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MEASURES OF NONCOMPACTNESS IN $\bar{N}(p,q)$ SUMMABLE SEQUENCE SPACES

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Abstract. In this paper, we first define the $\bar{N}(p,q)$ summable sequence spaces and obtain some basic results related to these spaces. The necessary and sufficient conditions for an infinite matrix A to map these spaces into the spaces c_0, c and ℓ_∞ is obtained and Hausdorff measure of non-compactness is then used to obtain the necessary and sufficient conditions for the compactness of linear operators defined on these spaces.

1. Introduction and preliminaries

Measures of non-compactness is very useful tool in Banach spaces. The degree of non-compactness of a set is measured by means of functions called measures of non-compactness. Kuratowski [13] first introduced this concept, after that many measures of non-compactness have been defined and studied as in [2, 3]. Many researcher have used the concept of measure of non-compactness to characterize the linear operator between sequence spaces like [11, 12, 14, 16, 17, 18].

By ω we denote the set of all complex sequences $x=(x_k)_{k=0}^{\infty}$ and ϕ, c_0, c and ℓ_∞ denotes the sets of all finite sequences, sequences convergent to zero, convergent sequences and bounded sequences respectively. By e we denote the sequence of 1's, $e=(1,1,1,\ldots)$ and by $e^{(n)}$ the sequence with 1 as only nonzero term at the nth place for each $n\in\mathbb{N}$, where $\mathbb{N}=\{0,1,2,\ldots\}$. Further by cs and ℓ_1 we denote the convergent and absolutely convergent series respectively. If $x=(x_k)_{k=0}^{\infty}\in\omega$ then $x^{[m]}=\sum_{k=0}^{m}x_k e^{(k)}$ denotes the $m-$th section of x.

If X and Y are Banach Spaces, then by $\mathscr{B}(X,Y)$ we denote the set of all bounded (continuous) linear operators $L:X\to Y$, which is itself a Banach space with the operator norm $\|L\|=\sup_x\{\|L(x)\|_Y:\|x\|=1\}$ for all $L\in\mathscr{B}(X,Y)$. The linear operator $L:X\to Y$ is said to be compact if its domain is all of X and for every bounded sequence $(x_n)\in X$, the sequence $(L(x_n))$ has a subsequence which converges in Y. The operator $L\in\mathscr{B}(X,Y)$ is said to be of finite rank if $\dim R(L)<\infty$, where $R(L)$ denotes the range space of L.

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DEFINITION 1. A sequence space X is a linear subspace of ω, such a space is called a BK space if it is a Banach space with continuous coordinates
$P_n : X \to \mathbb{C}$ $(n = 0, 1, 2, \dots)$, where

$$P_n(x) = x_n,\ x = (x_k)_{k=0}^{\infty} \in X.$$

The BK space X is said to have AK if every $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$ [15, Definition 1.18].
The spaces c_0, c and ℓ_∞ are BK spaces with respect to the norm

$$\|x\|_\infty = \sup_k \{|x_k| : k \in \mathbb{N}\}.$$

DEFINITION 2. The $\beta-$dual of a subset X of ω is defined by

$$X^\beta = \{a \in \omega : ax = (a_k x_k) \in cs, \text{ for all } x = (x_k) \in X\}.$$

Let $(X, \|\cdot\|)$ be a Banach space, for any $E \subset X$, $\bar{E}$ denotes closure of E and $conv(E)$ denotes the closed convex hull of E. We denote the family of non-empty bounded subsets of X by M_X and family of non-empty and relatively compact subsets of X by N_X. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}$ the set of real numbers for $\mathbb{R}_+ = [0, \infty)$ the axiomatic definition of measures of noncompactness is

DEFINITION 3. [3] The measure of noncompactness on X is a function $\psi : \mathsf{M}_X \to \mathbb{R}_+$ the accompanying conditions hold:

(i) The family $\mathrm{Ker}\ \psi = \{E \in \mathsf{M}_X : \psi(E) = 0\}$ is non-empty and $\mathrm{Ker}\ \psi \subset \mathsf{N}_X$;

(ii) $E_1 \subset E_2 \Rightarrow \psi(E_1) \leqslant \psi(E_2)$;

(iii) $\psi(\bar{E}) = \psi(E)$;

(iv) $\psi(convE) = \psi(E)$;

(v) $\psi[\lambda E_1 + (1-\lambda)E_2] \leqslant \lambda \psi(E_1) + (1-\lambda)\psi(E_2)$ for $0 \leqslant \lambda \leqslant 1$;

(vi) Given a sequence (E_n) of closed set of M_X such that $E_{n+1} \subset E_n$ and $\lim_{n\to\infty} \psi(E_n) = 0$ then the intersection set $E_\infty = \bigcap_{n=1}^{\infty} E_n$ is non-empty.

The measure of noncompactness ψ is said to be regular measure if following additional conditions are satisfied:

(vii) $\psi(E_1 \cup E_2) = \max\{\psi(E_1), \psi(E_2)\}$;

(viii) $\psi(E_1 + E_2) \leqslant \psi(E_1) + \psi(E_2)$;

(ix) $\psi(\lambda E) = |\lambda| \psi(E)$, for $\lambda \in \mathbb{R}$;

(x) $\mathrm{Ker}\ \psi = \mathsf{N}_X$.

More on different measures of noncompactness can be found in [1, 2, 3, 12].

In this paper, we first define $\bar{N}(p,q)$ summable sequence spaces as the matrix domains X_T of arbitrary triangle $\bar{N}_p^q$ and obtain some basic results related to these spaces. We then find out the necessary and sufficient condition for matrix transformations to map these spaces into c_0, c and ℓ_∞. Finally we characterize the classes of compact matrix operators from these spaces into c_0, c and ℓ_∞.

2. Matrix domains

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers, we write A_n for the sequence in the nth row of A, $A_n = (a_{nk})_{k=0}^\infty$. The $A-$transform of any $x = (x_k) \in \omega$ is given by $Ax = (A_n(x))_{n=0}^\infty$, where

$$A_n(x) = \sum_{k=0}^\infty a_{nk}x_k, \qquad n \in \mathbb{N},$$

the series on right must converge for each $n \in \mathbb{N}$.

If X and Y are subsets of ω, we denote by (X,Y), the class of all infinite matrices that map X into Y. So $A \in (X,Y)$ if and only if $A_n \in X^\beta$, $n = 0,1,2,\dots$ and $Ax \in Y$ for all $x \in X$. The matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in \omega : Ax \in X\}.$$

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation method has been studied by several authors see [4, 6, 7, 8, 9, 10].

For any two sequences x and y in ω the product xy is given by $xy = (x_k y_k)_{k=0}^\infty$ and for any subset X of ω

$$y^{-1} * X = \{a \in \omega : ay \in X\}.$$

We denote by $\mathfrak{U}$ the set of all sequences $u = (u_k)_{k=0}^\infty$ such that $u_k \neq 0$, $\forall\, k = 0,1,2,\dots$ and for any $u \in \mathfrak{U}$, $\frac{1}{u} = \left(\frac{1}{u_k}\right)_{k=0}^\infty$.

THEOREM 1. *a) Let X be a BK space with basis $(\alpha^{(k)})_{k=0}^\infty$, $u \in \mathfrak{U}$ and $\beta^{(k)} = (1/u)\alpha^{(k)}$, $k = 0,1,\dots$. Then $(\beta^{(k)})_{k=0}^\infty$ is a basis of $Y = u^{-1} * X$.*

b) Let $(p_k)_{k=0}^\infty$ be a positive sequence, $u \in \mathfrak{U}$ a sequence such that

$$|u_0| \leqslant |u_1| \leqslant \cdots \quad \text{and} \quad |u_n| \to \infty \ (n \to \infty),$$

and T a triangle with

$$t_{nk} = \begin{cases} \frac{p_{n-k}}{u_n}, & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}, \qquad n = 0,1,2,\dots.$$

Then $(c_0)_T$ has AK.

Proof.

a) Proof same as [11, Theorem 2].

b) $(c_0)_T$ is a BK space by [22, Theorem 4.3.12], the norm $\|x\|_{(c_0)_T}$ on it is defined as

$$\|x\|_{(c_0)_T} = \sup_n \left| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} x_k \right|.$$

Since, $|u_n| \to \infty$ $(n \to \infty)$ gives $\phi \subset (c_0)_T$. Let $\varepsilon > 0$ and $x \in (c_0)_T$ then there exists integer $N > 0$, such that $|T_n(x)| < \frac{\varepsilon}{2}$ for all $n \geqslant N$. Let $m > N$, then

$$\|x - x^{[m]}\|_{(c_0)_T} = \sup_{n \geqslant m+1} \left| \frac{1}{u_n} \sum_{k=m+1}^{n} p_{n-k} x_k \right|. \tag{1}$$

Now,

$$T_n(x) = \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} x_k, \quad T_m(x) = \frac{1}{u_n} \sum_{k=0}^{m} p_{n-k} x_k$$

$$\Rightarrow T_n(x) + T_m(x) = \frac{1}{u_n} \left[2(p_n x_0 + \cdots + p_{n-m} x_m) + \sum_{k=m+1}^{n} p_{n-k} x_k \right].$$

Then, by (1), we have

$$\|x - x^{[m]}\|_{(c_0)_T} \leqslant \sup_{n \geqslant m+1} (|T_n(x)| + |T_m(x)|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $x = \sum_{k=0}^{\infty} x_k \beta^{(k)}$.
This representation is obviously unique. □

3. $\bar{N}(p,q)$ summable sequence spaces

Let $(p_k)_{k=0}^{\infty}$, $(q_k)_{k=0}^{\infty}$ be positive sequences in $\mathfrak{U}$ and $(R_n)_{n=0}^{\infty}$ the sequence with $R_n = \sum_{j=0}^{n} p_{n-j} q_j$. The $\bar{N}(p,q)$ transform of the sequence $(x_k)_{k=0}^{\infty}$ is the sequence $(t_n)_{n=0}^{\infty}$ defined as

$$t_n = \frac{1}{R_n} \sum_{j=0}^{n} p_{n-j} q_j x_j \,.$$

The matrix $\bar{N}_p^q$ for this transformation is

$$(\bar{N}_p^q)_{nk} = \begin{cases} \frac{p_{n-k} q_k}{R_n}, & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}. \tag{2}$$

We define the spaces $(\bar{N}_p^q)_0$, $(\bar{N}_p^q)$ and $(\bar{N}_p^q)_\infty$ that are $\bar{N}(p,q)$ summable to zero, summable and bounded respectively as

$$(\bar{N}_p^q)_0 = (c_0)_{\bar{N}_p^q} = \left\{ x \in \omega : \bar{N}_p^q x = \left(\frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right)_{n=0}^{\infty} \in c_0 \right\},$$
$$(\bar{N}_p^q) = (c)_{\bar{N}_p^q} = \left\{ x \in \omega : \bar{N}_p^q x = \left(\frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right)_{n=0}^{\infty} \in c \right\},$$
$$(\bar{N}_p^q)_\infty = (\ell_\infty)_{\bar{N}_p^q} = \left\{ x \in \omega : \bar{N}_p^q x = \left(\frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right)_{n=0}^{\infty} \in \ell_\infty \right\}.$$

For any sequence $x = (x_k)_{k=0}^\infty$, define $\tau = \tau(x)$ as the sequence with nth term given by

$$\tau_n = (\bar{N}_p^q)_n(x) = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \qquad (n = 0, 1, 2, \ldots). \tag{3}$$

This sequence τ is called as *weighted means of* x.

THEOREM 2. *The spaces $(\bar{N}_p^q)_0$, $(\bar{N}_p^q)$ and $(\bar{N}_p^q)_\infty$ are BK spaces with respect to the norm $\|\ .\ \|_{\bar{N}_p^q}$ given by*

$$\|x\|_{\bar{N}_p^q} = \sup_n \left| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k x_k \right|.$$

If $R_n \to \infty$ ($n \to \infty$), then $(\bar{N}_p^q)_0$ has AK, and every sequence $x = (x_k)_{k=0}^\infty \in (\bar{N}_p^q)$ has unique representation

$$x = le + \sum_{k=0}^{\infty} (x_k - l) e^{(k)}, \tag{4}$$

where $l \in \mathbb{C}$ is such that $x - le \in (\bar{N}_p^q)_0$.

Proof. The sets $(\bar{N}_p^q)_0$, $(\bar{N}_p^q)$ and $(\bar{N}_p^q)_{\ell_\infty}$ are BK spaces [22, Theorem 4.3.12]. Let us consider the matrix $T = (t_{nk})$ defined by

$$t_{nk} = \begin{cases} \frac{p_{n-k}}{R_n}, & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}, \qquad n = 0, 1, 2, \ldots .$$

Then $(\bar{N}_p^q)_0 = q^{-1} * (c_0)_T$ has AK by Theorem 1.
Now if $x \in (\bar{N}_p^q)$, then there exists a $l \in \mathbb{C}$ such that $x - le \in (\bar{N}_p^q)_0$. Now $\tau(e) = (\tau_n)_{n=0}^\infty$ where

$$\tau_n = (\bar{N}_p^q)_n(e) = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k e_k \qquad (n = 0, 1, 2, \ldots)$$

$$= \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \qquad \text{as } e_k = 1 \;\forall\; (k = 0,1,2,\dots)$$
$$= 1.$$

Therefore, $\tau(e) = e$ which implies the uniqueness of l. Therefore, (4) follows from the fact that $(\bar{N}_p^q)_\infty$ has AK. $\square$

Now, $\bar{N}_p^q$ is a triangle, it has a unique inverse and the inverse is also a triangle [12]. Take $H_0^{(p)} = \frac{1}{p_0}$ and

$$H_n^{(p)} = \frac{1}{p_0^{n+1}} \begin{vmatrix} p_1 & p_0 & 0 & 0 & \dots & 0 \\ p_2 & p_1 & p_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & p_{n-4} & \cdots & p_0 \\ p_n & p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 \end{vmatrix}. \tag{5}$$

Then, the inverse of matrix defined in (2) is the matrix $S = (s_{nk})_{n,k=0}^{\infty}$ which is defined as see [19] in

$$s_{nk} = \begin{cases} (-1)^{n-k} \frac{H_{n-k}^{(p)}}{q_n} R_k, & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}. \tag{6}$$

3.1. β dual of $\bar{N}(p,q)$ sequence spaces

In order to find the β dual we need the following results:

LEMMA 1. [21] *If $A = (a_{nk})_{n,k=0}^{\infty}$, then $A \in (c_0, c)$ if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty, \tag{7}$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0, \qquad \textit{for every } k. \tag{8}$$

LEMMA 2. [5] *If $A = (a_{nk})_{n,k=0}^{\infty}$, then $A \in (c,c)$ if and only if conditions (7), (8) hold and*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} a_{nk} \qquad \textit{exists for all } k. \tag{9}$$

LEMMA 3. [5] *If $A = (a_{nk})_{n,k=0}^{\infty}$, then $A \in (\ell_\infty, c)$ if and only if condition (8) holds and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n\to\infty} a_{nk} \right|. \tag{10}$$

THEOREM 3. *Let* $(p_k)_{k=0}^{\infty}$, $(q_k)_{k=0}^{\infty}$ *be positive sequences,* $R_n = \sum_{j=0}^{n} p_{n-j}q_j$ *and* $a = (a_k) \in \omega$ *, we define a matrix* $C = (c_{nk})_{n,k=0}^{\infty}$ *as*

$$c_{nk} = \begin{cases} R_k \left[\sum_{j=k}^{n} (-1)^{j-k} \left(\frac{H_{j-k}^{(p)}}{q_j} a_j \right) \right], & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}, \tag{11}$$

and consider the sets

$$c_1 = \left\{ a \in \omega : \sup_n \sum_k |c_{nk}| < \infty \right\}, \quad c_2 = \left\{ a \in \omega : \lim_{n\to\infty} c_{nk} \text{ exists for each } k \in \mathbb{N} \right\},$$

$$c_3 = \left\{ a \in \omega : \lim_{n\to\infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n\to\infty} c_{nk} \right| \right\}, \quad c_4 = \left\{ a \in \omega : \lim_{n\to\infty} \sum_k c_{nk} \text{ exists} \right\}.$$

Then $\left[\left(\bar{N}_p^q \right)_0 \right]^{\beta} = c_1 \cap c_2$, $\left[\left(\bar{N}_p^q \right) \right]^{\beta} = c_1 \cap c_2 \cap c_4$ *and* $\left[\left(\bar{N}_p^q \right)_{\infty} \right]^{\beta} = c_2 \cap c_3$.

Proof. We prove the result for $\left[\left(\bar{N}_p^q \right)_0 \right]^{\beta}$. Let $x \in \left(\bar{N}_p^q \right)_0$ then there exists a y such that $y = \bar{N}_p^q x$. Hence

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left(\bar{N}_p^q \right)^{-1} y_k = \sum_{k=0}^{n} a_k \left[\sum_{j=0}^{k} (-1)^{k-j} R_j \left(\frac{H_{k-j}^{(p)}}{q_k} \right) y_j \right]$$

$$= \sum_{k=0}^{n} R_k \left[\sum_{j=k}^{n} (-1)^{j-k} \left(\frac{H_{j-k}^{(p)}}{q_j} a_j \right) \right] y_k = (Cy)_n.$$

So, $ax = (a_n x_n) \in cs$ whenever $x \in \left(\bar{N}_p^q \right)_0$ if and only if $Cy \in cs$ whenever $y \in c_0$.
Using Lemma 1 we get $\left[\left(\bar{N}_p^q \right)_0 \right]^{\beta} = c_1 \cap c_2$.
Similarly, using Lemma 2 and Lemma 3 the β dual of $\left(\bar{N}_p^q \right)$ and $\left(\bar{N}_p^q \right)_{\infty}$ can be found same way we can show the other two results as well. □
Let $X \subset \omega$ be a normed space and $a \in \omega$. Then we write

$$\|a\|^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : \|x\| = 1 \right\},$$

provided the term on the right side exists and is finite, which is the case whenever X is a BK space and $a \in X^{\beta}$ [22, Theorem 7.2.9].

THEOREM 4. *For* $\left[\left(\bar{N}_p^q \right)_0 \right]^{\beta}$, $\left[\left(\bar{N}_p^q \right) \right]^{\beta}$ *and* $\left[\left(\bar{N}_p^q \right)_{\infty} \right]^{\beta}$ *the norm* $\| \ . \ \|^*$ *is defined as*

$$\|a\|^* = \sup_n \left\{ \sum_{k=0}^{n} R_k \left| \sum_{j=k}^{n} (-1)^{j-k} \frac{H_{j-k}^{(p)}}{q_j} a_j \right| \right\}.$$

Proof. If $x^{[n]}$ denotes the nth section of the sequence $x \in \left(\bar{N}_p^q\right)_0$ then using (3) we have

$$\tau_k^{[n]} = \tau_k(x^{[n]}) = \frac{1}{R_k}\sum_{j=0}^{k} p_{n-j}q_j x_j^{[n]}.$$

Let $a \in \left[\left(\bar{N}_p^q\right)_0\right]^\beta$, then for any non-negative integer n define the sequence $d^{[n]}$ as

$$d_k^{[n]} = \begin{cases} R_k\left[\sum_{j=k}^{n}(-1)^{j-k}\frac{H_{j-k}^{(p)}}{q_j}a_j\right], & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}.$$

Let $\|a\|_\Pi = \sup_n \|d^{[n]}\|_1 = \sup_n\left(\sum_{k=0}^{\infty}|d_k^{[n]}|\right)$, where $\Pi = \left[\left(\bar{N}_p^q\right)\right]^\beta$. Then

$$\begin{aligned}\left|\sum_{k=0}^{\infty}a_k x_k^{[n]}\right| &= \left|\sum_{k=0}^{n}a_k\left(\sum_{j=0}^{k}(-1)^{k-j}\frac{H_{k-j}^{(p)}}{q_k}R_j\tau_j^{[n]}\right)\right| \qquad \text{using (6)}\\ &= \left|\sum_{k=0}^{n}R_k\left(\sum_{j=k}^{n}(-1)^{j-k}\frac{H_{j-k}^{(p)}}{q_j}a_j\right)\tau_k^{[n]}\right| \\ &\leqslant \sup_k|\tau_k^{[n]}|\cdot\left(\sum_{k=0}^{n}R_k\left|\sum_{j=k}^{n}(-1)^{j-k}\frac{H_{j-k}^{(p)}}{q_j}a_j\right|\right) = \|x^{[n]}\|_{\bar{N}_p^q}\|d^{[n]}\|_1 \\ &= \|a\|_\Pi\|x^{[n]}\|_{\bar{N}_p^q}.\end{aligned}$$

Hence,

$$\|a\|^* \leqslant \|a\|_\Pi. \tag{12}$$

To prove the converse define the sequence $x^{(n)}$ for any arbitrary n by

$$\tau_k\left(x^{(n)}\right) = \operatorname{sign}\left(d_k^{[n]}\right) \qquad (k = 0,1,2,\ldots).$$

Then

$$\tau_k\left(x^{(n)}\right) = 0 \quad \text{for } k > n \quad \text{i.e } x^{(n)} \in \left(\bar{N}_p^q\right)_0, \qquad \|x^{(n)}\|_{\bar{N}_p^q} = \|\tau_k\left(x^{(n)}\right)\|_\infty \leqslant 1,$$

and

$$\left|\sum_{k=0}^{\infty}a_k x_k^{(n)}\right| = \left|\sum_{k=0}^{n}d_k^{[n]}x_k^{(n)}\right| \leqslant \sum_{k=0}^{n}\left|d_k^{[n]}\right| \leqslant \|a\|^*.$$

Since, n is arbitrarily choosen so

$$\|a\|_\Pi \leqslant \|a\|^*. \tag{13}$$

From (12) and (13) we get the required conclusion. □

Some well known results that are required for proving the compactness of operators are:

PROPOSITION 1. [17, Theorem 7] *Let X and Y be BK spaces, then $(X,Y) \subset \mathscr{B}(X,Y)$ that is every matrix A from X into Y defines an element L_A of $\mathscr{B}(X,Y)$ where*

$$L_A(x) = A(x), \qquad \forall\, x \in X.$$

Also $A \in (X,\ell_\infty)$ if and only if

$$\|A\|^* = \sup_n \|A_n\|^* = \|L_A\| < \infty.$$

If $\left(b^{(k)}\right)_{k=0}^{\infty}$ is a basis of X, Y and Y_1 are FK spaces with Y_1 a closed subspace of Y, then $A \in (X,Y_1)$ if and only if $A \in (X,Y)$ and $A\left(b^{(k)}\right) \in Y_1$ for all $k = 0,1,2,\dots$.

PROPOSITION 2. [18, Proposition 3.4] *Let T be a triangle.*

(i) If X and Y are subsets of ω, then $A \in (X,Y_T)$ if and only if $B = TA \in (X,Y)$.

(ii) If X and Y are BK spaces and $A \in (X,Y_T)$, then

$$\|L_A\| = \|L_B\|.$$

Using Proposition 1 and Theorem 4 we conclude the following corollary:

COROLLARY 1. *Let $(p_k)_{k=0}^{\infty}, (q_k)_{k=0}^{\infty}$ be given positive sequences, and $R_n = \sum_{k=0}^{n} p_{n-k} q_k$ then:*

i) $A \in \left(\left(N_p^q\right)_\infty, \ell_\infty\right)$ if and only if

$$\sup_{n,m} \left\{ \sum_{k=0}^{m} R_k \left| \sum_{j=k}^{m} (-1)^{j-k} \frac{H_{j-k}^{(p)}}{q_j} a_{nj} \right| \right\} < \infty, \tag{14}$$

and

$$\frac{A_n H_n^{(p)} R}{q} \in c_0, \quad \forall\, n = 0,1,\dots . \tag{15}$$

ii) $A \in \left(\left(\bar{N}_p^q\right), \ell_\infty\right)$ if and only if condition (14) holds and

$$\frac{A_n H_n^{(p)} R}{q} \in c, \qquad \forall\, n = 0,1,2,\dots . \tag{16}$$

iii) $A \in \left(\left(\bar{N}_p^q\right)_0, \ell_\infty\right)$ if and only if condition (14) holds.

iv) $A \in \left(\left(\bar{N}_p^q\right)_0, c_0\right)$ if and only if condition (14) holds and

$$\lim_{n\to\infty} a_{nk} = 0, \qquad \text{for all } k = 0,1,2\dots . \tag{17}$$

v) $A \in \left(\left(\bar{N}_p^q\right)_0, c\right)$ *if and only if condition* (14) *holds and*

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \qquad \textit{for all } k = 0,1,2\dots\,. \tag{18}$$

vi) $A \in \left(\left(\bar{N}_p^q\right), c_0\right)$ *if and only if conditions* (14), (15) *and* (17) *hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0, \qquad \textit{for all } k = 0,1,2\dots\,. \tag{19}$$

vii) $A \in \left(\left(\bar{N}_p^q\right), c\right)$ *if and only if conditions* (14), (15) *and* (18) *hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha, \qquad \textit{for all } k = 0,1,2\dots\,. \tag{20}$$

From Theorem 2, Theorem 4 and Proposition 2 we conclude the following corollary:

COROLLARY 2. *Let X be a BK-space and $(p_k)_{k=0}^{\infty}, (q_k)_{k=0}^{\infty}$ be positive sequences, $R_n = \sum_{k=0}^{n} p_{n-k} q_k$ then:*

i) $A \in \left(X, \left(\bar{N}_p^q\right)_\infty\right)$ *if and only if*

$$\sup_m \left\| \frac{1}{R_m} \sum_{n=0}^{m} p_{m-n} q_n A_n \right\|^* < \infty. \tag{21}$$

ii) $A \in \left(X, \left(\bar{N}_p^q\right)_0\right)$ *if and only if* (21) *holds and*

$$\lim_{m\to\infty} \left(\frac{1}{R_m} \sum_{n=0}^{m} p_{m-n} q_n A_n \left(c^{(k)}\right) \right) = 0, \quad \forall\, k = 0,1,2\dots\,, \tag{22}$$

where $\left(c^{(k)}\right)$ *is a basis of* X.

iii) $A \in \left(X, \left(\bar{N}_p^q\right)\right)$ *if and only if* (22) *holds and*

$$\lim_{m\to\infty} \left(\frac{1}{R_m} \sum_{n=0}^{m} p_{m-n} q_n A_n \left(c^{(k)}\right) \right) = \alpha_k, \quad \forall\, k = 0,1,2\dots\,. \tag{23}$$

4. Hausdorff measure of noncompactness

Let S and M be the subsets of a metric space (X,d) and $\varepsilon > 0$. Then S is called an $\varepsilon-$net of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. Further, if the set S is finite, then the $\varepsilon-$net S of M is called *finite $\varepsilon-$net* of M. A subset of

a metric space is said to be *totally bounded* if it has a finite $\varepsilon-$net for every $\varepsilon>0$ see [20].
If $\mathscr{M}_X$ denotes the collection of all bounded subsets of metric space (X,d) and $Q\in\mathscr{M}_X$ then the *Hausdorff measure of noncompactness* of the set Q is defined by

$$\chi(Q)=\inf\{\varepsilon>0: Q \text{ has a finite } \varepsilon-\text{net in } X\}.$$

The function $\chi:\mathscr{M}_X\to[0,\infty)$ is called *Hausdorff measure of noncompactness* [2].

DEFINITION 4. For a metric space (Ω,d), Hausdorff measure of noncompactness (also called as the ball measure) is defined as

$$\chi(A)=\inf\left\{\varepsilon>0: A\subset\bigcup_{i=1}^{n}B(x_i,r_i), x_i\in\Omega, r_i<\varepsilon\ (i=1,\ldots,n)\ , n\in\mathbb{N}\right\},$$

where $A\subset\Omega$ is bounded and $B(x_i,r_i)$ denotes closed ball with center at x_i and radius r_i.

The basic properties of *Hausdorff measure of noncompactness* can be found in ([2, 3, 15]). Some of those properties are:
If Q,Q_1 and Q_2 are bounded subsets of a metric space (X,d), then:

$$\begin{aligned}
\chi(Q)=0 &\Leftrightarrow Q \text{ is totally bounded set;}\\
\chi(Q)&=\chi(\bar{Q});\\
Q_1\subset Q_2 &\Rightarrow \chi(Q_1)\leqslant\chi(Q_2);\\
\chi(Q_1\cup Q_2)&=\max\{\chi(Q_1),\chi(Q_2)\};\\
\chi(Q_1\cap Q_2)&=\min\{\chi(Q_1),\chi(Q_2)\}.
\end{aligned}$$

Further if X is a normed space then *Hausdorff measure of noncompactness* χ has the following additional properties connected with the linear structure.

$$\begin{aligned}
\chi(Q_1+Q_2)&\leqslant\chi(Q_1)+\chi(Q_2);\\
\chi(\eta Q)&=|\eta|\chi(Q), \qquad \eta\in\mathbb{C}.
\end{aligned}$$

The most effective way of characterizing operators between Banach spaces is by applying Hausdorff measure of noncompactness. If X and Y are Banach spaces, and $L\in\mathscr{B}(X,Y)$, then the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$ is defined as

$$\|L\|_\chi=\chi\left(L(S_X)\right).$$

Where $S_X=\{x\in X:\|x\|=1\}$ is the unit ball in X.
From [12, Corollary 1.15] we know that

$$L \text{ is compact if and only if } \|L\|_\chi=0.$$

PROPOSITION 3. [2, Theorem 6.1.1, $X = c_0$] *Let $Q \in M_{c_0}$ and $P_r : c_0 \to c_0$ ($r \in \mathbb{N}$ be the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. Then we have*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|\right),$$

where I is the identity operator on c_0.

PROPOSITION 4. [2, Theorem 6.1.1] *Let X be a Banach space with a Schauder basis $\{e_1, e_2, \ldots\}$, and $Q \in M_X$ and $P_n : X \to X$ ($n \in \mathbb{N}$ be the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then we have*

$$\frac{1}{a}\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right) \leqslant \chi(Q) \leqslant \inf_n\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right)$$
$$\leqslant \limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right),$$

where $a = \lim_{n\to\infty}\sup\|I-P_n\|$, and I is the identity operator on c. If $X = c$ then $a = 2$ (see [2]).

5. Compact operators on the spaces $\left(\bar{N}_p^q\right)_0$, $\left(\bar{N}_p^q\right)$ and $\left(\bar{N}_p^q\right)_\infty$

THEOREM 5. *Consider the matrix A as in Corollary 1, and for any integers n, s, $n > s$ set*

$$\|A\|^{(s)} = \sup_{n>s}\sup_m\left\{\sum_{k=0}^m R_k\left|\sum_{j=k}^m(-1)^{j-k}\frac{H_{j-k}^{(p)}}{q_j}a_{nj}\right|\right\}. \tag{24}$$

If X be either $\left(\bar{N}_p^q\right)_0$ or $\left(\bar{N}_p^q\right)$ and $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \lim_{s\to\infty}\|A\|^{(s)}. \tag{25}$$

If X be either $\left(\bar{N}_p^q\right)_0$ or $\left(\bar{N}_p^q\right)$ and $A \in (X, c)$, then

$$\frac{1}{2}\cdot\lim_{s\to\infty}\|A\|^{(s)} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(s)}, \tag{26}$$

and if X be either $\left(\bar{N}_p^q\right)_0$, $\left(\bar{N}_p^q\right)$ or $\left(\bar{N}_p^q\right)_\infty$ and $A \in (X, \ell_\infty)$, then

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{s\to\infty}\|A\|^{(s)}. \tag{27}$$

Proof. Let $F=\{x\in X:\|x\|\leqslant 1\}$ if $A\in(X,c_0)$ and X is one of the spaces $\left(\bar{N}_p^q\right)_0$ or $\left(\bar{N}_p^q\right)$, then by Proposition 3

$$\|L_A\|_\chi=\chi(AF)=\lim_{s\to\infty}\left[\sup_{x\in F}\|(I-P_s)Ax\|\right]. \tag{28}$$

Again using Proposition 1 and Corollary 1, we have

$$\|A\|^s=\sup_{x\in F}\|(I-P_s)Ax\|. \tag{29}$$

From (28) and (29) we get

$$\|L_A\|_\chi=\lim_{s\to\infty}\|A\|^{(s)}.$$

Since every sequence $x=(x_k)_{k=0}^\infty\in c$ has a unique representation

$$x=le+\sum_{k=0}^{\infty}(x_k-l)e^{(k)},\qquad \text{where } l\in\mathbb{C} \text{ is such that } x-le\in c_0.$$

We define $P_s:c\to c$ by $P_s(x)=le+\sum_{k=0}^{s}(x_k-l)e^{(k)}$, $s=0,1,2,\dots$.
Then $\|I-P_s\|=2$ and using (29) and Proposition 4 we get

$$\frac{1}{2}\cdot\lim_{s\to\infty}\|A\|^{(s)}\leqslant\|L_A\|_\chi\leqslant\lim_{s\to\infty}\|A\|^{(s)}.$$

Finally, we define $P_s:\ell_\infty\to\ell_\infty$ by $P_s(x)=(x_0,x_1,\dots,x_s,0,0\dots)$, $x=(x_k)\in\ell_\infty$.
Clearly, $AF\subset P_s(AF)+(I-P_s)(AF)$.
So, using the properties of χ we get

$$\chi(AF)\leqslant\chi[P_s(AF)]+\chi[(I-P_s)(AF)]=\chi[(I-P_s)(AF)]\leqslant\sup_{x\in F}\|(I-P_s)A(x)\|.$$

Hence, by Proposition 1 and Corollary 1 we get

$$0\leqslant\|L_A\|_\chi\leqslant\lim_{s\to\infty}\|A\|^{(s)}.\quad\square$$

A direct corollary of the above theorem is:

COROLLARY 3. *Consider the matrix A as in Corollary 1, and $X=\left(\bar{N}_p^q\right)_0$ or $X=\left(\bar{N}_p^q\right)$, then if $A\in(X,c_0)$ or $A\in(X,c)$ we have*

$$L_A \text{ is compact if and only if } \lim_{s\to\infty}\|A\|^{(s)}=0.$$

Further, for $X=\left(\bar{N}_p^q\right)_0$, $X=\left(\bar{N}_p^q\right)$ or $X=\left(\bar{N}_p^q\right)_\infty$, if $A\in(X,\ell_\infty)$ then we have

$$L_A \text{ is compact if } \lim_{s\to\infty}\|A\|^{(s)}=0. \tag{30}$$

In (30) it is possible for L_A to be compact although $\lim_{s\to\infty} \|A\|^{(s)} \neq 0$, that is the condition is only sufficient condition for L_A to be compact.
For example, let the matrix A be defined as $A_n = e^{(1)} \quad n = 0,1,2,\ldots$ and the positive sequences $q_n = 3^n$, $n = 0,1,2,\ldots$ and $p_0 = 1, p_1 = 1, p_k = 0$, $\forall\, k = 2,3,\ldots$.
Then by (14) we have

$$\sup_{n,m}\left\{\sum_{k=0}^{m} R_k \left|\sum_{j=k}^{m} (-1)^{j-k} \frac{H_{j-k}^{(p)}}{q_j} a_{nj}\right|\right\} = \sup_{m}\left(2 - \frac{2}{3^m}\right) = 2 < \infty.$$

Now, by Corollary 1 we know $A \in \left(\left(\bar{N}_p^q\right)_\infty, \ell_\infty\right)$.
But,

$$\|A\|^{(s)} = \sup_{n>s}\left[2 - \frac{2}{3^m}\right] = 2 - \frac{1}{2\cdot 3^s}, \qquad \forall\, s.$$

Which gives $\lim_{s\to\infty} \|A\|^{(s)} = 2 \neq 0$.
Since $A(x) = x_1$ for all $x \in \left(\bar{N}_p^q\right)_\infty$, so L_A is a compact operator.

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