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# POSITIVE DEFINITENESS OF PIECEWISE–LINEAR FUNCTION 2

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*Abstract.* Let $\alpha,\beta \in (0,1)$, $0<\alpha \leqslant \beta < 1$, $s \in \mathbb{R}$ and let $w_{\alpha,\beta,s}$ be an even function with the properties: $w_{\alpha,\beta,s}(x)=0$ for $x>1$, $w_{\alpha,\beta,s}(0)=1$, $w_{\alpha,\beta,s}(1)=0$, $w_{\alpha,\beta,s}(x)=s$ for $x\in[\alpha,\beta]$ ($[\alpha,\alpha]:=\{\alpha\}$), $w_{\alpha,\beta,s}$ is linear over the intervals $[0,\alpha]$ and $[\beta,1]$. In this paper we prove that $w_{\alpha,\beta,s}$ is positive definite on $\mathbb{R} \iff m(\alpha,\beta)\leqslant s\leqslant M(\alpha,\beta)$, where $m(\alpha,\beta)\leqslant 0$, $M(\alpha,\beta)\geqslant 0$. If either $(1+\beta)/\alpha,(1-\beta)/\alpha\in\mathbb{N}$ or $1/\alpha\notin\mathbb{N}$, $\beta/\alpha\in\mathbb{N}$, then $M(\alpha,\beta)>0$, otherwise $M(\alpha,\beta)=0$. If either $(1+\beta)/\alpha,(1-\beta)/\alpha\in\mathbb{N}$ or $1/\alpha\in\mathbb{N}$, $\beta/\alpha\notin\mathbb{N}$, then $m(\alpha,\beta)<0$, otherwise $m(\alpha,\beta)=0$. Moreover, we find explicit values of $M(\alpha,\beta)$, $m(\alpha,\beta)$ for some $\alpha$ and $\beta$.

## 1. Introduction

A function $f:\mathbb{R}\to\mathbb{C}$ is said to be positive definite on $\mathbb{R}$ $(f\in\Phi(\mathbb{R}))$ if for every $n\in\mathbb{N}$, and for every choice of $x_1,\dots,x_n\in\mathbb{R}$, the $n\times n$ matrix $[f(x_i-x_j)]$ is positive semidefinite (see, e.g., [2, Chapter 7]).

It is well-known that positive definite functions and kernels have applications in various parts of mathematics: in approximation theory, probability theory [4, 6], operator theory [3] and other areas. In particular, positive definiteness of an integrable function on $\mathbb{R}$ is equivalent to positivity of some summation method of Fourier series (positive operator) (see, e.g., [10, Lemma 10]).

This paper considers the following problem. Let $\alpha,\beta\in(0,1)$, $0<\alpha\leqslant\beta<1$, $s\in\mathbb{R}$ and let $w_{\alpha,\beta,s}$ be an even function with the properties: $w_{\alpha,\beta,s}(x)=0$ for $x>1$, $w_{\alpha,\beta,s}(0)=1$, $w_{\alpha,\beta,s}(1)=0$, $w_{\alpha,\beta,s}(x)=s$ for $x\in[\alpha,\beta]$ ($[\alpha,\alpha]:=\{\alpha\}$), $w_{\alpha,\beta,s}$ is linear over the intervals $[0,\alpha]$ and $[\beta,1]$. For each pair of $\alpha,\beta\in(0,1)$, $0<\alpha\leqslant\beta<1$ find the set of all values of $s\in\mathbb{R}$ for which $w_{\alpha,\beta,s}\in\Phi(\mathbb{R})$.

If $0<\alpha\leqslant\beta<1$ and $s=0$, then $w_{\alpha,\beta,0}\in\Phi(\mathbb{R})$. Indeed, in this case $w_{\alpha,\beta,0}(x)=(1-|x/\alpha|)_+$ and it follows from the definition of positive definite functions and the easily verified relation

$$(1-|x|)_+=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\sin^2(t/2)}{(t/2)^2}e^{itx}\,dt,\qquad x\in\mathbb{R}$$

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that $w_{\alpha,\beta,0}$ is positive definite. This also follows from Pólya's theorem (see, e.g., [4, Theorem 4.3.1], [6, Theorem 3.9.11]). Notice that if $\alpha \neq \beta$ and $s \neq 0$, then $w_{\alpha,\beta,s}$ is not convex on $(0,+\infty)$, and therefore Pólya's theorem does not apply.

The case $\alpha = \beta$ is known as Trigub problem, for which a solution has been provided by Zastavnyi and Manov in [5]. The following result provides necessary and sufficient conditions for the function $w_{\alpha,\alpha,s}$ to be positive definite.

THEOREM 1. ([5]) *Let* $\alpha \in (0,1)$, $s \in \mathbb{R}$. *Then* $w_{\alpha,\alpha,s} \in \Phi(\mathbb{R}) \iff m(\alpha) \leqslant s \leqslant 1-\alpha$, *where* $m(\alpha) = 0$ *if* $1/\alpha \notin \mathbb{N}$, *and* $m(\alpha) = -\alpha$ *if* $1/\alpha \in \mathbb{N}$.

In [8] Zastavnyi and Manov found necessary and sufficient conditions for positive definiteness of $w_{\alpha,\alpha,c}$ when $c := s+ih$, for $s,h \in \mathbb{R}$. In this case the function $w_{\alpha,\alpha,c}$ is defined by the relations: 1) $w_{\alpha,\alpha,c}$ is hermitian, i.e., $w_{\alpha,\alpha,c}(-x) = \overline{w_{\alpha,\alpha,c}(x)}$, $x \in \mathbb{R}$; 2) $w_{\alpha,\alpha,c}(x) = 0$ for $x > 1$, $w_{\alpha,\alpha,c}$ is linear over the intervals $[0,\alpha]$ and $[\alpha,1]$, $w_{\alpha,\alpha,c}(0) = 1$, $w_{\alpha,\alpha,c}(\alpha) = c$, $w_{\alpha,\alpha,c}(1) = 0$. Furthermore, in [8] new Bernstein type inequalities for trigonometric polynomials were obtained by using Theorem 1. For more details about the connection between sharp inequalities for trigonometric polynomials and positive definite functions the reader is referred to [9].

The main result of this paper is the following theorem.

THEOREM 2. *Let* $0 < \alpha \leqslant \beta < 1$ *and* $s \in \mathbb{R}$. *Then* $w_{\alpha,\beta,s} \in \Phi(\mathbb{R}) \iff m(\alpha,\beta) \leqslant s \leqslant M(\alpha,\beta)$, *where*

$$M(\alpha,\beta) := \frac{1-\beta}{1-\beta-\alpha\mathfrak{m}_1\left(\frac{1+\beta}{\alpha},\frac{1-\beta}{\alpha}\right)},\ m(\alpha,\beta) := \frac{1-\beta}{1-\beta-\alpha\mathfrak{m}_2\left(\frac{1+\beta}{\alpha},\frac{1-\beta}{\alpha}\right)}, \tag{1}$$

*and* $\mathfrak{m}_1(\nu_1,\nu_2)$, $\mathfrak{m}_2(\nu_1,\nu_2)$, $\nu_1,\nu_2 > 0$ *are defined by*

$$\mathfrak{m}_1(\nu_1,\nu_2) := \inf_{\mathbb{R}\setminus\pi\mathbb{Z}} \frac{\sin(\nu_1 t)\sin(\nu_2 t)}{\sin^2(t)},\ \mathfrak{m}_2(\nu_1,\nu_2) := \sup_{\mathbb{R}\setminus\pi\mathbb{Z}} \frac{\sin(\nu_1 t)\sin(\nu_2 t)}{\sin^2(t)}. \tag{2}$$

*Moreover:*

1) *If either* $(1+\beta)/\alpha, (1-\beta)/\alpha \in \mathbb{N}$ *or* $1/\alpha \notin \mathbb{N}$, $\beta/\alpha \in \mathbb{N}$, *then* $M(\alpha,\beta) > 0$, *otherwise* $M(\alpha,\beta) = 0$.

2) *If either* $(1+\beta)/\alpha, (1-\beta)/\alpha \in \mathbb{N}$ *or* $1/\alpha \in \mathbb{N}$, $\beta/\alpha \notin \mathbb{N}$, *then* $m(\alpha,\beta) < 0$, *otherwise* $m(\alpha,\beta) = 0$.

In the following we find explicit values of $\mathfrak{m}_1(\nu_1,\nu_2)$, $\mathfrak{m}_2(\nu_1,\nu_2)$ for some values of $\nu_1$ and $\nu_2$.

THEOREM 3. *Let* $\nu_1,\nu_2 > 0$ *and* $\mathfrak{m}_1(\nu_1,\nu_2)$, $\mathfrak{m}_2(\nu_1,\nu_2)$ *as defined in equation* (2). *Then the following assertions hold.*

1) *If* $|\nu_1-\nu_2| = 2$, *then* $\mathfrak{m}_1(\nu_1,\nu_2) = -1$.

2) *If* $\nu_1,\nu_2\in\mathbb{N}$, *then* $\mathfrak{m}_2(\nu_1,\nu_2)=\nu_1\nu_2$ *and* $\mathfrak{m}_1(\nu_1,\nu_2)\geqslant-\nu_1\nu_2$. *If, in addition,* $\nu_1$ *and* $\nu_2$ *have opposite parity, then* $\mathfrak{m}_1(\nu_1,\nu_2)=-\nu_1\nu_2$.

3) *If* $\nu_1=p_1/q$, $\nu_2=p_2/q$, *where* $p_1,p_2,q\in\mathbb{N}$, *then*

$$\mathfrak{m}_1(\nu_1,\nu_2)=\inf_{[-1,1]\setminus A}\frac{U_{p_1-1}(x)U_{p_2-1}(x)}{[U_{q-1}(x)]^2},\ \mathfrak{m}_2(\nu_1,\nu_2)=\sup_{[-1,1]\setminus A}\frac{U_{p_1-1}(x)U_{p_2-1}(x)}{[U_{q-1}(x)]^2},$$

*where* $A:=\{x:U_{q-1}(x)=0\}$, *and* $U_p(\cos(t)):=\sin((p+1)t)/\sin(t)$, $t\in[0,\pi]$, $p\in\mathbb{Z}_+$ *are the Chebyshev polynomials of the second kind.*

4) $\mathfrak{m}_1(1,1)=1$, $\mathfrak{m}_1(1,3)=-1$, $\mathfrak{m}_1(1,5)=-5/4$, $\mathfrak{m}_1(1,7)=-(7+14\sqrt{7})/27$, $\mathfrak{m}_2(1/2,7/2)=7/4$.

Notice that Theorem 2 is proved in the same way as Theorem 1 in [5], but the result essentially depends on the extremal properties of a function of a certain type (see Proposition 1).

Let us note that combination of Theorem 2 and Theorem 3 provides the following sufficient conditions for positive definiteness: *let* $g\in C(\mathbb{R})$ *be an even function, which is nonnegative, nonincreasing and convex on* $(0,+\infty)$, *and let* $\alpha=2/(2m+2k+1)$, $\beta=(2k+1)/(2m+2k+1)$, $m,k\in\mathbb{N}$, $s\in\mathbb{R}$. *Define*

$$g_{\alpha,\beta,s}(x):=(1-s)g\left(\frac{x}{\alpha}\right)-s\frac{\beta}{1-\beta}g\left(\frac{x}{\beta}\right)+\frac{s}{1-\beta}g(x),\qquad x\in\mathbb{R}.$$

*If* $-1/(m+2k)\leqslant s\leqslant 1/(m+2k+2)$, *then* $g_{\alpha,\beta,s}\in\Phi(\mathbb{R})$. Indeed, in this case the function $g$ can be represented in the form (see, e.g., [6, 3.9.12]):

$$g(x)=\int_0^{+\infty}(1-|xu|)_+d\mu(u),\ x\in\mathbb{R},$$

where $\mu$ is a finite nonnegative Borel measure on $[0,+\infty)$. It is easy to prove that

$$w_{\alpha,\beta,s}(x)=(1-s)(1-|x/\alpha|)_+-s\frac{\beta}{1-\beta}(1-|x/\beta|)_++\frac{s}{1-\beta}(1-|x|)_+,\ x\in\mathbb{R},\quad (3)$$

and hence

$$g_{\alpha,\beta,s}(x)=\int_0^{+\infty}w_{\alpha,\beta,s}(xu)d\mu(u),\ x\in\mathbb{R}.$$

For $\alpha,\beta$ and $s$ as above we have $w_{\alpha,\beta,s}\in\Phi(\mathbb{R})$ (see Example 3), and so $g_{\alpha,\beta,s}\in\Phi(\mathbb{R})$ (see, e.g., [10, Lemma 1]).

Let us note one more corollary of Theorem 2. A function $f$ is said to be completely monotone on the interval $(0,+\infty)$ ($f\in\mathscr{CM}$) if $f\in C^\infty(0,+\infty)$ and $(-1)^n f^{(n)}(x)\geqslant 0$ for all $k\in\mathbb{Z}_+$ and $x>0$. Let $\alpha,\beta\in(0,1)$, $a,b,c\in\mathbb{R}$ and define the function $f^{\alpha,\beta}_{a,b,c}$ by

$$f^{\alpha,\beta}_{a,b,c}(x):=\frac{a}{x(x^2+\alpha^2)}+\frac{b}{x(x^2+\beta^2)}+\frac{c}{x(x^2+1)},\ x>0.$$

If $\alpha = \beta$, then

$$f_{a,b,c}^{\alpha,\alpha}(x) := \frac{(a+b+c)x^2 + (a+b+\alpha^2 c)}{x(x^2+\alpha^2)(x^2+1)}, \; x > 0.$$

In this case, it follows from Theorem 4 in [8] that if $\alpha \in (0,1)$ and $(a+b+c)^2 + (a+b+\alpha^2 c) \neq 0$, then

$$f_{a,b,c}^{\alpha,\alpha} \in \mathscr{CM} \iff a+b+c \geqslant 0, \; a+b+\alpha^2 c > 0, \; m(\alpha) \leqslant \frac{\alpha(1-\alpha)c}{a+b+c\alpha} \leqslant 1-\alpha,$$

where $m(\alpha)$ is defined as in Theorem 1.

In the case $0 < \alpha < \beta < 1$ the following theorem holds true.

THEOREM 4. *Let* $0 < \alpha < \beta < 1$, $a,b,c \in \mathbb{R}$, *and suppose that*

$$1 - \frac{a}{\alpha} = \frac{b(\beta-1)}{\beta^2} = c(1-\beta) =: s. \tag{4}$$

*Then* $f_{a,b,c}^{\alpha,\beta} \in \mathscr{CM} \iff m(\alpha,\beta) \leqslant s \leqslant M(\alpha,\beta)$, *where* $m(\alpha,\beta)$, $M(\alpha,\beta)$ *are defined as in Theorem 2.*

This paper is organized as follows. Section 2 contains some auxiliary facts and statements. In Sections 3 and 4, we prove Theorems 2 and 3, respectively. In Section 5, we give some examples of application of Theorem 2, in particular we obtain Theorem 1. In Section 6, we prove Theorem 4.

## 2. Auxiliary facts and statements

Let $f \in \Phi(\mathbb{R})$. Then $f$ is continuous at the origin if and only if $f$ is continuous on $\mathbb{R}$. Furthermore, if $f,g \in \Phi(\mathbb{R})$ then $|f(x)| \leqslant f(0)$, $\overline{f(-x)} = f(x)$, $x \in \mathbb{R}$ and $\overline{f}$, $\Re f$, $fg \in \Phi(\mathbb{R})$. The following theorem was proved independently by S. Bochner and A. Khinchin in 1932.

THEOREM 5. (Bochner-Khinchin) $f \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ *if and only if there is a finite nonnegative Borel measure* $\mu$ *on* $\mathbb{R}$ *such that*

$$f(x) = \int_{\mathbb{R}} e^{ixt} d\mu(t), \; x \in \mathbb{R}.$$

The proof can be found in [4, 6, 7, 1]. As a consequence, we obtain the following criterion for positive definiteness in terms of the Fourier transform.

COROLLARY 1. *If* $f \in C(\mathbb{R}) \cap L_1(\mathbb{R})$, *then* $f \in \Phi(\mathbb{R}) \iff \widehat{f}(t) \geqslant 0$, $t \in \mathbb{R}$, *where*

$$\widehat{f}(t) := \int_{\mathbb{R}} f(x) e^{-itx} dx, \; t \in \mathbb{R}.$$

The following lemma is needed in the sequel.

LEMMA 1. ([5]) *Let $V$ be an arbitrary nonempty set and $h \in \mathbb{R}$, let $G : V \to \mathbb{R}$, $M := \sup_V G(t)$, and $m := \inf_V G(t)$. If $M > 0$ and $m < 0$, then the inequality $1 - hG(t) \geqslant 0$ is satisfied for any $t \in V \iff 1/m \leqslant h \leqslant 1/M$.*

For completeness, we give a proof here.

*Proof.* Let us find all $h \in \mathbb{R}$ such that the inequality

$$1 - hG(t) \geqslant 0 \tag{5}$$

is satisfied for any $t \in V$. Solve this problem for each set of the following partition of $V$:

$$V_0 := \{t \in V : G(t) = 0\}, \; V_- := \{t \in V : G(t) < 0\}, \; V_+ := \{t \in V : G(t) > 0\}.$$

From the conditions of the lemma it follows that $V_-$ and $V_+$ are nonempty and the following equalities are true:

$$M = \sup_V G(t) = \sup_{V_+} G(t) \text{ and } m = \inf_V G(t) = \inf_{V_-} G(t).$$

For $t \in V_0$ inequality (5) is satisfied for any $h$. For $t \in V_-$ inequality (5) holds if and only if $h \geqslant \sup_{V_-} 1/G(t) = 1/\inf_{V_-} G(t) = 1/m$. For $t \in V_+$ inequality (5) holds if and only if $h \leqslant \inf_{V_+} 1/G(t) = 1/\sup_{V_+} G(t) = 1/M$. Lemma 1 is proved. $\square$

Define the function $K_{\nu_1,\nu_2}$ by

$$K_{\nu_1,\nu_2}(t) := \frac{\sin(\nu_1 t)\sin(\nu_2 t)}{\sin^2(t)}, \; \nu_1, \nu_2 > 0. \tag{6}$$

The values of $K_{\nu_1,\nu_2}$ at points $t = \pi n$, $n \in \mathbb{Z}$ are defined by continuity, whenever possible. The following properties are easily obtained: 1) $K_{\nu_1,\nu_2}$ is an even function; 2) $K_{\nu_1,\nu_2}$ is symmetric with respect to $\nu_1$, $\nu_2$, i.e. $K_{\nu_1,\nu_2} = K_{\nu_2,\nu_1}$; 3) $K_{\nu_1,\nu_2}$ is nonnegative on its domain if and only if $\nu_1 = \nu_2$; 4) the supremum (the infimum) of $K_{\nu_1,\nu_2}$ over its domain is the same as that over $\mathbb{R} \setminus \pi\mathbb{Z}$. In addition, $\mathfrak{m}_1(\nu_1,\nu_2) = \inf_{\mathbb{R}\setminus\pi\mathbb{Z}} K_{\nu_1,\nu_2}(t)$ and $\mathfrak{m}_2(\nu_1,\nu_2) = \sup_{\mathbb{R}\setminus\pi\mathbb{Z}} K_{\nu_1,\nu_2}(t)$.

PROPOSITION 1. *Let $\nu_1, \nu_2 > 0$. Then the following assertions hold.*

1) *$K_{\nu_1,\nu_2}$ is bounded on $\mathbb{R}$ if and only if $\nu_1, \nu_2 \in \mathbb{N}$.*

2) *$K_{\nu_1,\nu_2}$ is bounded below but not bounded above on its domain if and only if $\nu_1, \nu_2 \notin \mathbb{N}$ and $|\nu_1 - \nu_2| = 2n$ for some $n \in \mathbb{Z}_+$.*

3) *$K_{\nu_1,\nu_2}$ is bounded above but not bounded below on its domain if and only if $\nu_1, \nu_2 \notin \mathbb{N}$ and $\nu_1 + \nu_2 = 2n$ for some $n \in \mathbb{N}$.*

4) *Otherwise,* $K_{\nu_1,\nu_2}(t)$ *is neither bounded above nor bounded below on its domain. In particular, when a)* $\nu_1 \in \mathbb{Q}$, $\nu_2 \notin \mathbb{Q}$; *b)* $\nu_1, \nu_2 \in \mathbb{Q}$ *and* $\nu_1 = p_1/q_1$, $\nu_2 = p_2/q_2$, $p_i, q_i \in \mathbb{N}$, $q_1 \neq q_2$, *where* $p_i$ *and* $q_i$ *are relatively prime.*

Let $\nu_1, \nu_2 > 0$ and denote by $S(\nu_1, \nu_2)$ the set of all $k \in \mathbb{Z}$ for which the values of $K_{\nu_1,\nu_2}$ at points $t = \pi k$ can be defined by continuity. It is obvious that

$$S(\nu_1, \nu_2) = \{k \in \mathbb{Z} : \sin(\nu_1 \pi k) = 0 \text{ and } \sin(\nu_2 \pi k) = 0\}.$$

Since $0 \in S(\nu_1, \nu_2)$ for all $\nu_1, \nu_2 > 0$, we have $S(\nu_1, \nu_2) \neq \varnothing$. It is easily seen that if $k \in S(\nu_1, \nu_2)$, then $|K_{\nu_1,\nu_2}(\pi k)| = \nu_1 \nu_2$. From the definition of $S(\nu_1, \nu_2)$ it follows that the domain of $K_{\nu_1,\nu_2}$ is $\mathbb{R} \setminus \pi(\mathbb{Z} \setminus S(\nu_1, \nu_2))$.

The following lemmas are needed to prove Proposition 1.

LEMMA 2. *Let* $\nu_1, \nu_2 > 0$. *Then:*

1) *If the function* $K_{\nu_1,\nu_2}(t)$ *is bounded below but not bounded above on its domain, then* $S(\nu_1, \nu_2) \neq \mathbb{Z}$ *and the inequality* $\sin(\nu_1 \pi k) \sin(\nu_2 \pi k) > 0$ *holds for every* $k \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$.

2) *If the function* $K_{\nu_1,\nu_2}(t)$ *is bounded above but not bounded below on its domain, then* $S(\nu_1, \nu_2) \neq \mathbb{Z}$ *and the inequality* $\sin(\nu_1 \pi k) \sin(\nu_2 \pi k) < 0$ *holds for every* $k \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$.

*Proof.*

1) Suppose $K_{\nu_1,\nu_2}(t)$ is bounded below but not bounded above on $\mathbb{R} \setminus \pi(\mathbb{Z} \setminus S(\nu_1, \nu_2))$. Hence, there is a point $t_0 = \pi k_0$, $k_0 \in \mathbb{Z}$ such that $K_{\nu_1,\nu_2}(t)$ is unbounded in a neighborhood of $t_0$, and so $S(\nu_1, \nu_2) \neq \mathbb{Z}$. Let us show that $\sin(\nu_1 \pi k) \neq 0$ and $\sin(\nu_2 \pi k) \neq 0$ for all $k \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$. Assume that $\sin(\nu_1 \pi k_0) = 0$ and $\sin(\nu_2 \pi k_0) \neq 0$ for some $k_0 \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$. Hence, the function $\sin(\nu_2 t)$ does not change sign in the neighborhood of $t_0 = \pi k_0$, but $\sin(\nu_1 t)$ changes sign in the same neighborhood. Therefore, $K_{\nu_1,\nu_2}(t)$ is neither bounded above nor bounded below, so we have a contradiction. Since $K_{\nu_1,\nu_2}(t)$ is bounded below, we have $\sin(\nu_1 \pi k) \sin(\nu_2 \pi k) > 0$, $k \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$.

2) Assertion 2) is proved analogously. □

REMARK 1. If the function $K_{\nu_1,\nu_2}(t)$ is bounded below (above) but not bounded above (below) on its domain, we may assume that either $\nu_1, \nu_2 \notin \mathbb{Q}$ or $\nu_1 = p_1/q$, $\nu_2 = p_2/q$, $p_1, p_2, q \in \mathbb{N}$, $q > 1$, where $p_i$ and $q$ are relatively prime. In these cases, $S(\nu_1, \nu_2) = \{0\}$ and $S(\nu_1, \nu_2) = q\mathbb{Z}$, respectively.

Indeed:

1) If $\nu_1, \nu_2 \in \mathbb{N}$, then $S(\nu_1, \nu_2) = \mathbb{Z}$.

2) If $\nu_1 \in \mathbb{Q}$, $\nu_2 \notin \mathbb{Q}$ and $\nu_1 = p/q$, $p, q \in \mathbb{N}$, then $k = q \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$ and $\sin(\nu_1 \pi k) \sin(\nu_2 \pi k) = 0$.

3) If $\nu_1 = p_1/q_1$, $\nu_2 = p_2/q_2$, $p_i, q_i \in \mathbb{N}$, $q_1 \neq q_2$, where $p_i$ and $q_i$ are relatively prime, then $k = \min\{q_1, q_2\} \in \mathbb{Z} \setminus S(\nu_1, \nu_2)$ and $\sin(\nu_1 \pi k)\sin(\nu_2 \pi k) = 0$.

The formulation of the following lemma is due to Zaraisky.

LEMMA 3. *Let $h > 0$, $x_0 \in \mathbb{R}$ and $T(x) := \sum_{j=0}^{n} c_j e^{i\lambda_j x}$, $x \in \mathbb{R}$, where $\lambda_j, c_j \in \mathbb{R}$. If $e^{i\lambda_j h} \neq 1$, $j = 0, \ldots, n$ and $e^{i(\lambda_j - \lambda_p)h} \neq 1$ for $j \neq p$ and $T(x_0 + kh) \geqslant 0$ for every $k \in \mathbb{Z}_+$, then $c_j = 0$ for $j = 0, \ldots, n$.*

*Proof.* Consider the partial sums

$$S_{m+1} := \sum_{k=0}^{m} T(x_0 + kh) = \sum_{k=0}^{m} \sum_{j=0}^{n} c_j e^{i\lambda_j (x_0 + kh)} = \sum_{j=0}^{n} c_j e^{i\lambda_j x_0} \sum_{k=0}^{m} e^{ik\lambda_j h}.$$

Since $e^{i\lambda_j h} \neq 1$, $j = 0, \ldots, n$, we have

$$\left| \sum_{k=0}^{m} e^{ik\lambda_j h} \right| = \left| \frac{1 - e^{i(m+1)\lambda_j h}}{1 - e^{i\lambda_j h}} \right| \leqslant \frac{2}{|1 - e^{i\lambda_j h}|}.$$

Therefore,

$$S_{m+1} = |S_{m+1}| \leqslant \sum_{j=0}^{n} |c_j| \frac{2}{|1 - e^{i\lambda_j h}|}.$$

Since $T(x_0 + kh) \geqslant 0$ holds for every $k \in \mathbb{Z}_+$, it follows that the series $\sum_{k=0}^{\infty} T(x_0 + kh)$ is convergent. Hence $T(x_0 + kh) \to 0$ as $k \to \infty$.

Let $p \in \{0, \ldots, n\}$. Consider the sequence

$$t_k(p) := e^{-i\lambda_p kh} T(x_0 + kh) = c_p e^{i\lambda_p x_0} + \sum_{j=0, j\neq p}^{n} c_j e^{i\lambda_j x_0} e^{i(\lambda_j - \lambda_p)kh}.$$

Since $t_k(p) \to 0$ as $k \to \infty$, it follows that the arithmetic mean of the sequence $\{t_k(p)\}_{k=0}^{\infty}$ converges to zero as well. On the other hand, since $e^{i(\lambda_j - \lambda_p)h} \neq 1$ for $j \neq p$, we have

$$\begin{aligned}\left| \sum_{k=0}^{m} \sum_{j=0, j\neq p}^{n} c_j e^{i\lambda_j x_0} e^{i(\lambda_j - \lambda_p)kh} \right| &= \left| \sum_{j=0, j\neq p}^{n} c_j e^{i\lambda_j x_0} \frac{1 - e^{i(m+1)(\lambda_j - \lambda_p)h}}{1 - e^{i(\lambda_j - \lambda_p)h}} \right| \\ &\leqslant \sum_{j=0, j\neq p}^{n} |c_j| \frac{2}{|1 - e^{i(\lambda_j - \lambda_p)h}|}.\end{aligned}$$

Therefore,

$$\frac{1}{m+1} (t_0(p) + \cdots + t_m(p)) \to c_p e^{i\lambda_p x_0}, \ k \to \infty,$$

and hence $c_p = 0$. $\square$

*Proof of Proposition* 1. Let $\nu_1,\nu_2>0$. Let us prove Assertion *1*). *Necessity*. Since $K_{\nu_1,\nu_2}(t)$ is bounded on $\mathbb{R}$, we have $S(\nu_1,\nu_2)=\mathbb{Z}$. Then, $1\in S(\nu_1,\nu_2)$, i.e., $\sin(\nu_1\pi)=\sin(\nu_2\pi)=0$, hence there are $k_1,k_2\in\mathbb{Z}$ such that $\nu_1=k_1,\ \nu_2=k_2$. Since $\nu_1,\nu_2>0$, we have $k_1,k_2\in\mathbb{N}$.

*Sufficiency.* Let $\nu_1,\nu_2\in\mathbb{N}$. Then $\sin(\nu_1\pi k)=\sin(\nu_2\pi k)=0$ for all $k\in\mathbb{Z}$. Therefore, $S(\nu_1,\nu_2)=\mathbb{Z}$. Since $K_{\nu_1,\nu_2}$ is continuous on $[-\pi,\pi]$, $K_{\nu_1,\nu_2}$ is bounded on $[-\pi,\pi]$. Since $K_{\nu_1,\nu_2}$ has period $2\pi$, $K_{\nu_1,\nu_2}$ is bounded on $\mathbb{R}$.

Let us prove Assertion *2*). *Necessity*. Suppose the function $K_{\nu_1,\nu_2}(t)$ is bounded below but not bounded above on its domain. Then from lemma 2 it follows that $S(\nu_1,\nu_2)\neq\mathbb{Z}$ and the inequality

$$\sin(\nu_1\pi k)\sin(\nu_2\pi k)>0 \tag{7}$$

holds for every $k\in\mathbb{Z}\setminus S(\nu_1,\nu_2)$. If $k\in S(\nu_1,\nu_2)$, then $\sin(\nu_1\pi k)\sin(\nu_2\pi k)=0$. Therefore,

$$\cos((\nu_1-\nu_2)\pi k)-\cos((\nu_1+\nu_2)\pi k)=2\sin(\nu_1\pi k)\sin(\nu_2\pi k)\geqslant 0,\ k\in\mathbb{Z}_+.$$

Let $x_0=0$, $h=\pi$ and

$$\begin{aligned}T(x)&:=\cos((\nu_1-\nu_2)x)-\cos((\nu_1+\nu_2)x)\\&=\frac{1}{2}e^{i(\nu_1-\nu_2)x}+\frac{1}{2}e^{-i(\nu_1-\nu_2)x}-\frac{1}{2}e^{i(\nu_1+\nu_2)x}-\frac{1}{2}e^{-i(\nu_1+\nu_2)x}.\end{aligned}$$

Let $\lambda_1=\nu_1-\nu_2$, $\lambda_2=-\nu_1+\nu_2$, $\lambda_3=\nu_1+\nu_2$, $\lambda_4=-\nu_1-\nu_2$. Since $\nu_1,\nu_2\notin\mathbb{N}$, we have $e^{\pm i2\nu_1\pi}\neq 1$, $e^{\pm i2\nu_2\pi}\neq 1$. It follows from Lemma 3 that at least one of the following equalities holds: $e^{\pm i(\lambda_1-\lambda_2)\pi}=e^{\pm i2(\nu_1-\nu_2)\pi}=1$, $e^{\pm i(\lambda_3-\lambda_4)\pi}=e^{\pm i2(\nu_1+\nu_2)\pi}=1$, $e^{\pm i\lambda_1\pi}=e^{\pm i(\nu_1-\nu_2)\pi}=1$, $e^{\pm i\lambda_3\pi}=e^{\pm i(\nu_1+\nu_2)\pi}=1$, so either $\nu_1-\nu_2=m$ or $\nu_1+\nu_2=m$, where $m\in\mathbb{Z}$.

If $\nu_1+\nu_2=2n$, $n\in\mathbb{Z}$ then $\sin(\nu_1\pi k)\sin(\nu_2\pi k)=-\sin^2(\nu_2\pi k)\leqslant 0$ holds for every $k\in\mathbb{Z}_+$. This is a contradiction, since $S(\nu_1,\nu_2)\neq\mathbb{Z}$ and inequality (7) holds for some $k\in\mathbb{Z}\setminus S(\nu_1,\nu_2)$.

Suppose $\nu_1,\nu_2\notin\mathbb{Q}$. Then $\mathbb{Z}\setminus S(\nu_1,\nu_2)=\mathbb{Z}\setminus\{0\}$ (see Remark 1). If $\nu_1+\nu_2=2n+1$, $n\in\mathbb{Z}$, then there is an even $k\in\mathbb{Z}\setminus S(\nu_1,\nu_2)$; hence $\sin(\nu_1\pi k)\sin(\nu_2\pi k)=-\sin^2(\nu_2\pi k)$ and we have a contradiction, since the inequality (7) does not hold. If $\nu_1-\nu_2=2n+1$, $n\in\mathbb{Z}$, then there is an odd $k\in\mathbb{Z}\setminus S(\nu_1,\nu_2)$; hence $\sin(\nu_1\pi k)\sin(\nu_2\pi k)=-\sin^2(\nu_2\pi k)$ and we have a contradiction. Therefore, $\nu_1-\nu_2=2n$, $n\in\mathbb{Z}$, and so $|\nu_1-\nu_2|=2n$, $n\in\mathbb{Z}_+$.

Suppose $\nu_1=p_1/q$, $\nu_2=p_2/q$, $p_1,p_2,q\in\mathbb{N}$, $q>1$, where $p_i$ and $q$ are relatively prime. Then $\mathbb{Z}\setminus S(\nu_1,\nu_2)=\mathbb{Z}\setminus q\mathbb{Z}$ (see Remark 1). If $q>2$, then $1,2\in\mathbb{Z}\setminus S(\nu_1,\nu_2)$, so the same arguments as above shows that $|\nu_1-\nu_2|=2n$, $n\in\mathbb{Z}_+$.

In case of $q=2$ we have $\nu_1=p_1/2$, $\nu_2=p_2/2$, where $p_1,p_2$ are odd. If $\nu_1+\nu_2=2m+1$, $m\in\mathbb{Z}$, then $p_1=2(2m+1)-p_2$; hence $\nu_1-\nu_2=(2m+1)-p_2$. Since $p_2$ is odd, so $\nu_1-\nu_2$ is even. Therefore, $|\nu_1-\nu_2|=2n$, $n\in\mathbb{Z}_+$.

*Sufficiency.* Let $\nu_1,\nu_2\notin\mathbb{N}$ and $|\nu_1-\nu_2|=2n$, $n\in\mathbb{Z}_+$. Without loss of a generality we may assume that $\nu_2=\nu_1+2n$ for some $n\in\mathbb{Z}_+$. Since $\nu_1,\nu_2\notin\mathbb{N}$, $K_{\nu_1,\nu_2}(t)$

is unbounded on its domain. From the inequality

$$\begin{aligned}
&\frac{\sin(\nu_1 t)\sin((\nu_1+2n)t)}{\sin^2(t)} \\
&=\frac{\sin(\nu_1 t)(\sin(\nu_1 t)\cos(2nt)+\cos(\nu_1 t)\sin(2nt))-\sin^2(\nu_1 t)+\sin^2(\nu_1 t)}{\sin^2(t)} \\
&=-\sin^2(\nu_1 t)\frac{2\sin^2(nt)}{\sin^2(t)}+\frac{(\sin(\nu_1 t)+(1/2)\cos(\nu_1 t)\sin(2nt))^2}{\sin^2(t)}-\frac{1}{4}\frac{\cos^2(\nu_1 t)\sin^2(2nt)}{\sin^2(t)} \\
&\geqslant -3n^2,\ t\neq \pi k,\ k\in\mathbb{Z},
\end{aligned}$$

it follows that $K_{\nu_1,\nu_2}(t)$ is bounded below but not bounded above on its domain.

Assertion 3) is proved analogously. Assertion 4) follows from Assertions 1)–3). □

## 3. Proof of Theorem 2

Let us prove the main assertion of Theorem 2. Let $\alpha,\beta\in(0,1)$, $0<\alpha\leqslant\beta<1$ and $s\in\mathbb{R}$. The function $w_{\alpha,\beta,s}$ has the following explicit form:

$$w_{\alpha,\beta,s}(x)=\begin{cases}-\frac{1-s}{\alpha}|x|+1,\ |x|\in[0,\alpha],\\ s,\ |x|\in[\alpha,\beta],\\ -\frac{s}{1-\beta}|x|+\frac{s}{1-\beta},\ |x|\in[\beta,1],\\ 0,\ |x|\geqslant 1,\end{cases}\qquad 0<\alpha\leqslant\beta<1,\ s\in\mathbb{R}.$$

Since $w_{\alpha,\beta,s}$ is a continuous function with compact support, it follows from Corollary 1 that $w_{\alpha,\beta,s}\in\Phi(\mathbb{R}) \iff \widehat{w_{\alpha,\beta,s}}(t)\geqslant 0$ for all $t\in\mathbb{R}$, where

$$\widehat{w_{\alpha,\beta,s}}(t)=\frac{4\sin^2\left(\frac{\alpha t}{2}\right)}{\alpha t^2}-s\left[\frac{4\sin^2\left(\frac{\alpha t}{2}\right)}{\alpha t^2}-\frac{4\sin\left(\frac{1+\beta}{2}t\right)\sin\left(\frac{1-\beta}{2}t\right)}{(1-\beta)t^2}\right],\ t\in\mathbb{R}.$$

Let $E_{\alpha,\beta}:=\{s\in\mathbb{R}: w_{\alpha,\beta,s}\in\Phi(\mathbb{R})\}$. Since $\widehat{w_{\alpha,\beta,s}}(t)$ is linear with respect to $s$, it follows that for fixed $t\in\mathbb{R}$ the inequality $\widehat{w_{\alpha,\beta,s}}(t)\geqslant 0$ holds for every $s\in[\gamma,+\infty)$ or $s\in(-\infty,\gamma]$, where $\gamma\in\mathbb{R}$. Obviously, $E_{\alpha,\beta}$ is the intersection of closed, convex sets, so $E_{\alpha,\beta}$ is convex and closed. If $s\in E_{\alpha,\beta}$, then $w_{\alpha,\beta,s}\in\Phi(\mathbb{R})$ and hence $|w_{\alpha,\beta,s}(x)|\leqslant w_{\alpha,\beta,s}(0)=1$ for all $x\in\mathbb{R}$. For $x=\alpha$ we have $|s|\leqslant 1$. Since $E_{\alpha,\beta}$ is a convex, closed, bounded subset of $\mathbb{R}$, it follows that $E_{\alpha,\beta}$ is a closed interval (or a degenerate interval), i.e., $E_{\alpha,\beta}=[a,b]$. In addition, $E_{\alpha,\beta}$ contains the origin, and hence $a\leqslant 0$, $b\geqslant 0$.

It is obvious that $\widehat{w_{\alpha,\beta,s}}(t)\geqslant 0,\ t\in\mathbb{R} \iff (1/\alpha)\widehat{w_{\alpha,\beta,s}}(2t/\alpha)\geqslant 0,\ t\in\mathbb{R}$. Since $\widehat{w_{\alpha,\beta,s}}$ is continuous on $\mathbb{R}$, it follows that the last inequality is equivalent to the following inequality: $(1/\alpha)\widehat{w_{\alpha,\beta,s}}(2t/\alpha)\geqslant 0,\ t\in\mathbb{R}\setminus A$, where $A:=\pi\mathbb{Z}\setminus\{0\}$.

Let us find all $s\in\mathbb{R}$ such that the inequality

$$(1/\alpha)\widehat{w_{\alpha,\beta,s}}\left(\frac{2t}{\alpha}\right)=\frac{\sin^2(t)}{t^2}-s\left(\frac{\sin^2(t)}{t^2}-\frac{\alpha}{1-\beta}\frac{\sin\left(\frac{1+\beta}{\alpha}t\right)\sin\left(\frac{1-\beta}{\alpha}t\right)}{t^2}\right)\geqslant 0 \quad (8)$$

holds for all $t\in\mathbb{R}\setminus A$. Since $\sin^2(t)/t^2>0$ (at the origin the function is defined by continuity) for all $t\in\mathbb{R}\setminus A$, we can divide inequality (8) by $\sin^2(t)/t^2$:

$$1-s\left(1-\frac{\alpha}{1-\beta}\frac{\sin\left(\frac{1+\beta}{\alpha}t\right)\sin\left(\frac{1-\beta}{\alpha}t\right)}{\sin^2(t)}\right)\geqslant 0. \quad (9)$$

Consider the function

$$G(t):=1-\frac{\alpha}{1-\beta}\frac{\sin\left(\frac{1+\beta}{\alpha}t\right)\sin\left(\frac{1-\beta}{\alpha}t\right)}{\sin^2(t)}=1-\frac{\alpha}{1-\beta}K_{\nu_1,\nu_2}(t),\ t\in\mathbb{R}\setminus A,$$

where

$$\nu_1=\nu_1(\alpha,\beta):=\frac{1+\beta}{\alpha},\ \nu_2=\nu_2(\alpha,\beta):=\frac{1-\beta}{\alpha}. \quad (10)$$

It is easily seen that $G(0)=(\alpha-1-\beta)/\alpha<0$. On the other hand, since $\nu_1\neq\nu_2$, from properties 3), 4) of $K_{\nu_1,\nu_2}$ (see Introduction) it follows that there is a $t_0\in\mathbb{R}\setminus A$ such that $K_{\nu_1,\nu_2}(t_0)<0$, and hence $G(t_0)>0$. Therefore, we can apply Lemma 1 to the function $G$ on $V:=\mathbb{R}\setminus A$. From Lemma 1 it follows that inequality (9) holds for all $t\in\mathbb{R}\setminus A$ if and only if

$$\frac{1}{\inf\limits_{\mathbb{R}\setminus A}G(t)}=\frac{1}{1-\frac{\alpha}{1-\beta}\sup\limits_{\mathbb{R}\setminus A}K_{\nu_1,\nu_2}(t)}\leqslant s\leqslant\frac{1}{\sup\limits_{\mathbb{R}\setminus A}G(t)}=\frac{1}{1-\frac{\alpha}{1-\beta}\inf\limits_{\mathbb{R}\setminus A}K_{\nu_1,\nu_2}(t)}.$$

Obviously,

$$M(\alpha,\beta)=\frac{1}{\sup\limits_{\mathbb{R}\setminus A}G(t)},\ m(\alpha,\beta)=\frac{1}{\inf\limits_{\mathbb{R}\setminus A}G(t)},$$

where $M(\alpha,\beta)$ and $m(\alpha,\beta)$ are defined by (1). Thus inequality (9) holds for every $t\in\mathbb{R}\setminus A\iff m(\alpha,\beta)\leqslant s\leqslant M(\alpha,\beta)$. We have proved that $E_{\alpha,\beta}=[m(\alpha,\beta),M(\alpha,\beta)]$. The main assertion of Theorem 2 is proved.

Let us prove *1)*. If $K_{\nu_1,\nu_2}$ is bounded below on $\mathbb{R}\setminus\pi\mathbb{Z}$, then $\mathfrak{m}_1(\nu_1,\nu_2)$ attains finite, negative value, and hence $M(\alpha,\beta)>0$. If $K_{\nu_1,\nu_2}$ is not bounded below, then $M(\alpha,\beta)=0$. By Proposition 1, $K_{\nu_1,\nu_2}$ is bounded below on $\mathbb{R}\setminus\pi\mathbb{Z}$ if and only if either $\nu_1,\nu_2\in\mathbb{N}$ or $|\nu_1-\nu_2|=2n$, $n\in\mathbb{Z}_+$, $\nu_1,\nu_2\notin\mathbb{N}$ (we may assume that $n\in\mathbb{N}$). Taking into account equation (10), we obtain

$$\nu_1,\nu_2\in\mathbb{N}\iff(1+\beta)/\alpha,(1-\beta)/\alpha\in\mathbb{N}.$$

If $|\nu_1-\nu_2|=2n$, $n\in\mathbb{N}$, then $\beta=n\alpha$. In this case $\nu_1=1/\alpha+n$, $\nu_2=1/\alpha-n$. The condition $\nu_1,\nu_2\notin\mathbb{N}$ is equivalent to $1/\alpha\notin\mathbb{N}$. Therefore,

$$|\nu_1-\nu_2|=2n,\ n\in\mathbb{N},\ \nu_1,\nu_2\notin\mathbb{N} \iff 1/\alpha\notin\mathbb{N},\ \beta/\alpha\in\mathbb{N}.$$

Let us prove *2)*. If $K_{\nu_1,\nu_2}$ is bounded above on $\mathbb{R}\setminus\pi\mathbb{Z}$, then $\mathfrak{m}_2(\nu_1,\nu_2)$ attains finite, positive value and we have

$$\mathfrak{m}_2(\nu_1,\nu_2)\geqslant K_{\nu_1,\nu_2}(0)=\nu_1\nu_2=\frac{1-\beta^2}{\alpha^2}.$$

It easily seen that $m(\alpha,\beta)<0$. If $K_{\nu_1,\nu_2}$ is not bounded above, then $m(\alpha,\beta)=0$. By Proposition 1, $K_{\nu_1,\nu_2}$ is bounded above on $\mathbb{R}\setminus\pi\mathbb{Z}$ if and only if either $\nu_1,\nu_2\in\mathbb{N}$ or $\nu_1+\nu_2=2n$, $n\in\mathbb{N}$, $\nu_1,\nu_2\notin\mathbb{N}$. If $\nu_1+\nu_2=2n$, $n\in\mathbb{N}$, then $1/\alpha=n$. In this case $\nu_1=n+n\beta=n+\beta/\alpha$, $\nu_2=n-n\beta=n-\beta/\alpha$. The condition $\nu_1,\nu_2\notin\mathbb{N}$ is equivalent to $\beta/\alpha\notin\mathbb{N}$. Therefore,

$$\nu_1+\nu_2=2n,\ n\in\mathbb{N},\ \nu_1,\nu_2\notin\mathbb{N} \iff 1/\alpha\in\mathbb{N},\ \beta/\alpha\notin\mathbb{N}.$$

Theorem 2 is proved. $\square$

## 4. Proof of Theorem 3

Let $\nu_1,\nu_2>0$ and $\mathfrak{m}_1(\nu_1,\nu_2)$, $\mathfrak{m}_2(\nu_1,\nu_2)$ be defined by (2).

Let us prove 1). Let $|\nu_1-\nu_2|=2$. Without loss of generality we may assume that $\nu_1>\nu_2$. In this case $\nu_1=\nu_2+2$ and

$$\frac{\sin((\nu_2+2)t)\sin(\nu_2 t)}{\sin^2(t)}=\frac{\cos(2t)-\cos(2(\nu_2+1)t)}{2\sin^2(t)}=\frac{-2\sin^2(t)+2\sin^2((\nu_2+1)t)}{2\sin^2(t)}$$
$$=\frac{\sin^2((\nu_2+1)t)}{\sin^2(t)}-1\geqslant-1.$$

Since $0<1/(\nu_2+1)<1$ we have $\sin(\pi/(\nu_2+1))\neq0$. Therefore, for $t=\pi/(\nu_2+1)$ the last inequality is an equality, and so $\mathfrak{m}_1(\nu_1,\nu_2)=-1$.

Let us prove 2). Let $\nu_1,\nu_2\in\mathbb{N}$. From the inequality $|\sin(mt)|\leqslant m|\sin t|$, which is satisfied for any $m\in\mathbb{N}$ and $t\in\mathbb{R}$ it follows that $|K_{\nu_1,\nu_2}(t)|\leqslant\nu_1\nu_2$. On the other hand, $K_{\nu_1,\nu_2}(0)=\nu_1\nu_2$. Therefore, $\mathfrak{m}_2(\nu_1,\nu_2)=\sup_{\mathbb{R}}K_{\nu_1,\nu_2}(t)=\nu_1\nu_2$ and $\mathfrak{m}_1(\nu_1,\nu_2)\geqslant-\nu_1\nu_2$. If, in addition, $\nu_1$ and $\nu_2$ have different parity, then $K_{\nu_1,\nu_2}(\pi)=-\nu_1\nu_2$ and $\mathfrak{m}_1(\nu_1,\nu_2)=\inf_{\mathbb{R}}K_{\nu_1,\nu_2}(t)=-\nu_1\nu_2$.

Let us prove 3). Let $\nu_1=p_1/q$, $\nu_2=p_2/q$, where $p_1,p_2,q\in\mathbb{N}$. Obviously, the infimum (the supremum) of $\sin(p_1t/q)\sin(p_2t/q)/\sin^2(t)$ over $\mathbb{R}\setminus\pi\mathbb{Z}$ is same as the infimum (the supremum) of $\sin(p_1t)\sin(p_2t)/\sin^2(qt)$ over $[0,\pi]\setminus(\pi/q)\mathbb{Z}$, and so Assertion 3) follows from formula:

$$U_p(\cos(t)):=\frac{\sin((p+1)t)}{\sin(t)},\ t\in[0,\pi],\ p\in\mathbb{Z}_+,$$

where $U_p$ is the Chebyshev polynomial of the second kind.

Assertion 4) follows from Assertion 3). $\square$

## 5. Examples

EXAMPLE 1. Let $\alpha \in (0,1)$, $\beta = \alpha$, $s \in \mathbb{R}$. Then $w_{\alpha,\alpha,s} \in \Phi(\mathbb{R}) \iff m(\alpha,\alpha) \leqslant s \leqslant 1-\alpha$, where $m(\alpha,\alpha) = -\alpha$, if $1/\alpha \in \mathbb{N}$ and $m(\alpha,\alpha) = 0$, if $1/\alpha \notin \mathbb{N}$.

*Proof.* From Assertion 1) of Theorem 2 it follows that $M(\alpha,\alpha) > 0$. From Assertion 2) of Theorem 2 it follows that $m(\alpha,\alpha) < 0$ if and only if $1/\alpha \in \mathbb{N}$.

It follows from Theorem 3 that $\mathfrak{m}_1((1+\beta)/\alpha,(1-\beta)/\alpha) = -1$ and $\mathfrak{m}_2((1+\beta)/\alpha,(1-\beta)/\alpha) = (1-\alpha^2)/\alpha^2$, if $1/\alpha \in \mathbb{N}$. Therefore, $M(\alpha,\alpha) = 1-\alpha$ and $m(\alpha,\alpha) = -\alpha$, if $1/\alpha \in \mathbb{N}$ and $m(\alpha,\alpha) = 0$, if $1/\alpha \notin \mathbb{N}$. $\square$

EXAMPLE 2. Let $\alpha \in (0,1/2)$, $\beta = 1-\alpha$, $s \in \mathbb{R}$. Then:

1) If $2/\alpha \notin \mathbb{N}$, then $w_{\alpha,1-\alpha,s} \in \Phi(\mathbb{R}) \iff s = 0$.

2) If $2/\alpha \in \mathbb{N}$, then

$$w_{\alpha,1-\alpha,s} \in \Phi(\mathbb{R}) \iff -\frac{\alpha}{2(1-\alpha)} \leqslant s \leqslant \frac{1}{1-\mathfrak{m}_1(2/\alpha-1,1)}.$$

Moreover, if $2/\alpha$ is odd, then $1/(1-\mathfrak{m}_1(2/\alpha-1,1)) = \alpha/2$.

*Proof.* From Assertions 1), 2) of Theorem 2 it follows that

$$M(\alpha,1-\alpha) > 0 \iff (2-\alpha)/\alpha \in \mathbb{N} \iff 2/\alpha \in \mathbb{N}$$

and

$$m(\alpha,1-\alpha) < 0 \iff (2-\alpha)/\alpha \in \mathbb{N} \iff 2/\alpha \in \mathbb{N}.$$

From Theorem 3 it follows that if $2/\alpha \in \mathbb{N}$, the $\mathfrak{m}_2((2-\alpha)/\alpha,1) = (2-\alpha)/\alpha$ and if $2/\alpha$ is even, then $\mathfrak{m}_1((2-\alpha)/\alpha,1) = -(2-\alpha)/\alpha$. $\square$

EXAMPLE 3. Let $\alpha = 2/(2m+2k+1)$, $\beta = (2k+1)/(2m+2k+1)$, where $m,k \in \mathbb{N}$. Then $w_{\alpha,\beta,s} \in \Phi(\mathbb{R}) \iff -1/(m+2k) \leqslant s \leqslant 1/(m+2k+2)$.

*Proof.* It is easily shown that $(1+\beta)/\alpha = m+2k+1$ and $(1-\beta)/\alpha = m$. Since the last two number have opposite parity, it follows from Theorem 2 and Theorem 3 that $M(\alpha,\beta) = 1/(m+2k+2)$ and $m(\alpha,\beta) = -1/(m+2k)$. $\square$

EXAMPLE 4. Let $\alpha = 1/2$, $\beta = 3/4$, $s \in \mathbb{R}$. Then $w_{\alpha,\beta,s} \in \Phi(\mathbb{R}) \iff s \in [-2/5,0]$.

## 6. Proof of Theorem 4

The following theorem (see, e.g., [6]) is needed to prove Theorem 4.

THEOREM 6. (Hausdorff–Bernstein–Widder) $f \in \mathscr{CM} \iff$

$$f(x) = \int_0^{+\infty} e^{-xt}\, d\mu(t),\ x > 0 \tag{11}$$

*where $\mu$ is a nonnegative Borel measure on $[0,+\infty)$ such that the integral (11) converges for all $x > 0$.*

Let us prove Theorem 4. From equation (3) we see that

$$(1/2)\widehat{w_{\alpha,\beta,s}}(t) = (1-s)\frac{1-\cos(\alpha t)}{\alpha t^2} - \frac{s}{1-\beta}\frac{1-\cos(\beta t)}{t^2} + \frac{s}{1-\beta}\frac{1-\cos(t)}{t^2},\ t \in \mathbb{R}.$$

It is obvious that $(1/2)\widehat{w_{\alpha,\beta,s}}(t) \geqslant 0,\ t \in \mathbb{R} \iff g(t) := (t^2/2)\widehat{w_{\alpha,\beta,s}}(t) \geqslant 0,\ t \in [0,+\infty)$. From Theorem 6 it follows that $g(t) \geqslant 0,\ t \in [0,+\infty) \iff \mathscr{L}[g] \in \mathscr{CM}$, where $\mathscr{L}[g]$ is the Laplace transform of $g$, i. e.

$$\mathscr{L}[g](x) := \int_0^{+\infty} g(t)e^{-xt}\, dt,\ x \in (0,+\infty).$$

It is easy to verify that

$$\mathscr{L}[g](x) = \frac{(1-s)\alpha}{x(x^2+\alpha^2)} - \frac{s\beta^2}{1-\beta}\frac{1}{x(x^2+\beta^2)} + \frac{s}{1-\beta}\frac{1}{x(x^2+1)},\ x > 0.$$

Taking into account conditions (4), we obtain $\mathscr{L}[g] = f^{\alpha,\beta}_{a,b,c}$, so the assertion of Theorem 4 follows from Theorem 2 and Corollary 1. Theorem 4 is proved. □

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