Operators
and
Matrices
Volume 14, Number 1 (2020), 1–7
doi:10.7153/oam-2020-14-01

INVARIANCE OF DISTRIBUTIONAL CHAOS FOR BACKWARD SHIFTS

XINXING WU* AND YANG LUO

(*Communicated by R. Curto*)

Abstract. A sufficient and necessary condition ensuring that the backward shift operator on the Köthe sequence space admits an invariant distributionally ε-scrambled set for some $\varepsilon > 0$ is obtained, improving the main results in [10].

Let $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{Z}^+ = \{0,1,2,\ldots\}$. According to [7], an infinite matrix $A = (a_{j,k})_{j,k\in\mathbb{N}}$ is called a *Köthe matrix* if, for every $j \in \mathbb{N}$, there exists some $k \in \mathbb{N}$ with $a_{j,k} > 0$ and $0 \leqslant a_{j,k} \leqslant a_{j,k+1}$ for all $j,k \in \mathbb{N}$.

Consider the *backward shift* defined by

$$B(x_1,x_2,x_3,\ldots) = (x_2,x_3,x_4,\ldots)$$

on the Köthe sequence space $\lambda_p(A)$ determined by a Köthe matrix A, where, for $1 \leqslant p < +\infty$,

$$\lambda_p(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \|x\|_k := \left(\sum_{j=1}^{\infty} |x_j a_{j,k}|^p \right)^{1/p} < \infty, \forall k \in \mathbb{N} \right\},$$

and, for $p = 0$,

$$\lambda_0(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \lim_{j\to\infty} x_j a_{j,k} = 0, \|x\|_k := \sup_{j\in\mathbb{N}} |x_j a_{j,k}|, \forall k \in \mathbb{N} \right\}.$$

It is possible to define a complete metric on $\lambda_p(A)$ which is invariant by translation:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

Mathematics subject classification (2010): 47A16.
Keywords and phrases: Backward shift, Köthe sequence space, distributional chaos, invariant set.
This work was supported by the National Natural Science Foundation of China (No. 11601449), the National Nature Science Foundation of China (Key Program) (No. 51534006), the Science and Technology Innovation Team of Education Department of Sichuan for Dynamical System and its Applications (No. 18TD0013), and the Youth Science and Technology Innovation Team of Southwest Petroleum University for Nonlinear Systems (No. 2017CXTD02).
* Corresponding author.

The operator $B:\lambda_p(A)\longrightarrow\lambda_p(A)$ is continuous and well-defined if and only if the following condition on the matrix A is satisfied:

$$\forall n\in\mathbb{N},\ \exists m>n \text{ such that } \sup_{j\in\mathbb{N}}\left|\frac{a_{j,n}}{a_{j+1,m}}\right|<+\infty, \tag{1}$$

where in the case of $a_{j+1,m}=0$, one has $a_{j,n}=0$ and we consider $\frac{0}{0}$ as 1 (see [7]).

For simplicity, throughout this paper, for any $x=(x_1,x_2,x_3,\dots)\in\lambda_p(A)$ and any $k,n\in\mathbb{Z}^+$, denote

$$(x)_k := x_k,\ x(k) := B^k(x) = (x_{k+1},x_{k+2},x_{k+3},\dots),$$
$$x(k,n) := (x_{k+1},x_{k+2},\dots,x_{k+n},0,0,\dots),$$

and

$$x[k,n] := (\underbrace{0,0,\dots,0}_{n},x_{k+n+1},x_{k+n+2},\dots).$$

The notion of distributional chaos was introduced by Schweizer and Smítal [15]. Let $f:X\longrightarrow X$ be a continuous map defined on a metric space (X,d). For any $x,y\in X$, $n\in\mathbb{N}$ and $t\in\mathbb{R}$, let

$$\Phi_{x,y}^{(n)}(t)=\left|\{0\leqslant i<n : d(f^i(x),f^i(y))<t\}\right|.$$

Define *lower* and *upper distributional functions*, $\mathbb{R}\longrightarrow[0,1]$ generated by f, x and y, as follows:

$$\Phi_{x,y}(t)=\liminf_{n\to\infty}\frac{1}{n}\Phi_{x,y}^{(n)}(t),$$

and

$$\Phi_{x,y}^{*}(t)=\limsup_{n\to\infty}\frac{1}{n}\Phi_{x,y}^{(n)}(t),$$

respectively, where $|A|$ denotes the cardinality of set A. A subset $D\subset X$ is *distributionally ε-scrambled* if for any distinct points $x,y\in D$, $\Phi_{x,y}^{*}(t)=1$ for any $t>0$ and $\Phi_{x,y}(\varepsilon)=0$. A pair satisfying the above condition is called a *distributionally ε-chaotic pair*.

During the last decades, many research works were devoted to the 'chaotic behavior' of the backward shift operator on the Köthe sequence space (more generally, Banach or Fréchet space) (see, e.g., [1, 8, 9, 10, 11, 16, 21, 22, 23, 20, 25]). For example, Martínez-Giménez and Peris [9] obtained some characterizations for hypercyclicity and Devaney chaos under backward shift on the Köthe sequence space. Martínez-Giménez [8] provided some sufficient conditions for the operator $f(B_{\mathrm{w}})$ to be chaotic in the sense of Devaney. Then, Bermúdez et al. [1] proved some useful equivalent conditions for Li-Yorke chaos and a few sufficient criteria for distributionally chaotic operators. By employing methods developed in [1], Wu and Zhu [22] proved that for a bounded operator defined on a Banach space, Li-Yorke chaos, Li-Yorke sensitivity, spatiotemporal chaos, and distributional chaos in a sequence are all equivalent, and they are all strictly stronger than sensitivity. Further results of [11] were extended to maximal distributional chaos for the annihilation operator of a quantum harmonic oscillator in [21, 17]. In 2009, Martínez-Giménez et al. [10] provided sufficient conditions for uniform distributional chaos under backward shift. Very recently, we [16, 18, 19, 23] provided a class of characterizations for uniform Li-Yorke chaos and a sufficient condition for maximal distributional chaos under backward shift on the Köthe sequence space. Bernardes et al. [2] characterized distributional chaos for linear operators on Fréchet spaces and obtained a sufficient condition to ensure the existence of dense uniformly distributionally irregular manifolds.

For quite a long time, operator theorists have been studying the so-called cyclic vectors in connection with the (invariant) subspace problem [3, 5, 6, 13, 14]. The invariant subspace problem, which is open to this day, asks whether every Hilbert space operator possesses an invariant closed subspace other than the two trivial ones given by $\{0\}$ and the whole space. Du [4] proved that an interval map is turbulent if and only if there is an invariant scrambled set in 2005. Later, Oprocha [12] extended this approach and proved that exactly the same characterization is valid for distributional chaos. Very recently, for the full shift (Σ_2, σ) on two symbols, we [24] constructed an invariant distributionally ε-scrambled set for any $0 < \varepsilon < \dim(\Sigma_2)$, in which each point is transitive but is not weakly almost periodic.

In [10], Martínez-Giménez et al. proved the following:

THEOREM 1. [10, Theorem 5] *Let A be a Köthe matrix satisfying (1), $1 \leqslant p < +\infty$ (or, $p = 0$). If there exist $x, y \in \lambda_p(A)$ such that $\Phi_{x,y}(\delta) = 0$ holds for some $\delta > 0$, then $B : \lambda_p(A) \longrightarrow \lambda_p(A)$ has a distributionally ε-scrambled subset for some $\varepsilon > 0$.*

Combining this with [18, Theorem 3.3], Wu et al. [18] provided the following question:

QUESTION 1. [18, Question 3.5] Does the hypothesis in Theorem 1 imply that B has an invariant distributionally scrambled linear manifold?

Being a partial answer to Question 1, this paper shall prove that the hypothesis in Theorem 1 can ensure that B admits an invariant distributionally ε-scrambled subset for any $0 < \varepsilon < \delta$ (see Theorem 2).

THEOREM 2. *Let A be a Köthe matrix satisfying (1), $1 \leqslant p < +\infty$ (or, $p = 0$). If there exist $x, y \in \lambda_p(A)$ such that $\Phi_{x,y}(\delta) = 0$ holds for some $\delta > 0$, then $B : \lambda_p(A) \longrightarrow \lambda_p(A)$ has an invariant distributionally ε-scrambled subset for any $0 < \varepsilon < \delta$.*

Proof. By the invariability of the metric d, we may assume that $x = 0$ and $y = (y_1, y_2, y_3, \ldots)$. Since $\Phi_{x,y}(\delta) = 0$, there exists an increasing sequence $\{\mathrm{N}_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{1}{\mathrm{N}_k} \left|\{0 \leqslant i < \mathrm{N}_k : d(B^i(x), B^i(y)) \geqslant \delta\}\right| = 1. \tag{2}$$

It is not difficult to check that there exists a subsequence $\{\mathrm{M}_k\}_{k\in\mathbb{N}}$ of $\{\mathrm{N}_k\}_{k\in\mathbb{N}}$ such that for any $k\in\mathbb{N}$,

$$\mathrm{M}_{k+1}-\mathrm{M}_k \geqslant 4^{\mathrm{M}_k}, \tag{3}$$

$$\sum_{j=\mathrm{M}_k}^{\infty} \left|y_j a_{j,k}\right|^p < \frac{1}{2^k}, \tag{4}$$

and that for any $\mathrm{M}_{2k} < j \leqslant \mathrm{M}_{2k+1}$,

$$d\left(0, y\left(j, \mathrm{M}_{2k+2}-\mathrm{M}_{2k+1}\right)\right) \geqslant d(0, B^j(y)) - \frac{1}{2k}. \tag{5}$$

Define $v=(v_1,v_2,v_3,\ldots)$ by

$$v_j = \begin{cases} k\cdot|y_j|, & \mathrm{M}_{4k} < j \leqslant \mathrm{M}_{4k+3},\ k\in\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Because

$$\begin{aligned} \sum_{j=\mathrm{M}_k}^{\infty} |v_j a_{j,k}|^p &= \sum_{l\geqslant k}\sum_{j=\mathrm{M}_l+1}^{\mathrm{M}_{l+1}} |v_j a_{j,k}|^p \leqslant \sum_{l\geqslant k}\sum_{j=\mathrm{M}_l+1}^{\mathrm{M}_{l+1}} l^p |y_j a_{j,k}|^p \leqslant \sum_{l\geqslant k}\sum_{j=\mathrm{M}_l+1}^{\mathrm{M}_{l+1}} l^p |y_j a_{j,l}|^p \\ &\leqslant \sum_{l\geqslant k} \frac{l^p}{2^l} < +\infty, \end{aligned}$$

one has $v\in\lambda_p(A)$. Applying the method of induction, it can be verified that there exists a subsequence $\{\mathrm{M}_{k_n}\}_{n\in\mathbb{N}}$ of $\{\mathrm{M}_k\}_{k\in\mathbb{N}}$ such that for any $n\in\mathbb{N}$,

$$\{k_{2n} : n\in\mathbb{N}\} \subset \{4k : k\in\mathbb{N}\}, \quad k_1 = 1, \quad k_{2n+1} = k_{2n}+3,$$

and that for any $\mathrm{M}_{k_{2n-1}} < j \leqslant \mathrm{M}_{k_{2n-1}} + 2^{M_{k_{2n-1}}}$,

$$d\left(0, v\left[j, \mathrm{M}_{k_{2n}} - 4^{\mathrm{M}_{k_{2n-1}}}\right]\right) \leqslant \frac{1}{2n}. \tag{6}$$

Arrange all odd prime numbers by the natural order '$<$' and denote them as $\mathrm{P}_1, \mathrm{P}_2, \ldots$. For any $n,l\in\mathbb{N}$, set

$$\begin{aligned} \mathscr{C}_{n,l}^0 = \Bigg\{ j\in\mathbb{N} : \mathrm{M}_{k_{\mathrm{P}_n^l+1}} + {} & (2u)l < j \leqslant \mathrm{M}_{k_{\mathrm{P}_n^l+1}} + (2u+1)l, \\ & 0 \leqslant 2u \leqslant \left[\frac{\mathrm{M}_{k_{\mathrm{P}_n^l+1}+3} - \mathrm{M}_{k_{\mathrm{P}_n^l+1}}}{l}\right] - 1 \Bigg\}, \end{aligned}$$

and

$$\mathscr{C}_{n,l}^1 = \left\{ j\in\mathbb{N} : \mathrm{M}_{k_{\mathrm{P}_n^l+1}} < j \leqslant \mathrm{M}_{k_{\mathrm{P}_n^l+1}+3} \right\} - \mathscr{C}_{n,l}^0.$$

Take $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots) \in \lambda_p(A)$ with

$$\bar{v}_j = \begin{cases} |v_j|, & j \in \mathscr{C}^0_{n,l},\ n,l \in \mathbb{N}, \\ -|v_j|, & j \in \mathscr{C}^1_{n,l},\ n,l \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

and set

$$\mathscr{D} = \bigcup_{n=0}^{\infty} B^n\left(\{\alpha\bar{v} : \alpha \in (0,1)\}\right).$$

Clearly, $B(\mathscr{D}) \subset \mathscr{D}$ and $\mathscr{D}$ is uncountable. Given any two fixed points $a,\ b \in \mathscr{D}$ with $a \neq b$, there exist $\alpha, \beta \in (0,1)$ and $p,q \in \mathbb{Z}^+$ such that $a = B^p(\alpha\bar{v})$ and $b = B^q(\beta\bar{v})$. Without loss of generality, assume that $p \leqslant q$.

Now, we assert that (a,b) is a distributionally ε-chaotic pair for any $0 < \varepsilon < \delta$.

Firstly, for any $\mathrm{M}_{k_{2n-1}} < j \leqslant \mathrm{M}_{k_{2n-1}} + 2^{M_{k_{2n-1}}} - q$, noting that $\mathrm{M}_{k_{2n}} - 4^{\mathrm{M}_{k_{2n-1}}} \leqslant \mathrm{M}_{k_{2n}} - (j+q)$, and applying (6), it follows that

$$\begin{aligned} d(B^j(a), B^j(b)) &\leqslant d(0, B^j(a)) + d(0, B^j(b)) \leqslant d(0, B^{j+p}(v)) + d(0, B^{j+q}(v)) \\ &\leqslant \frac{1}{n} \longrightarrow 0,\ (n \longrightarrow \infty). \end{aligned}$$

Then, given any $t > 0$, there exists some $\mathrm{N} \in \mathbb{N}$ such that for any $n \geqslant \mathrm{N}$ and any $\mathrm{M}_{k_{2n-1}} < j \leqslant \mathrm{M}_{k_{2n-1}} + 2^{\mathrm{M}_{k_{2n-1}}} - q$,

$$d(B^j(a), B^j(b)) < t,$$

implying that

$$\begin{aligned} &\Phi^*_{a,b}(t) \\ &= \limsup_{n\to\infty} \frac{1}{n} \left|\{0 \leqslant i < n : d(B^i(a), B^i(b)) < t\}\right| \\ &\geqslant \limsup_{n\to\infty} \frac{1}{\mathrm{M}_{k_{2n-1}} + 2^{\mathrm{M}_{k_{2n-1}}} - q} \left|\left\{0 \leqslant i < \mathrm{M}_{k_{2n-1}} + 2^{\mathrm{M}_{k_{2n-1}}} - q : d(B^i(a), B^i(b)) < t\right\}\right| \\ &\geqslant \limsup_{n\to\infty} \frac{2^{\mathrm{M}_{k_{2n-1}}} - q}{\mathrm{M}_{k_{2n-1}} + 2^{\mathrm{M}_{k_{2n-1}}} - q} = 1. \end{aligned}$$

Second, to prove $\Phi_{a,b}(\varepsilon) = 0$ for any $0 < \varepsilon < \delta$, we consider two cases as follows:

Case 1. $p = q$ and $\alpha \neq \beta$. Noting that for any $\mathrm{M}_{k_{\mathrm{P}_{n+1}}} < j \leqslant \mathrm{M}_{k_{\mathrm{P}_{n+1}+1}}$ and any $1 \leqslant i \leqslant \mathrm{M}_{k_{\mathrm{P}_{n+1}+2}} - \mathrm{M}_{k_{\mathrm{P}_{n+1}+1}}$, $\left|\left(B^j\left((\alpha-\beta)\bar{v}\right)\right)_i\right| = \left|\left((\alpha-\beta)\frac{k_{\mathrm{P}_{n+1}}}{4} y(j)\right)_i\right|$, and applying (5), it follows that for all $n \in \mathbb{N}$ with $|\alpha-\beta|\frac{k_{\mathrm{P}_{n+1}}}{4} > 1$ and any $\mathrm{M}_{k_{\mathrm{P}_{n+1}}} < j \leqslant \mathrm{M}_{k_{\mathrm{P}_{n+1}+1}}$,

$$\begin{aligned} d(B^j(\alpha\bar{v}), B^j(\beta\bar{v})) = d(0, B^j((\alpha-\beta)\bar{v})) &\geqslant d\left(0, y\left(j, \mathrm{M}_{k_{\mathrm{P}_{n+1}+2}} - \mathrm{M}_{k_{\mathrm{P}_{n+1}+1}}\right)\right) \\ &\geqslant d(0, B^j(y)) - \frac{1}{k_{\mathrm{P}_{n+1}}}. \end{aligned}$$

This, together with (2) and (3), implies that for any $0<\varepsilon<\delta$,

$$\begin{aligned}\Phi_{a,b}(\varepsilon) &= 1-\limsup_{n\to\infty}\frac{1}{n}\left|\{0\leqslant i<n : d(B^i(a),B^i(b))\geqslant\varepsilon\}\right| \\ &\leqslant 1-\limsup_{n\to\infty}\frac{1}{\mathrm{M}_{k_{\mathrm{P}_{n+1}}+1}}\left|\{0\leqslant i<\mathrm{M}_{k_{\mathrm{P}_{n+1}}+1} : d(B^i(a),B^i(b))\geqslant\varepsilon\}\right| \\ &\leqslant 1-\limsup_{n\to\infty}\frac{\Phi_{0,y}^{(\mathrm{M}_{k_{\mathrm{P}_{n+1}}+1})}(\delta)-\mathrm{M}_{k_{\mathrm{P}_{n+1}}}}{\mathrm{M}_{k_{\mathrm{P}_{n+1}}+1}}=0.\end{aligned}$$

Case 2. $q>p$. Fix any $n\in\mathbb{N}$ with $\alpha\frac{k_{\mathrm{P}_n^{q-p}+1}}{4}>1$. For any $\mathrm{M}_{k_{\mathrm{P}_n^{q-p}+1}}<i\leqslant\mathrm{M}_{k_{\mathrm{P}_n^{q-p}+1}+1}$ and any $i+1\leqslant j<i+(\mathrm{M}_{k_{\mathrm{P}_n^{q-p}}+3}-\mathrm{M}_{k_{\mathrm{P}_n^{q-p}}+2})$, noting that $\left|(\alpha\bar{v}-B^{q-p}(\beta\bar{v}))_j\right|\geqslant\left|(\alpha\bar{v})_j\right|$ (as $(\alpha\bar{v})_j$ and $(B^{q-p}(\beta\bar{v}))_j$ are of different signs), it follows that

$$d\big(B^{i-p}(a),B^{i-p}(b)\big)=d(0,B^i(\alpha\bar{v}-B^{q-p}(\beta\bar{v})))\geqslant d(0,\alpha\bar{v}(i,\mathrm{M}_{k_{\mathrm{P}_n^{q-p}}+3}-\mathrm{M}_{k_{\mathrm{P}_n^{q-p}}+2})).$$

Similarly to the proof of Case 1, it can be verified that for any $0<\varepsilon<\delta$, $\Phi_{a,b}(\varepsilon)=0$.

Therefore, $\mathscr{D}$ is an invariant distributionally ε-scrambled subset for any $0<\varepsilon<\delta$.

REMARK 1. (1) Given a sequence $\{\mathrm{w}_i\}_{i\geqslant 2}$ of strictly positive scalars, consider its associated *weighted backward shift*

$$B_{\mathrm{w}}(x_1,x_2,\ldots):=(\mathrm{w}_2x_2,\mathrm{w}_3x_3,\ldots).$$

According to the discussions in [8, 10], the study of chaos under a weighted backward shift can be reduced to the unweighted case, with a suitable Köthe matrix. So, for weighted backward shift, we actually have also obtained similar result.

(2) Applying Theorem 2, it is easy to verify that all examples in [10] admit an invariant ε-scrambled subset for some $\varepsilon>0$.

(3) Combining Theorem 2 with [16, Theorem 2.1], it follows that [19, Theorem 3.1] holds trivially.

REFERENCES

[1] T. BERMÚDEZ, A. BONILLA, F. MARTÍNEZ-GIMÉNEZ, A. PERIS, *Li-Yorke and distributionally chaotic operators*, J. Math. Anal. Appl. **373**, (2011), 83–93.

[2] N. C. BERNARDES JR., A. BONILLA, V. MÜLLER, A. PERIS, *Distributional chaos for linear operators*, J. Funct. Anal. **265**, (2013), 2143–2163.

[3] J. BÈS, *Invariant manifolds of hypercyclic vectors for the real scalar case*, Proc. Amer. Math. Soc. **127**, (1999), 1801–1804.

[4] B. DU, *On the invariance of Li-Yorke chaos of interval maps*, J. Difference Equ. Appl. **11**, (2005), 823–828.

[5] M. GOLIŃSKI, *Invariant subspace problem for classical spaces of functions*, J. Funct. Anal. **262**, (2012), 1251–1273.

[6] G. GODEFROY, J. H. SHAPIRO, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98**, (1991), 229–269.

[7] G. KÖTHE, *Topological Vector Space. I*, translated from German by D. J. H. Garling, Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag New York Inc., New York, 1969.

[8] F. MARTÍNEZ-GIMÉNEZ, *Chaos for power series of backward shift operators*, Proc. Amer. Math. Soc. **135**, (2007), 1741–1752.

[9] F. MARTÍNEZ-GIMÉNEZ, A. PERIS, *Chaos for backward shift operators*, Int. J. Bifurcation and Chaos **12**, (2002), 1703–1715.

[10] F. MARTÍNEZ-GIMÉNEZ, P. OPROCHA, A. PERIS, *Distributional chaos for backward shift*, J. Math. Anal. Appl. **351**, (2009), 607–615.

[11] P. OPROCHA, *A quantum harmonic oscillator and strong chaos*, J. Phys. A: Math. Gen. **39**, (2006), 14559–14565.

[12] P. OPROCHA, *Invariant scrambled sets and distributional chaos*, Dyn. Syst. **24**, (2009), 31–43.

[13] C. J. READ, *A solution to the invariant subspace problem*, Bull. London Math. Soc. **16**, (1984), 337–401.

[14] C. J. READ, *The invariant subspace problem for a class of Banach spaces. II. Hypercyclic operators*, Israel J. Math. **63**, (1988), 1–40.

[15] B. SCHWEIZER, J. SMÍTAL, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. **344**, (1994), 737–754.

[16] X. WU, *Maximal distributional chaos of backward shift operators on Köthe sequence spaces*, Czechoslovak Math. J. **64**, (2014), 105–114.

[17] X. WU, G. CHEN, *On the invariance of maximal distributional chaos under an annihilation operator*, Appl. Math. Lett. **26**, (2013), 1134–1140.

[18] X. WU, G. CHEN, P. ZHU, *Invariance of chaos from backward shift on the Köthe sequence space*, Nonlinearity **27**, (2014), 271–288.

[19] X. WU, L. WANG, G. CHEN, *Weighted backward shift operators with invariant distributionally scrambled subsets*, Ann. Funct. Anal. **8**, (2017), 199–210.

[20] X. WU, X. ZHANG, X. MA, *Various shadowing in linear dynamical systems*, Int J. Bifurcation and Chaos **29**, (2019), 1950042 (10 pages).

[21] X. WU, P. ZHU, *The principal measure of a quantum harmonic oscillator*, J. Phys. A: Math. Theor. **44**, (2011), 505101 (6pp).

[22] X. WU, P. ZHU, *On the equivalence of four chaotic operators*, Appl. Math. Lett. **25**, (2012), 545–549.

[23] X. WU, P. ZHU, *Li-Yorke chaos for backward shift operators on Köthe sequence spaces*, Topology Appl. **160**, (2013), 924–929.

[24] X. WU, P. ZHU, *Invariant scrambled set and maximal distributional chaos*, Ann. Polon. Math. **109**, (2013), 271–278.

[25] Z. YIN, Z. CHEN, Y. CHEN, X. WU, *Perturbation of distributionally chaotic operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, accepted for publication.

(Received October 10, 2017)

Xinxing Wu
School of Sciences
Southwest Petroleum University
Chengdu, Sichuan, 610500, P.R. China
Institute for Artificial Intelligence
Southwest Petroleum University
Chengdu, Sichuan, 610500, P.R. China
e-mail: wuxinxing5201314@163.com

Yang Luo
School of Sciences
Southwest Petroleum University
Chengdu, Sichuan, 610500, P.R. China
e-mail: 1321488002@qq.com