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G–DRAZIN INVERSES FOR OPERATOR MATRICES

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Abstract. Additive results for the generalized Drazin inverse of Banach space operators are presented. Suppose the bounded linear operators a and b on an arbitrary complex Banach space have generalized Drazin inverses. If $b^{\pi}aba = 0$ and $ab^2 = 0$, then $a+b$ has generalized Drazin inverse. This extends the main results of Djordjević and Wei (J. Austral. Math. J., **73**(2002), 115–125). Then we apply our results to 2×2 operator matrices and thereby generalize the results of Deng, Cvetković-Ilić and Wei (Linear and Multilinear Algebra, **58**(2010), 503–521).

1. Introduction

Let X be an arbitrary complex Banach space and $\mathscr{L}(X)$ denote the Banach algebra of all bounded operators on X. Set $\mathscr{A} = \mathscr{L}(X)$. The commutant of $a \in \mathscr{A}$ is defined by $comm(a) = \{x \in \mathscr{A} \mid xa = ax\}$. The double commutant of $a \in \mathscr{A}$ is defined by $comm^2(a) = \{x \in \mathscr{A} \mid xy = yx \text{ for all } y \in comm(a)\}$. An element a in $\mathscr{A}$ has g-Drazin inverse, i.e., a is GD-invertible if and only if there exists $b \in comm(a)$ such that $b = bab$ and $a - a^2b \in \mathscr{A}^{qnil}$. Such b, if exists, is unique, and is denoted by a^d. We call a^d the g-Drazin inverse of a. As is well known, a is GD-invertible if and only if it is quasipolar, i.e., there exists $e^2 = e \in comm^2(a)$ such that $a+e \in \mathscr{A}$ is invertible and $ae \in \mathscr{A}^{qnil}$. Here, $\mathscr{A}^{qnil} = \{a \in \mathscr{A} \mid 1 + ax \in U(\mathscr{A}) \text{ for every } x \in comm(a)\}$. As is well known,

$$a \in \mathscr{A}^{qnil} \Leftrightarrow \lim_{n\to\infty} \parallel a^n \parallel^{\frac{1}{n}} = 0.$$

Suppose the bounded linear operators a and b on an arbitrary complex Banach space have g-Drazin inverses. In Section 2, we prove that if $b^{\pi}aba = 0$ and $ab^2 = 0$ then $a+b$ has g-Drazin inverse. This extends the results of Djordjevic and Wei (see [7, Theorem 2.3]).

We next consider the g-Drazin inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{*}$$

where $A \in \mathscr{L}(X), D \in \mathscr{L}(Y)$ are GD-invertible and X, Y are complex Banach spaces. Here, M is a bounded operator on $X \oplus Y$. The g-Drazin inverses have various applications in singular differential and differential equations, Markov chains, and iterative

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methods (see [1, 2, 4, 14]). In Section 3, we present some g-Drazin inverses for a 2×2 operator matrix M under a number of different conditions, which generalize [15, Theorem 2.1 and Theorem 2.2].

If $a\in\mathscr{A}$ has g-Drazin inverse a^d. The element $p=1-aa^d$ is called the spectral idempotent of a. In Section 4, we further consider the g-Drazin inverse of a 2×2 operator matrix M under the conditions on spectral idempotents. These also extends [5, Theorem 6 and Theorem 7] to wider cases.

Throughout the paper, X is a Banach space and $\mathscr{A}=\mathscr{L}(X)$. We use $U(\mathscr{A})$ to denote the set of all units in $\mathscr{A}$. $\mathscr{A}^d$ indicates the set of all GD-invertible elements in $\mathscr{A}$. $\mathbb{N}$ stands for the set of all natural numbers.

2. Additive results

The purpose of this section is to establish the generalized Drazin inverse of $P+Q$ in the case $PQP=0$ and $PQ^2=0$. The explicit formula for the generalized Drazin inverse of $P+Q$ is illustrated as well. We start by

LEMMA 2.1. *Let $a,b\in\mathscr{A}$ and $ab=0$. If $a,b\in\mathscr{A}^d$, then $a+b\in\mathscr{A}^d$.*

Proof. See [7, Theorem 2.3].

LEMMA 2.2. *Let $a\in\mathscr{A}$ and $n\in\mathbb{N}$. Then $a^n\in\mathscr{A}^d$ if and only if $a\in\mathscr{A}^d$.*

Proof. See [10, Theorem 2.7].

Let $x\in\mathscr{A}$. Then we have Pierce decomposition

$$x=pxp+px(1-p)+(1-p)xp+(1-p)x(1-p).$$

For further use, we induce a representation given by the matrix

$$x=\begin{pmatrix} pxp & px(1-p)\\ (1-p)xp & (1-p)x(1-p)\end{pmatrix}_p.$$

We have

LEMMA 2.3. *Let $\mathscr{A}$ be a Banach algebra, let $a\in\mathscr{A}$ and let*

$$x=\begin{pmatrix} a & c\\ 0 & b\end{pmatrix}_p,$$

relative to $p^2=p\in\mathscr{A}$. If $a\in(p\mathscr{A}p)^d$ and $b\in((1-p)\mathscr{A}(1-p))^d$, then $x\in\mathscr{A}^d$.

Proof. See [3, Theorem 2.3].

We are now ready to extend [15, Theorem 2.1 and Theorem 2.2] and prove:

THEOREM 2.4. *Let $a,b\in \mathscr{A}^d$. If $b^{\pi}aba=0$ and $ab^2=0$, then $a+b\in \mathscr{A}^d$.*

Proof. Let $p=bb^d$. Then we have

$$a=\begin{pmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{pmatrix}_p,\ b=\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p,$$

where $b_2=(1-bb^d)b(1-bb^d)=b-b^2b^d\in \mathscr{A}^{qnil}$. Since $ab^2=0$, we see that $ap=(ab^2)(b^d)^2=0$, and so $a_{11}=a_{21}=0$. Hence, we have

$$a+b=\begin{pmatrix} b_1 & a_1 \\ 0 & a_2+b_2 \end{pmatrix}_p.$$

Since $b_1=pbp=b(bb^d)$ and $bb^d=b^db$, we easily see that $b_1\in (p\mathscr{A}p)^d$. By using Cline's formula, we have $a_2=b^{\pi}ab^{\pi}\in \mathscr{A}^d$. By hypothesis, we check that

$$a_2b_2a_2=0, a_2b_2^2=0.$$

We will suffice to prove $a_2+b_2\in (1-p)\mathscr{A}(1-p))^d$.

Set

$$M=\begin{pmatrix} a_2^2+a_2b_2 & a_2^2b_2 \\ a_2+b_2 & a_2b_2+b_2^2 \end{pmatrix}.$$

Then

$$M=\begin{pmatrix} a_2b_2 & a_2^2b_2 \\ 0 & a_2b_2 \end{pmatrix}+\begin{pmatrix} a_2^2 & 0 \\ a_2+b_2 & b_2^2 \end{pmatrix}:=G+F.$$

We see that $G^2=0$ and $GF=0$.

$$F=\begin{pmatrix} a_2^2 & 0 \\ a_2+b_2 & b_2^2 \end{pmatrix}=\begin{pmatrix} a_2^2 & 0 \\ a_2 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ b_2 & b_2^2 \end{pmatrix}:=H+K.$$

One easily check that

$$H=\begin{pmatrix} a_2^2 & 0 \\ a_2 & 0 \end{pmatrix}=\begin{pmatrix} a_2 \\ 1 \end{pmatrix}(a_2,0).$$

Since $(a_2,0)\begin{pmatrix} a_2 \\ 1 \end{pmatrix}=a_2^2\in \mathscr{A}^d$, it follows by Cline's formula, we see that

$$\begin{aligned} H^d &= \begin{pmatrix} a_2 \\ 1 \end{pmatrix}((a_2^2)^d)^2(a_2,0)=\begin{pmatrix} a_2 \\ 1 \end{pmatrix}(a_2^d)^4(a_2,0) \\ &= \begin{pmatrix} a_2(a_2^d)^4a_2 & 0 \\ (a_2^d)^4a_2 & 0 \end{pmatrix}=\begin{pmatrix} (a_2^d)^2 & 0 \\ (a_2^d)^3 & 0 \end{pmatrix}. \end{aligned}$$

Likewise, we have

$$K^d=\begin{pmatrix} 0 \\ b_2 \end{pmatrix}(b_2^d)^4(1,b_2)=\begin{pmatrix} 0 & 0 \\ (b_2^d)^3 & (b_2^d)^2 \end{pmatrix}.$$

Clearly, $HK = 0$. In light of Lemma 2.1,

$$F^d = (I - KK^d)\left[\sum_{n=0}^{\infty} K^n (H^d)^n\right] H^d + K^d \left[\sum_{n=0}^{\infty} (K^d)^n H^n\right](I - HH^d).$$

In light of [15, Theorem 2.1], we see that

$$M^d = F^d + G(F^d)^2.$$

Clearly, $M = \left(\begin{pmatrix} a_2 \\ 1 \end{pmatrix}(1, b_2)\right)^2$. By virtue of Lemma 2.1,

$$(a_2 + b_2)^d = \left((1, b_2)\begin{pmatrix} a_2 \\ 1 \end{pmatrix}\right)^d = (1, b_2) M^d \begin{pmatrix} a_2 \\ 1 \end{pmatrix}$$

as asserted.

As an immediate consequence, we can derive the following which was given in [9, Lemma 5].

COROLLARY 2.5. *Let $a, b \in \mathscr{A}^{qnil}$. If $aba = 0$ and $ab^2 = 0$, then $a + b \in \mathscr{A}^{qnil}$.*

Proof. Since $a, b \in \mathscr{A}^{qnil}$, we see that $a^d = b^d = 0$. In light of Theorem 2.4, $(a+b)^d = 0$, and therefore $a + b \in \mathscr{A}^{qnil}$, as required.

Let $P, Q \in \mathscr{A}(X)^d$. In [8, Theorem 4.2.2], Guo proved that $P + Q \in \mathscr{A}(X)^d$ if $PQP = 0$ and $Q^2P = 0$ by a different route, and so it is worth noting the following examples.

EXAMPLE 2.6. Let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Then $aba = 0$ and $ab^2 = 0$, while $b^2a \neq 0$.

EXAMPLE 2.7. Let $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Then $aba = 0$ and $b^2a = 0$, while $ab^2 \neq 0$.

3. Splitting approach

Let $A \in \mathscr{L}(X), D \in \mathscr{L}(Y)$ be GD-invertible and M be given by $(*)$. The aim of this section is to consider a GD-invertible 2×2 operator matrix M. Using different splitting of the operator matrix M as $M = P + Q$, we will apply Theorem 2.4 to obtain various conditions for a GD-invertible M, which extend [15, Theorem 2.1 and Theorem 2.2].

THEOREM 3.1. *If $BCA = 0$, $BCB = 0$, $DCA = 0$ and $DCB = 0$, then M is GD-invertible.*

Proof. We easily see that

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}=p+q,$$

where

$$p=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix},\ q=\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

By virtue of [7, Lemma 2.2] p and q are GD-invertible. Obviously, $pq^2=0, pqp=0$ and then we complete the proof by Theorem 2.4.

COROLLARY 3.2. *If $BC=0$ and $DC=0$, then M is GD-invertible.*

Proof. If $BC=0$ then $BCA=0$ and $BCB=0$. If $DC=0$, then $DCA=0$ and $DCB=0$. So we get the result by Theorem 3.1.

COROLLARY 3.3. *If $CA=0$ and $CB=0$, then M is GD-invertible.*

Proof. If $CA=0$ then $BCA=0$ and $DCA=0$. If $CB=0$, then $DCB=0$ and $BCB=0$. So we get the result by Theorem 3.1.

THEOREM 3.4. *If $ABC=0$, $ABD=0$, $CBC=0$, $CBD=0$, then M is GD-invertible.*

Proof. Clearly, we have

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}=p+q,$$

where

$$p=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix},\ q=\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.4, we complete the proof as in Theorem 3.1.

COROLLARY 3.5. (1) *If $BC=0$ and $BD=0$, then M is GD-invertible.*

(2) *If $AB=0$ and $CB=0$, then M is GD-invertible.*

EXAMPLE 3.6. Let A,B,C,D be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:

$$\begin{aligned} A(x_1,x_2,x_3,x_4,\cdots) &= (x_1,x_1,x_3,x_4,\cdots), \\ B(x_1,x_2,x_3,x_4,\cdots) &= (x_1,-x_1,x_3,x_4,\cdots), \\ C(x_1,x_2,x_3,x_4,\cdots) &= (x_1+x_2,x_1-x_2,0,0,\cdots), \\ D(x_1,x_2,x_3,x_4,\cdots) &= (-x_2,x_2,0,0,\cdots). \end{aligned}$$

Set $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $BCA=0, BCB=0, DCA=0$ and $DCB=0$. By virtue of Theorem 3.4, M is GD-invertible.

It is convenient this stage to include the following spiliting theorem.

THEOREM 3.7. *If* $BCA = 0, BCB = 0, BDC = 0$ *and* $BD^2 = 0$, *then* M *is GD-invertible.*

Proof. Let

$$p = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},\ q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$

Then $M = p+q$. In view of [7, Lemma 2.2,] p and q are GD-invertible. By hypothesis, we easily verify that $pqp = 0$ and $pq^2 = 0$. This completes the proof, by Theorem 2.4.

4. Spectral conditions

The goal of this section is to consider another splitting of the block matrix M and present alternative theorems on spectral idempotents. Let $A \in \mathscr{L}(X), D \in \mathscr{L}(Y)$ be GD-invertible and M be given by $(*)$. We derive

THEOREM 4.1. (1) *If* $BCA = 0, BCB = 0, D^dC = 0$ *and* $BDD^\pi = 0$, *then* M *is GD-invertible.*

(2) *If* $CBC = 0, CBD = 0, CA^d = 0$ *and* $AA^\pi B = 0$, *then* M *is GD-invertible.*

Proof.

(1) Let

$$P = \begin{pmatrix} A^2A^d & B \\ 0 & D^2D^d \end{pmatrix},\ Q = \begin{pmatrix} AA^\pi & 0 \\ C & DD^\pi \end{pmatrix}.$$

Then P and Q are GD-invertible by Theorem 3.1. We compute that

$$PQP = \begin{pmatrix} BCA^2A^d & BCB \\ D^2D^dCA^2A^d & D^2D^dCB \end{pmatrix},$$

$$PQ^2 = \begin{pmatrix} BCAA^\pi & BD^2D^\pi \\ D^2D^dCAA^\pi & 0 \end{pmatrix}.$$

By hypothesis, $PQP = 0$ and $PQ^2 = 0$. In light of Lemma 2.1, $M = P+Q$ is GD- invertible.

(2) Choosing the same P and Q as in (1) we have $QPQ = 0$ and $QP^2 = 0$. As P and Q are GD- invertible, we complete the proof by Theorem 2.4.

COROLLARY 4.2. ([5, Theorem 6])

(1) *If* $BC = 0$, $D^dC = 0$ *and* $BDD^\pi = 0$, *then* M *is GD-invertible.*

(2) *If* $CB = 0$, $CA^d = 0$ *and* $AA^\pi B = 0$, *then* M *is GD-invertible.*

THEOREM 4.3. *If* $BCA = 0$, $BCB = 0$, $BD^d = 0$ *and* $D^\pi DC = 0$, *then* M *is GD-invertible.*

Proof. Let

$$P = \begin{pmatrix} A(I-A^\pi) & B \\ 0 & DD^\pi \end{pmatrix}, \quad Q = \begin{pmatrix} AA^\pi & 0 \\ C & D(I-D^\pi) \end{pmatrix}.$$

In view of Theorem 3.1, P and Q are GD- invertible. Moreover, we have

$$PQP = \begin{pmatrix} BCA(I-A^\pi) & BCB \\ DD^\pi CA(I-A^\pi) & DD^\pi CB \end{pmatrix},$$

$$PQ^2 = \begin{pmatrix} BCAA^\pi + BD(I-D^\pi)C & BD^2(I-D^\pi) \\ DD^\pi CAA^\pi & 0 \end{pmatrix}.$$

By hypothesis, we get $PQP = 0$ and $PQ^2 = 0$. According to Theorem 2.4, $M = P+Q$ is G-Drazin invertible, as asserted.

COROLLARY 4.4. ([5, Theorem 7]) *If* $BC = 0, BD^d = 0$ *and* $D^\pi DC = 0$, *then* M *is GD-invertible.*

EXAMPLE 4.5. Let A,B,C,D be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:

$$\begin{aligned} A(x_1,x_2,x_3,x_4,\cdots) &= (0,x_1,x_3,x_4,\cdots), \\ B(x_1,x_2,x_3,x_4,\cdots) &= (0,x_1,x_3,x_4,\cdots), \\ C(x_1,x_2,x_3,x_4,\cdots) &= (x_1,x_1+x_2,0,0,\cdots), \\ D(x_1,x_2,x_3,x_4,\cdots) &= (x_1,x_1,0,0,\cdots). \end{aligned}$$

Then

$$D^\pi(x_1,x_2,x_3,x_4,\cdots) = (0,-x_1+x_2,x_3,x_4,\cdots).$$

Moreover, we have $BCA = 0, BCB = 0, BD(I-D^\pi) = 0$ and $D^\pi DC = 0$. In light of Theorem 4.3, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is GD-invertible. In this case, $BC \neq 0$.

THEOREM 4.6. *Let* $A \in \mathscr{L}(X)$ *be GD-invertible,* $D \in \mathscr{L}(Y)$ *and* M *be given by* $(*)$. *Let* $W = AA^d + A^dBCA^d$. *If* AW *is GD-invertible,*

$$ACA^\pi BC = 0, CA^\pi BC = 0, D = CA^dB,$$

then M *is GD-invertible.*

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P+Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, \; Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, we easily check that $PQP = 0$ and $PQ^2 = 0$ and Q is GD-invertible. Moreover, we have

$$P = P_1 + P_2, \; P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \; P_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}$$

and $P_2P_1 = 0$. In light of [7, Lemma 2.2,] P_2 is GD-invertible. It is easy to verify that

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A, AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = AW$$

is GD-invertible. In light of the Cline's formula, we see that P_1 is GD-invertible. According to [7, Theorem 2.3] P is GD-invertible. This completes the proof.

Similarly with application of Theorem 2.4 we deduce the following result.

THEOREM 4.7. *Let $A \in \mathscr{L}(X)$ be GD-invertible, $D \in \mathscr{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW is GD-invertible,*

$$BCAA^\pi = 0, \; BCA^\pi B = 0, \; D = CA^dB,$$

then M is GD-invertible.

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