilon)$, one gets $d^3 = 1$.

Now, write $\Psi(E_{22}) = \gamma z \otimes z$ for some unit vector $z = (z_1, z_2) \in \mathbb{C}^2$ and some scalar $\gamma \in \mathbb{C}$ with $\gamma^3 = 1$. We have

$$\begin{aligned} D(0,\varepsilon) &= \sigma_\varepsilon(E_{11}E_{22}E_{11}) = \sigma_\varepsilon(\Psi(E_{11})\Psi(E_{22})\Psi(E_{11})) = \sigma_\varepsilon(E_{11}\Psi(E_{22})E_{11}) \\ &= \sigma_\varepsilon(\gamma|z_1|^2 E_{11}), \end{aligned}$$

implying that $\gamma|z_1|^2 E_{11} = 0$; which shows that $z_1 = 0$, $|z_2| = 1$ and

$$\Psi(E_{22}) = \gamma E_{22}.$$

We assert that $\gamma = 1$. To do that, observe firstly that $\gamma = d$ is a consequence of the following equality

$$\begin{aligned} D(0,\varepsilon) \cup D(1,\varepsilon) &= \sigma_\varepsilon(E_{22}IE_{22}) = \sigma_\varepsilon(\Psi(E_{22})\Psi(I)\Psi(E_{22})) = \sigma_\varepsilon(\gamma^2 dE_{22}) \\ &= D(0,\varepsilon) \cup D(\gamma^2 d,\varepsilon), \end{aligned}$$

since $\gamma^3 = 1$. Next, set $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathscr{M}_2$, and write $\Psi(A) = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. We have $E_{11}AE_{11} = 0$, and so

$$D(0,\varepsilon) = \sigma_\varepsilon(\Psi(E_{11})\Psi(A)\Psi(E_{11}) = \sigma_\varepsilon(E_{11}\Psi(A)E_{11}) = \sigma_\varepsilon(xE_{11});$$

which yields that $x = 0$. Similarly, we have $t = 0$ since $E_{22}AE_{22} = 0$, and thus $\Psi(A) = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$. We claim that $|y| = 1$. Indeed, the fact that

$$D(1,\varepsilon) \cup D(-1,\varepsilon) = \sigma_\varepsilon(A) = \sigma_\varepsilon(\Psi(I)\Psi(A)\Psi(I)) = \sigma_\varepsilon(\gamma\Psi(A))$$

together with Lemma 1 and the fourth statement of Lemma 2 imply that $\gamma\Psi(A)$ is a self-adjoint matrix and $\sigma(\gamma\Psi(A)) = \{-1,1\}$. This proves that $\overline{\gamma y} = \gamma z$, $z = \gamma\bar{y}$ and $\gamma\Psi(A) = \begin{pmatrix} 0 & \gamma y \\ \bar{\gamma}\bar{y} & 0 \end{pmatrix}$. Thus,

$$\{-1,1\} = \sigma(\gamma\Psi(A)) = \{-|\gamma y|, |\gamma y|\} = \{-|y|, |y|\},$$

and so $|y| = 1$ as claimed.

In order to complete the proof of the step, observe that $A^2 = I$ and $\Psi(A)\Psi(I)\Psi(A) = \begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{pmatrix}$ since $|y| = 1$ and $d = \gamma$, and so

$$D(1,\varepsilon) = \sigma_\varepsilon(AIA) = \sigma_\varepsilon(\Psi(A)\Psi(I)\Psi(A)) = \sigma_\varepsilon(\begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{pmatrix});$$

which implies that $\gamma = 1$ as asserted.

Thus, $\Psi(E_{22}) = E_{22}$ and $\Psi(I) = E_{11} + dE_{22} = E_{11} + \gamma E_{22} = I$; which achieves the proof of the step.

STEP 3. There exists a unimodular scalar $\beta \in \mathbb{C}$ such that either

$$\Psi(E_{12}) = \beta E_{12} \quad \text{and} \quad \Psi(E_{21}) = \bar{\beta} E_{21}, \tag{3}$$

or

$$\Psi(E_{12}) = \beta E_{21} \quad \text{and} \quad \Psi(E_{21}) = \bar{\beta} E_{12}. \tag{4}$$

Write $\Psi(E_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and note that

$$D(0,\varepsilon) = \sigma_\varepsilon(E_{11}E_{12}E_{11}) = \sigma_\varepsilon(\Psi(E_{11})\Psi(E_{12})\Psi(E_{11})) = \sigma_\varepsilon(aE_{11}),$$

implying that $a = 0$. Similarly, we have $d = 0$ since $E_{22}E_{12}E_{22} = 0$. Furthermore, it follows from the equality $D(0,\varepsilon) = \sigma_\varepsilon(E_{12}^3) = \sigma_\varepsilon(\Psi(E_{12})^3)$ that $\Psi(E_{12})^3 = 0$ and $bc = 0$, and thus

$$\Psi(E_{12}) = bE_{12} \text{ or } \Psi(E_{12}) = cE_{21}.$$

Similar to the above argument, we also get

$$\Psi(E_{21}) = b'E_{12} \text{ or } \Psi(E_{21}) = c'E_{21}$$

for some scalars $b', c' \in \mathbb{C}$. We claim that either

$$\Psi(E_{12}) = bE_{12} \quad \text{and} \quad \Psi(E_{21}) = c'E_{21} \quad \text{with} \quad |b| = |c'| = 1,$$

or

$$\Psi(E_{12}) = cE_{21} \quad \text{and} \quad \Psi(E_{21}) = b'E_{12} \quad \text{with} \quad |c| = |b'| = 1.$$

Indeed, if $\Psi(E_{12}) = bE_{12}$ and $\Psi(E_{21}) = b'E_{12}$, we have

$$\sigma_\varepsilon(E_{12}) = \sigma_\varepsilon(E_{12}E_{21}E_{12}) = \sigma_\varepsilon(\Psi(E_{12})\Psi(E_{21})\Psi(E_{12})) = \sigma_\varepsilon(b^2b'E_{12}^3) = D(0,\varepsilon).$$

However this implies that $E_{12} = 0$, a contradiction. Thus, $\Psi(E_{12}) = bE_{12}$ and $\Psi(E_{21}) = c'E_{21}$. Observe that

$$\sigma_\varepsilon(\Psi(X)) = \sigma_\varepsilon(X),$$

for every $X \in \mathscr{M}_2$ since $\Psi(I) = I$. This together with the first statement of Lemma 2 implies that

$$D(0,\sqrt{\varepsilon^2+\varepsilon}) = \sigma_\varepsilon(E_{12}) = \sigma_\varepsilon(\Psi(E_{12})) = \sigma_\varepsilon(bE_{12}) = D(0,\sqrt{\varepsilon^2+|b|\,\varepsilon});$$

which shows that $|b| = 1$. Similarly, we have, in this case, $|c'| = 1$. The remainder case can be treated analogously, and the claim is proved.

Now, set $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and let us prove that $\Psi(B) = \begin{pmatrix} 1 & y \\ \bar{y} & 1 \end{pmatrix}$ for some scalar $y \in \mathbb{C}$ with $|y| = 1$. To do so, write $\Psi(B) = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, and note that $E_{11}BE_{11} = E_{11}$; which yields that

$$\begin{aligned} D(0,\varepsilon) \cup D(1,\varepsilon) &= \sigma_\varepsilon(E_{11}BE_{11}) = \sigma_\varepsilon(\Psi(E_{11})\Psi(B)\Psi(E_{11})) = \sigma_\varepsilon(xE_{11}) \\ &= D(0,\varepsilon) \cup D(x,\varepsilon) \end{aligned}$$

and shows that $x = 1$. Similarly, we get $t = 1$ since $E_{22}BE_{22} = E_{22}$. As

$$\sigma_\varepsilon(\Psi(B)) = \sigma_\varepsilon(B) = D(0,\varepsilon) \cup D(2,\varepsilon),$$

we get from the forth statement of Lemma 2 that $\Psi(B)$ is self-adjoint and $\sigma(\Psi(B)) = \{0,2\}$, which imply that $\bar{y} = z$ and $|y| = 1$, as desired.

In order to complete the proof of the step, assume firstly that $\Psi(E_{12}) = bE_{12}$ and $\Psi(E_{21}) = c'E_{21}$ with $|b| = |c'| = 1$, and let us prove that $c' = \bar{b}$. We have

$$BE_{12}B = B \text{ and } \Psi(B)\Psi(E_{12})\Psi(B) = \begin{pmatrix} b\bar{y} & b \\ b\bar{y}^2 & b\bar{y} \end{pmatrix},$$

and so

$$D(0,\varepsilon) \cup D(2,\varepsilon) = \sigma_\varepsilon(BE_{12}B) = \sigma_\varepsilon(\Psi(B)\Psi(E_{12})\Psi(B)). \tag{5}$$

This implies that $\begin{pmatrix} b\bar{y} & b \\ b\bar{y}^2 & b\bar{y} \end{pmatrix}$ is a self-adjoint matrix having 0 and 2 as eigenvalues. In particular, we have $\bar{b} = b\bar{y}^2$ and so $b^2 = y^2$ since $|b| = |y| = 1$; which shows that $b = y$ or $b = -y$. The case when $b = -y$ cannot occur since otherwise we have

$$\Psi(B)\Psi(E_{12})\Psi(B) = \begin{pmatrix} -1 & b \\ \bar{b} & -1 \end{pmatrix},$$

and so from (5) we deduce that

$$D(0,\varepsilon) \cup D(2,\varepsilon) = \sigma_\varepsilon\left(\begin{pmatrix} -1 & b \\ \bar{b} & -1 \end{pmatrix}\right) = D(0,\varepsilon) \cup D(-2,\varepsilon),$$

a contradiction. Consequently, $b = y$. Similarly, we also have $y = \overline{c'}$ since

$$BE_{21}B = B \text{ and } \Psi(B)\Psi(E_{21})\Psi(B) = \begin{pmatrix} cy & cy^2 \\ c & cy \end{pmatrix};$$

which proves the step in this case.

In the remainder case when $\Psi(E_{12}) = cE_{21}$ and $\Psi(E_{21}) = b'E_{12}$, replacing Ψ by the mapping $X \mapsto \Psi(A^{tr})$, similar to the above argument, we also get $c = \overline{b'}$; which proves the step in this case too.

STEP 4. If Ψ satisfies the the condition (3), then

$$\Psi\left(\begin{pmatrix}1 & x\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix}1 & \beta x\\ 0 & 0\end{pmatrix} \text{ and } \Psi\left(\begin{pmatrix}1 & 0\\ x & 0\end{pmatrix}\right)=\begin{pmatrix}1 & 0\\ \bar{\beta}x & 0\end{pmatrix},$$

for all $x\in\mathbb{C}$.

Set $S:=\begin{pmatrix}1 & x\\ 0 & 0\end{pmatrix}$, and write $\Psi(S)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. From the equalities

$$E_{11}SE_{11}=E_{11},\ E_{22}SE_{22}=0 \text{ and } E_{12}SE_{12}=0$$

together with (1), we deduce that

$$a=1,\ d=0 \text{ and } c=0,$$

respectively. To prove the first part of the step it suffices to show that $b=\beta x$. To do so, let $B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, and note that, by the proof of the above step, $\Psi(B)=\begin{pmatrix}1 & \beta\\ \bar{\beta} & 1\end{pmatrix}$. It easy to check that

$$BSB=(1+x)B \text{ and } \Psi(B)\Psi(S)\Psi(B)=\begin{pmatrix}1+\bar{\beta}b & \beta+b\\ \bar{\beta}+\bar{\beta}^2 b & 1+\bar{\beta}b\end{pmatrix},$$

and so

$$\sigma_\varepsilon\left(\begin{pmatrix}1+\bar{\beta}b & \beta+b\\ \bar{\beta}+\bar{\beta}^2 b & 1+\bar{\beta}b\end{pmatrix}\right)=\sigma_\varepsilon\left((1+x)B\right)=D(0,\varepsilon)\cup D(2(1+x),\varepsilon);$$

which implies that

$$\sigma\left(\begin{pmatrix}1+\bar{\beta}b & \beta+b\\ \bar{\beta}+\bar{\beta}^2 b & 1+\bar{\beta}b\end{pmatrix}\right)=\{0,2(1+x)\}.$$

On the other hand, straightforward computations give that

$$\sigma\left(\begin{pmatrix}1+\bar{\beta}b & \beta+b\\ \bar{\beta}+\bar{\beta}^2 b & 1+\bar{\beta}b\end{pmatrix}\right)=\left\{0,2(1+\bar{\beta}b)\right\},$$

and thus

$$2(1+\bar{\beta}b)=2(1+x).$$

This implies that $b=\beta x$ and $\Psi\left(\begin{pmatrix}1 & x\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix}1 & \beta x\\ 0 & 0\end{pmatrix}$ as desired.

To prove the second part of the step, set $L:=\begin{pmatrix}1 & 0\\ x & 0\end{pmatrix}$, and write $\Psi(L)=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$. Again from the condition (1) together with the fact that

$$E_{11}LE_{11}=E_{11},\ E_{22}LE_{22}=0 \text{ and } E_{12}LE_{12}=0,$$

one gets that

$$a' = 1,\ d' = 0 \quad \text{and} \quad b' = 0.$$

Similar augment as above together with the fact that

$$BLB = (1+x)B,\ \Psi(B)\Psi(L)\Psi(B) = \begin{pmatrix} 1+\beta c' & \beta+\beta^2 c' \\ \bar{\beta}+c' & 1+\beta c' \end{pmatrix}$$

and

$$\sigma\left(\begin{pmatrix} 1+\beta c' & \beta+\beta^2 c' \\ \bar{\beta}+c' & 1+\beta c' \end{pmatrix}\right) = \{0, 2(1+\beta c')\}$$

allows to get that $c' = \bar{\beta}x$; which concludes the proof of the step.

STEP 5. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a matrix in $\mathscr{M}_2$. If Ψ satisfies the condition (3), then

$$\Psi(A) = \begin{pmatrix} a_{11} & \beta a_{12} \\ \bar{\beta} a_{21} & a_{22} \end{pmatrix}.$$

Set $\Psi(A) = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$. We claim that

$$a_{ij} = 0 \Longleftrightarrow a'_{ij} = 0,$$

for all $1 \leqslant i, j \leqslant 2$. Observe that, for any $1 \leqslant i, j \leqslant 2$,

$$E_{ij}AE_{ij} = \begin{cases} a_{ij}E_{ij}, & \text{if } i = j, \\ a_{ji}E_{ij}, & \text{if } i \neq j, \end{cases}$$

and

$$\Psi(E_{ij})\Psi(A)\Psi(E_{ij}) = \begin{cases} a'_{ij}E_{ij}, & \text{if } i = j, \\ \beta^2 a'_{ji}E_{ij}, & \text{if } i = 1 \text{ and } j = 2, \\ \bar{\beta}^2 a'_{ji}E_{ij}, & \text{if } i = 2 \text{ and } j = 1. \end{cases}$$

This and the condition (1) yield that

$$\Psi(E_{ij})\Psi(A)\Psi(E_{ij}) = 0 \Longleftrightarrow E_{ij}AE_{ij} = 0;$$

which implies that $a'_{ij} = 0$ if and only if $a_{ij} = 0$ for all $1 \leqslant i, j \leqslant 2$, as claimed.

We assert that if $a_{ii} \neq 0$, then $a'_{ii} = a_{ii}$ for every $1 \leqslant i \leqslant 2$. Indeed, we have

$$D(0,\varepsilon) \cup D(a_{ii},\varepsilon) = \sigma_\varepsilon(E_{ii}AE_{ii}) = \sigma_\varepsilon(\Psi(E_{ii})\Psi(A)\Psi(E_{ii})) = D(0,\varepsilon) \cup D(a'_{ii},\varepsilon),$$

and so $a'_{ii} = a_{ii}$ as asserted. Thus

$$\Psi(A) = \begin{pmatrix} a_{11} & a'_{12} \\ a'_{21} & a_{22} \end{pmatrix}.$$

Now, let $x \in \mathbb{C}$ be a nonzero scalar, and set $X := \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$. By the above step, we have $\Psi(X) = \begin{pmatrix} 1 & \beta x \\ 0 & 0 \end{pmatrix}$. Note that the matrices X and $\Psi(X)$ are nontrivial projections, and thus from the condition (1) together with the fact that

$$XAX = (a_{11} + xa_{21})X$$

and

$$\Psi(X)\Psi(A)\Psi(X) = \left(a_{11} + \beta x a'_{21}\right)\Psi(X),$$

we deduce that

$$a_{11} + xa_{21} = a_{11} + \beta x a'_{21};$$

which implies that $a'_{21} = \bar{\beta} a_{21}$.

Next, let $x \in \mathbb{C}$ be a nonzero scalar, and set $Y = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$. Again, by the above step, we have $\Psi(Y) = \begin{pmatrix} 1 & 0 \\ \bar{\beta} x & 0 \end{pmatrix}$. Since the matrices Y and $\Psi(Y)$ are nontrivial projections satisfying

$$YAY = (a_{11} + xa_{12})Y$$

and

$$\Psi(Y)\Psi(A)\Psi(Y) = \left(a_{11} + \bar{\beta} x a'_{12}\right)\Psi(Y),$$

similar argument as above allows to get that $a'_{12} = \beta a_{12}$, and finishes the proof of the step.

In order to complete the proof of the theorem, set

$$W := \begin{pmatrix} 1 & 0 \\ 0 & \bar{\beta} \end{pmatrix},$$

and note that W is a unitary matrix. Assume firstly that Ψ satisfies the condition (3), and observe that

$$\Psi(A) = \begin{pmatrix} a_{11} & \beta a_{12} \\ \bar{\beta} a_{21} & a_{22} \end{pmatrix} = W \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} W^* = WAW^*$$

for every $A = (a_{ij}) \in \mathcal{M}_2$. This proves, in this case, by letting $U = VW$, that Φ has the form

$$\Phi : A \mapsto \alpha UAU^*,$$

as required.

In the remainder case when Ψ satisfies the condition (4), set $\chi(A) = \Psi(A^{tr})$, for every $A = (a_{ij}) \in \mathcal{M}_2$, and note that the mapping χ satisfies the condition (1),

$$\chi(E_{12}) = \bar{\beta} E_{12}, \chi(E_{21}) = \beta E_{21} \text{ and } \chi(X) = X \text{ for every } X \in \{I, E_{11}, E_{22}\}.$$

Similar to the above argument, we get

$$\chi\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & \bar{\beta} a_{12} \\ \beta a_{21} & a_{22} \end{pmatrix},$$

for every $A = (a_{ij}) \in \mathcal{M}_2$. Thus

$$\Psi\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & \bar{\beta} a_{21} \\ \beta a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & \beta a_{12} \\ \bar{\beta} a_{21} & a_{22} \end{pmatrix}^{tr},$$

and so

$$\Psi(A) = (WAW^*)^{tr} = \left(W^{tr}\right)^* A^{tr} W^{tr} = W^* A^{tr} W,$$

for every $A = (a_{ij}) \in \mathcal{M}_2$. This shows, in this case, by letting $U = VW^*$, that Φ has the form

$$\Phi : A \mapsto \alpha U A^{tr} U^*,$$

as asserted. The proof of the theorem is therefore complete. □

3. Pseudo spectra preservers of matrix skew products

In this section, we give the structure of non linear maps on $\mathcal{M}_2$ that preserve the pseudo spectrum of Jordan semi-triple product of matrices.

THEOREM 2. *Let $\varepsilon > 0$. A map $\Phi : \mathcal{M}_2 \to \mathcal{M}_2$ satisfies*

$$\sigma_\varepsilon\left(\Phi(A)\Phi(B)^*\right) = \sigma_\varepsilon\left(AB^*\right) \qquad (A, B \in \mathcal{M}_2) \tag{6}$$

if and only if there exist unitary matrices $U, V \in \mathcal{M}_2$ such that Φ has the form

$$\Phi : A \mapsto UAV.$$

Proof. The sufficiency condition can be readily checked. To prove the necessity, assume that

$$\sigma_\varepsilon(\Phi(A)\Phi(B)^*) = \sigma_\varepsilon(AB^*) \qquad (A, B \in \mathcal{M}_2).$$

We divide the proof of it into several claims.

CLAIM 1. There are unitary matrices $U, V \in \mathcal{M}_2$ such that

$$\Phi(E_{11}) = UE_{11}V.$$

Proof. For every nonzero vector $x \in \mathbb{C}^2$, we have

$$D(0,\varepsilon) \cup D(\|x\|^2, \varepsilon) = \sigma_\varepsilon((x \otimes x)(x \otimes x)^*) = \sigma_\varepsilon\left(\Phi(x \otimes x)\Phi(x \otimes x)^*\right).$$

The third statement of Lemma 2 tells us that there is a nontrivial projection $P \in \mathcal{M}_2$ such that

$$\Phi(x \otimes x)\Phi(x \otimes x)^* = \|x\|^2 P,$$

and then

$$\sigma\left(\Phi(x\otimes x)\Phi(x\otimes x)^*\right)=\left\{0,\|x\|^2\right\}.$$

Hence, the singular values of $\Phi(x\otimes x)$ are 0 and $\|x\|$, and thus there are unitary matrices $U_x,V_x\in\mathscr{M}_2$ such that

$$\Phi(x\otimes x)=U_x\begin{pmatrix}\|x\| & 0\\ 0 & 0\end{pmatrix}V_x=\|x\|\,U_xE_{11}V_x.$$

In particular, for $x=e_1$, there are unitary matrices $U_1,V_1\in\mathscr{M}_2$ such that

$$\Phi(E_{11})=U_1E_{11}V_1. \quad \square$$

Set

$$\Psi(A)=U_1^*\Phi(A)V_1^*,$$

for every $A\in\mathscr{M}_2$, and note that the map Ψ preserves the pseudo spectrum of skew-product of matrices and satisfies $\Psi(E_{11})=E_{11}$.

CLAIM 2. There is a unimodular scalar $\lambda\in\mathbb{C}$ such that

$$\Psi(I)=\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\ \text{ and }\ \Psi(E_{22})=\begin{pmatrix}0 & 0\\ 0 & \lambda\end{pmatrix}.$$

Proof. Write $\Psi(I)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. We have

$$\sigma_\varepsilon\left(\begin{pmatrix}\bar a & \bar c\\ 0 & 0\end{pmatrix}\right)=\sigma_\varepsilon\left(\Psi(E_{11})\Psi(I)^*\right)=\sigma_\varepsilon(E_{11}I^*)=D(0,\varepsilon)\cup D(1,\varepsilon).$$

This proves that $\begin{pmatrix}\bar a & \bar c\\ 0 & 0\end{pmatrix}$ is a nontrivial projection and shows that $a=1$ and $c=0$. On the other hand, we have

$$\sigma_\varepsilon\left(\begin{pmatrix}1+|b|^2 & b\bar d\\ d\bar b & |d|^2\end{pmatrix}\right)=\sigma_\varepsilon(\Psi(I)\Psi(I)^*)=\sigma_\varepsilon(II^*)=D(1,\varepsilon).$$

Thus, by the second statement of Lemma 2, we have $b=0$ and $|d|=1$.

Now, write $\Psi(E_{22})=\begin{pmatrix}x & y\\ z & t\end{pmatrix}$, and note that

$$\sigma_\varepsilon\left(\begin{pmatrix}\bar x & \bar z\\ 0 & 0\end{pmatrix}\right)=\sigma_\varepsilon(\Psi(E_{11})\Psi(E_{22})^*)=\sigma_\varepsilon(E_{11}E_{22}^*)=D(0,\varepsilon),$$

implying that $x=z=0$. Furthermore, the fact that

$$\sigma_\varepsilon\left(\begin{pmatrix}0 & y\bar d\\ 0 & t\bar d\end{pmatrix}\right)=\sigma_\varepsilon(\Psi(E_{22})\Psi(I)^*)=\sigma_\varepsilon(E_{22}I^*)=D(0,\varepsilon)\cup D(1,\varepsilon)$$

proves that the matrix $\begin{pmatrix} 0 & y\bar{d} \\ 0 & t\bar{d} \end{pmatrix}$ is a nontrivial projection, and shows that $y = 0$ and $t = d$ since $|d| = 1$. Thus, by letting $\lambda = d$, we get the desired conclusion in the claim. □

Set

$$\chi(A) := \Psi(I)^*\Psi(A),$$

for every $A \in \mathscr{M}_2$. Observe that $\Psi(I)$ is a unitary matrix, and so the mapping χ satisfies

$$\sigma_\varepsilon(\chi(A)\chi(B)^*) = \sigma_\varepsilon(\Psi(A)\Psi(B)^*) = \sigma_\varepsilon(AB^*) \qquad (A, B \in \mathscr{M}_2)$$

and

$$\chi(A) = A, \qquad \text{for all } A \in \{E_{11}, E_{22}, I\}.$$

CLAIM 3. There is a unimodular scalar $\xi \in \mathbb{C}$ such that

$$\chi(E_{12}) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \ \chi(E_{21}) = \begin{pmatrix} 0 & 0 \\ \bar{\xi} & 0 \end{pmatrix} \text{ and } \chi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & \xi \\ \bar{\xi} & 0 \end{pmatrix}.$$

Proof. Write $\chi(E_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$\sigma_\varepsilon\left(\begin{pmatrix} \bar{a} & \bar{c} \\ 0 & 0 \end{pmatrix}\right) = \sigma_\varepsilon(\chi(E_{11})\chi(E_{12})^*) = \sigma_\varepsilon(E_{11}E_{12}^*) = D(0,\varepsilon),$$

implying that $a = c = 0$. The fact that

$$\sigma_\varepsilon\left(\begin{pmatrix} |b|^2 & b\bar{d} \\ d\bar{b} & |d|^2 \end{pmatrix}\right) = \sigma_\varepsilon(\chi(E_{12})\chi(E_{12})^*) = \sigma_\varepsilon(E_{12}E_{12}^*) = D(0,\varepsilon) \cup D(1,\varepsilon)$$

yields that

$$\sigma\left(\begin{pmatrix} |b|^2 & b\bar{d} \\ d\bar{b} & |d|^2 \end{pmatrix}\right) = \{0, 1\},$$

and so

$$|b|^2 + |d|^2 = Trace\left(\begin{pmatrix} |b|^2 & b\bar{d} \\ d\bar{b} & |d|^2 \end{pmatrix}\right) = 1;$$

where *Trace* denotes the usual trace function on matrices. Since $\chi(I) = I$, we have $\sigma_\varepsilon(E_{12}) = \sigma_\varepsilon(\chi(E_{12}))$, and then, by Lemma 2, we have

$$\sqrt{\varepsilon^2+\varepsilon} = r_\varepsilon(E_{12}) = r_\varepsilon(\chi(E_{12})) = r_\varepsilon((be_1+de_2)\otimes e_2) = \frac{1}{2}\left(\sqrt{|d|^2+4\varepsilon^2+4\varepsilon}+|d|\right);$$

which proves that $d = 0$. Consequently, $|b| = 1$ and $\chi(E_{12}) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

Again, since

$$\sigma_\varepsilon(\chi(E_{22})\chi(E_{21})^*) = \sigma_\varepsilon(E_{22}E_{21}^*) = D(0,\varepsilon),$$

we have

$$E_{22}\chi(E_{21})^* = \chi(E_{22})\chi(E_{21})^* = 0;$$

which implies that $\chi(E_{21}) = b'E_{11} + c'E_{21}$ for some $b', c' \in \mathbb{C}$. Similar to the above argument, we also get $b' = 0$, $|c'| = 1$ and $\chi(E_{21}) = \begin{pmatrix} 0 & 0 \\ c' & 0 \end{pmatrix}$.

Now, let $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and write $\chi(A) = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. We have

$$D(1,\varepsilon) \cup D(-1,\varepsilon) = \sigma_\varepsilon(AI^*) = \sigma_\varepsilon(\chi(A)\chi(I)^*) = \sigma_\varepsilon(\chi(A)),$$

implying that $\chi(A)$ is a self-adjoint matrix and $\sigma(\chi(A)) = \{-1;1\}$. Thus,

$$z = \bar{y},\ x + t = Trace(\chi(A)) = 0 \text{ and } xt - yz = \det(\chi(A)) = -1;$$

which shows that

$$t = -x \text{ and } x^2 + |y|^2 = 1.$$

Furthermore, we have

$$\sigma_\varepsilon(E_{12}) = \sigma_\varepsilon(E_{11}A^*) = \sigma_\varepsilon(\chi(E_{11})\chi(A)^*) = \sigma_\varepsilon(\bar{x}E_{11} + \bar{y}E_{12}) = \sigma_\varepsilon(e_1 \otimes (xe_1 + ye_2)),$$

and so

$$\sqrt{\varepsilon^2 + \varepsilon} = r_\varepsilon(E_{12}) = r_\varepsilon(e_1 \otimes (xe_1 + ye_2)) = \frac{1}{2}(\sqrt{|x|^2 + 4\varepsilon^2 + 4\varepsilon} + |x|);$$

which yields that $x = 0$, $|y| = 1$ and $\chi(A) = \begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix}$.

To complete the proof of the claim, it suffices to show that $b = y = \overline{c'}$. Observe that

$$\sigma_\varepsilon(E_{11}) = \sigma_\varepsilon(E_{12}A^*) = \sigma_\varepsilon(\chi(E_{12})\chi(A)^*) = \sigma_\varepsilon(b\bar{y}E_{11}),$$

and so $b\bar{y} = 1$ and $b = y$. Similarly, we have

$$\sigma_\varepsilon(E_{22}) = \sigma_\varepsilon(E_{21}A^*) = \sigma_\varepsilon(\chi(E_{21})\chi(A)^*) = \sigma_\varepsilon(c'yE_{22}),$$

implying that $c'y = 1$ and $y = \overline{c'}$. Thus, by letting $\xi = y$, we get the desired conclusion in the claim. □

Let $W = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Set

$$\varphi(A) = W\chi(A)W^*,$$

for every $A \in \mathscr{M}_2$, and note that W is a unitary matrix and the mapping φ satisfies

$$\sigma_\varepsilon(\varphi(A)\varphi(B)^*) = \sigma_\varepsilon(\chi(A)\chi(B)^*) = \sigma_\varepsilon(AB^*) \qquad (A,B \in \mathscr{M}_2)$$

and

$$\varphi(A) = A \text{ for every } A \in \{E_{11}, E_{12}, E_{21}, E_{22}, I, T\}.$$

CLAIM 4. $\varphi\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ and $\varphi\left(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}$ for all nonzero scalars $x, y \in \mathbb{C}$.

Proof. Let $X = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$, and write $\varphi(X) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$D(0,\varepsilon) \cup D(x,\varepsilon) = \sigma_\varepsilon\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = \sigma_\varepsilon(XE_{11}^*) = \sigma_\varepsilon(\varphi(X)\varphi(E_{11})^*) = \sigma_\varepsilon\left(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right).$$

This proves that $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = xP$ for some nontrivial projection $P \in \mathscr{M}_2$, and shows that $a = x$ and $c = 0$. Since

$$\sigma_\varepsilon\left(\begin{pmatrix} |x|^2+|y|^2 & 0 \\ 0 & 0 \end{pmatrix}\right) = \sigma_\varepsilon(XX^*) = \sigma_\varepsilon(\varphi(X)\varphi(X)^*) = \sigma_\varepsilon\left(\begin{pmatrix} |x|^2+|b|^2 & b\overline{d} \\ d\overline{b} & |d|^2 \end{pmatrix}\right),$$

we deduce that

$$\sigma\left(\begin{pmatrix} |x|^2+|y|^2 & 0 \\ 0 & 0 \end{pmatrix}\right) = \sigma(XX^*) = \sigma(\varphi(X)\varphi(X)^*) = \sigma\left(\begin{pmatrix} |x|^2+|b|^2 & b\overline{d} \\ d\overline{b} & |d|^2 \end{pmatrix}\right).$$

In particular, we have

$$0 = \det(XX^*) = \det(\varphi(X)\varphi(X)^*) = (|x|^2+|b|^2)\,|d|^2 - |bd|^2$$

and

$$|x|^2+|y|^2 = Trace(XX^*) = Trace(\varphi(X)\varphi(X)^*) = |x|^2+|b|^2+|d|^2;$$

which yield that $d = 0$, $|y| = |b|$. On the other hand, the fact that

$$\sigma_\varepsilon(yE_{11}) = \sigma_\varepsilon(XE_{12}^*) = \sigma_\varepsilon(\varphi(X)\varphi(E_{12})^*) = \sigma_\varepsilon(bE_{11})$$

proves that $b = y$, and consequently $\varphi(X) = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ as required.

Now, let $Y = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}$, and write $\varphi(Y) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. We have

$$\sigma_\varepsilon(yE_{22}) = \sigma_\varepsilon(YE_{22}^*) = \sigma_\varepsilon(\varphi(Y)\varphi(E_{22})^*) = \sigma_\varepsilon(b'E_{12}+d'E_{22}),$$

implying that $b' = 0$ and $d' = y$. Using the equality $\sigma_\varepsilon(YY^*) = \sigma_\varepsilon(\varphi(Y)\varphi(Y)^*)$, similar to the above argument we get that $a' = 0$ and $|x| = |c'|$, and so $\varphi(Y) = \begin{pmatrix} 0 & 0 \\ c' & y \end{pmatrix}$. Finally, the fact that

$$\sigma_\varepsilon(xE_{22}) = \sigma_\varepsilon(YE_{21}^*) = \sigma_{\varepsilon\varepsilon}(\varphi(Y)\varphi(E_{21})^*) = \sigma_\varepsilon(c'E_{22})$$

ensures that $x = c'$ and $\varphi(Y) = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}$ as claimed. □

CLAIM 5. $\varphi(X) = X$ for all $X \in \mathscr{M}_2$.

Proof. Let $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathscr{M}_2$, and write $\varphi(X) = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$. For any $1 \leqslant i, j \leqslant 2$, we have

$$\sigma_\varepsilon\left(E_{ij}X^*\right) = \sigma_\varepsilon\left(\varphi(E_{ij})\varphi(X)^*\right),$$

implying that

$$x_{ij} = 0 \Longleftrightarrow y_{ij} = 0.$$

Thus, we may assume that $x_{ij} \neq 0$ and $y_{ij} \neq 0$ for all $1 \leqslant i, j \leqslant 2$. Let $a, b \in \mathbb{C}$ be nonzero scalars and set $Y = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathscr{M}_2$. The fact that

$$YX^* = \begin{pmatrix} a\bar{x}_{11} + b\bar{x}_{12} & a\bar{x}_{21} + b\bar{x}_{22} \\ 0 & 0 \end{pmatrix}$$

and

$$\varphi(Y)\varphi(X)^* = Y\varphi(X)^* = \begin{pmatrix} a\bar{y}_{11} + b\bar{y}_{12} & a\bar{y}_{21} + b\bar{y}_{22} \\ 0 & 0 \end{pmatrix}$$

together with the equality (6) imply that

$$a\bar{x}_{21} + b\bar{x}_{22} = 0 \Longleftrightarrow a\bar{y}_{21} + b\bar{y}_{22} = 0.$$

In particular, the vectors (a, b) and (x_{21}, x_{22}) are orthogonal if and only if the vectors (a, b) and (y_{21}, y_{22}) are. Consequently, there exists a nonzero $\alpha \in \mathbb{C}$ such that $y_{21} = \alpha x_{21}$ and $y_{22} = \alpha x_{22}$. Using the matrix $Y' = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ instead of Y, similar to the above argument allows to get that there is a nonzero $\mu \in \mathbb{C}$ such that $y_{11} = \beta x_{11}$ and $y_{12} = \beta x_{12}$, and thus

$$\varphi(X) = \begin{pmatrix} \beta x_{11} & \beta x_{12} \\ \alpha x_{21} & \alpha x_{22} \end{pmatrix}. \tag{7}$$

First consider the case when X is invertible. Then the row vectors (x_{21}, x_{22}) and (x_{11}, x_{12}) of the matrix X are linearly independent, and so there is $(c, d) \in \mathbb{C}^2$ such that $cx_{11}+dx_{12}\neq 0$ and $cx_{21}+dx_{22}=0$. Set $Z:=\begin{pmatrix}\bar{c} & \bar{d}\\ 0 & 0\end{pmatrix}$, and note that $\varphi(Z)=Z$ and

$$\sigma_\varepsilon((cx_{11}+dx_{12})E_{11})=\sigma_\varepsilon(XZ^*)=\sigma_\varepsilon(\varphi(X)\varphi(Z)^*)=\sigma_\varepsilon(\beta(cx_{11}+dx_{12})E_{11}).$$

This implies that $cx_{11}+dx_{12}=\beta(cx_{11}+dx_{12})$ and shows that $\beta=1$. Consider the column vectors of X and the matrix $Z':=\begin{pmatrix}0 & 0\\ \bar{c} & \bar{d}\end{pmatrix}$ instead of Z, similar to the above argument yields that $\alpha=1$, and thus $\varphi(X)=X$ for every invertible matrix $X\in\mathscr{M}_2$.

Next consider the case when X is of rank one. The equality (7) shows that there is a map $\tau:\mathscr{M}_2\to\mathscr{M}_2$ such that

$$\varphi(x\otimes f)=\tau(x)\otimes f,$$

for every rank one matrix $x\otimes f\in\mathscr{M}_2$. This together with the fact that $\sigma_\varepsilon(\varphi(X))=\sigma_\varepsilon(X)$ for every $X\in\mathscr{M}_2$ imply that, for every $x\in\mathbb{C}^2$,

$$r_\varepsilon(x\otimes f)=r_\varepsilon(\tau(x)\otimes f),\quad \forall f\in\mathbb{C}^2.$$

Lemma 3 tells us that, for every $x\in\mathbb{C}^2$, the vectors $\tau(x)$ and x are linearly dependent, and so there is a functional $\ell:\mathbb{C}^2\to\mathbb{C}$ such that $\tau(x)=\ell(x)x$ for every $x\in\mathbb{C}^2$. Furthermore, we have

$$\sigma_\varepsilon(\ell(x)x\otimes x)=\sigma_\varepsilon(\varphi(x\otimes x))=\sigma_\varepsilon(x\otimes x),$$

for every $x\in\mathbb{C}^2$, implying that $\ell(x)=1$ for every $x\in\mathbb{C}^2$. Therefore $\varphi(X)=X$ for every rank one matrix $X\in\mathscr{M}_2$. Consequently, $\varphi(X)=X$ for all $X\in\mathscr{M}_2$ and the claim is proved. □

By letting $U=U_1\psi(I)W^*$ and $V=WV_1$, the map Φ has the form

$$\Phi: A\mapsto UAV,$$

as asserted, and the proof of the theorem is therefore complete. □

Acknowledgement. The authors wish to express their thanks to the referee for carefully reading the paper and for given valuable comments and suggestions which improved the initial draft. The first author thanks the hospitality of the organizers of the Conferences "Presevers Everywhere, 19-23 June 2017, Szeged, Hungary" and "International Workshop on Operator Theory and its Applications, 22-26 July 2019, Lisbon, Portugal", where related problems of the ones treated in this paper were presented.

REFERENCES

[1] M. BENDAOUD, *Preservers of local spectrum of matrix Jordan triple products*, Linear Algebra Appl., 471, 1, 2015, 604–614.

[2] M. BENDAOUD, A. BENYOUNESS AND M. SARIH, *Preservers of pseudo spectral radius of operator products*, Linear Algebra Appl., 489, 15 January, 2016, 186–198.

[3] M. BENDAOUD, A. BENYOUNESS AND M. SARIH, *Preservers of pseudo spectra of operator Jordan triple products*, Oper. Matrices, 10, 1, 2016, 45–56.
[4] M. BENDAOUD, A. BENYOUNESS AND M. SARIH *Nonlinear maps preserving the pseudo spectral radius of skew semi-triple products of operators*, Acta Sci. Math. (Szeged), 84, 1-2, 2018, 39–47.
[5] M. BENDAOUD, M. JABBAR AND M. SARIH, *Preservers of local spectra of operator products*, Linear Multilinear Algebra, 63, 4, 2015, 806–819.
[6] R. BHATIA, P. ŠEMRL AND A. R. SOUROUR, *Maps on matrices that preserve the spectral radius distance*, Studia Math., 134, 1-3, 1999, 99–110.
[7] J. T. CHAN, C. K. LI AND N. S. SZE, *Mappings on matrices: invariance of functional values of matrix products*, J. Austral. Math. Soc., 81, 2, 2006, 165–184.
[8] J. T. CHAN, C. K. LI AND N. S. SZE, *Mappings preserving spectra of products of matrices*, Proc. Amer. Math. Soc., 135, 4, 2007, 977–986.
[9] J. CUI, V. FORSTALL, C.-K. LI AND V. YANNELLO, *Properties and preservers of the pseudospectrum*, Linear Algebra Appl., 436, 2, 2012, 316–325.
[10] J. CUI AND C.-K. LI, *Maps preserving peripheral spectrum of Jordan products of operators*, Oper. Matrices, 6, 6, 2012, 129–146.
[11] J. CUI, C.K. LI AND Y.T. POON, *Pseudospectra of special operators and pseudospectrum preservers*, J. Math. Anal. Appl., 419, 2, 2014, 1261–1273.
[12] J. CUI, C.K. LI AND Y.T. PONN, *Preservers of unitary similarity functions on Lie products of matrices*, Linear Algebra Appl., 498, 1 June, 2016, 160–180.
[13] G. DOLINAR, J. HOU, B. KUZMA AND X. QI, *Spectrum nonincreasing maps on matrices*, Linear Algebra Appl., 438, 8, 2013, 3504–3510.
[14] G. DOBOVIŠEK, B. KUZMA, G. LEŠNJAK, C.K. LI AND T. PETEK, *Mappings that preserve pairs of operators with zero triple Jordan product*, Linear Algebra Appl., 426, 2-3, 2007, 255–279.
[15] H. GAO, $*$*-Jordan-triple multiplicative surjective maps on B(H)*, J. Math. Anal. Appl., 401, 1, 2013, 397–403.
[16] J. HOU, K. HE AND X. ZHANG, *Nonlinear maps preserving numerical radius of indefinite skew products of operators*, Linear Algebra Appl., 430, 8-9, 2009, 2240–2253.
[17] J. C. HOU, C. K. LI AND N. C. WONG, *Maps preserving the spectrum of generalized Jordan product of operators*, Linear Algebra Appl., 432, 4, 2010, 1049–1069.
[18] L. MOLNÁR, *Some characterizations of the automorphisms of* $B(H)$ *and* $C(H)$, Proc. Amer. Math. Soc., 130, 1, 2001, 111–120.
[19] P. ŠEMRL, *Non-linear commutativity preserving maps*, Acta. Sci. Mah. (Szeged), 71, 3-4, 2005, 781–819.
[20] L.N. TREFETHEN AND M. EMBREE, *Spectra and Pseudospectra, The Behavior of Nonormal Matrices and Operators*, Princeton University Press, Princeton, 2005.
[21] W. ZHANG AND J. HOU, *Maps preserving peripheral spectrum of Jordan semi-triple products of operators*, Linear Algebra Appl., 435, 6, 2011, 1326–1335.

(Received March 10, 2019)

Mohamed Bendaoud
Department of Mathematics, ENSAM
Moulay Ismail University
Marjane II, B.P. 15290 Al Mansour, Meknès, Morocco
e-mail: m.bendaoud@ensam-umi.ac.ma

Ayyoub Benyouness
Department of Mathematics
Faculty of Sciences, Moulay Ismail University
BP 11201, Zitoune, Meknès, Morocco
e-mail: benyouness@fs-umi.ac.ma

Mustapha Sarih
Department of Mathematics
Faculty of Sciences, Moulay Ismail University
BP 11201, Zitoune, Meknès, Morocco
e-mail: sarih@fs-umi.ac.ma