Operators and Matrices
Volume 14, Number 1 (2020), 33–69 doi:10.7153/oam-2020-14-04

ON COUPLINGS OF SYMMETRIC OPERATORS WITH POSSIBLY UNEQUAL AND INFINITE DEFICIENCY INDICES

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(*Communicated by J. Behrndt*)

Abstract. In the paper the known results on couplings of symmetric operators A_j, $j\in\{1,2\}$, in the sense of A.V. Shtraus are extended to the case of operators A_j with arbitrary (possibly unequal and infinite) deficiency indices. In particular, we generalize to this case the coupling method based on the theory of boundary triplets for symmetric operators. This enables us to obtain the abstract Titchmarsh formula, which gives the representation of the Weyl function of the coupling in terms of Weyl functions of boundary triplets for A_1^* and A_2^*. In applications to differential operators on $\mathbb{R}$ this formula turns into the classical Titchmarsh formula, which gives a representation of the characteristic matrix $\Omega(\cdot)$ in terms of Titchmarsh-Weyl functions on semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$. Moreover, by using the coupling method we parameterize all Naimark exit space extensions $\widetilde{A}=\widetilde{A}^*$ of the second kind of a densely defined symmetric operator A with finite possibly unequal deficiency indices.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space and let $\widetilde{\mathscr{C}}(\mathfrak{H})$ be the set of all linear relations in $\mathfrak{H}$, i.e., the set of all closed subspaces in $\mathfrak{H}^2$. A relation $A\in\widetilde{\mathscr{C}}(\mathfrak{H})$ is called symmetric (self-adjoint) if $A\subset A^*$ (resp. $A=A^*$), where $A^*\in\widetilde{\mathscr{C}}(\mathfrak{H})$ is the adjoint relation to A. Identifying of a symmetric not necessarily densely defined (self-adjoint) operator A with its graph enables one to consider A as a symmetric (resp. self-adjoint) linear relation.

Let $\mathfrak{H}_j$ be a Hilbert space, let A_j be a symmetric relation in $\mathfrak{H}_j$, let $\widetilde{\mathfrak{H}}:=\mathfrak{H}_1\oplus\mathfrak{H}_2$ (so that $\widetilde{\mathfrak{H}}^2=\mathfrak{H}_1^2\oplus\mathfrak{H}_2^2$) and let $\widehat{P}_j$ be the orthoprojector in $\widetilde{\mathfrak{H}}^2$ onto $\mathfrak{H}_j^2$, $j\in\{1,2\}$. Recall [42, 11, 13] that a relation $\widetilde{A}=\widetilde{A}^*$ in $\widetilde{\mathfrak{H}}$ is called a coupling of relations A_j if $\widehat{P}_j\widetilde{A}=A_j^*$, $j\in\{1,2\}$. If $\widetilde{A}$ is a coupling of A_1 and A_2, then $A_j=\widetilde{A}\cap\mathfrak{H}_j^2$, $j\in\{1,2\}$.

In the paper by A.V. Shtraus [42] all couplings $\widetilde{A}$ of densely defined operators A_j with finite deficiency indices $n_\pm(A_j)$ are characterized in terms of boundary operators for A_j^*. In [11, 13] the coupling method for symmetric relations A with equal deficiency indices $n_+(A)=n_-(A)\leqslant\infty$ has been developed on a basis of the theory of boundary triplets. Recall that according to [19, 8] a collection $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ formed by a Hilbert space $\mathscr{H}$ and two linear mappings Γ_0 and Γ_1 from A^* to $\mathscr{H}$ is called a

Mathematics subject classification (2010): 34B20 47A06, 47A20,47A48, 47A56, 47B25.

Keywords and phrases: Symmetric relation (operator), boundary triplet, Weyl function, coupling of symmetric relations, Titchmarsh formula, exit space extension, generalized resolvent.

boundary triplet for A^* if the mapping $\Gamma=(\Gamma_0,\Gamma_1)^\top$ from A^* to $\mathscr{H}^2$ is surjective and the following abstract Green's identity holds:

$$(f',g)-(f,g')=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})_{\mathscr{H}}-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})_{\mathscr{H}},\quad \widehat{f}=\{f,f'\},\ \widehat{g}=\{g,g'\}\in A^*. \tag{1.1}$$

In the following theorem from [11, 13] a coupling construction in terms of boundary triplets is presented.

THEOREM 1.1. *Let A and A_r be symmetric relations in $\mathfrak{H}$ and $\mathfrak{H}_r$ respectively with equal deficiency indices $n_\pm(A)=n_\pm(A_r)=d\leqslant\infty$ and let $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ and $\Pi_r=\{\mathscr{H},\Gamma_0^r,\Gamma_1^r\}$ be boundary triplets for A^* and A_r^* respectively. Then the linear relation*

$$\widetilde{A}=\{\widehat{f}\oplus\widehat{f}_r\in A^*\oplus A_r^*:\Gamma_0\widehat{f}-\Gamma_0^r\widehat{f}_r=\Gamma_1\widehat{f}-\Gamma_1^r\widehat{f}_r=0\} \tag{1.2}$$

is a coupling of A and A_r. Conversely, let A be a symmetric relation in $\mathfrak{H}$ with $n_+(A)-n_-(A)$, let $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*, let $\widetilde{A}\supset A$ be a self-adjoint relation in the Hilbert space $\widetilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}_r$ satisfying

$$A=\widetilde{A}\cap\mathfrak{H}^2,\qquad A^*=\widehat{P}\widetilde{A} \tag{1.3}$$

($\widehat{P}$ is the orthoprojector in $\widetilde{\mathfrak{H}}^2$ onto $\mathfrak{H}^2$) and let $A_r=\widetilde{A}\cap\mathfrak{H}_r^2$. Then there exists a unique boundary triplet $\Pi_r=\{\mathscr{H},\Gamma_0^r,\Gamma_1^r\}$ for A_r^* such that $\widetilde{A}$ is of the form (1.2) (hence $\widetilde{A}$ is a coupling of A and A_r).

Let $B(\mathscr{H})$ be the set of all bounded operators in $\mathscr{H}$. According to [14, 30] with a boundary triplet $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ for A^* one associates the operator function $M(\cdot):\mathbb{C}\setminus\mathbb{R}\to B(\mathscr{H})$ (the Weyl function of Π) defined by

$$\Gamma_1\{f_\lambda,\lambda f_\lambda\}=M(\lambda)\Gamma_0\{f_\lambda,\lambda f_\lambda\},\quad f_\lambda\in\mathfrak{N}_\lambda(A):=\ker(A^*-\lambda),\ \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

It turns out that $M(\cdot)$ belongs to the class $R_u[\mathscr{H}]$ of uniformly strict Nevanlinna operator-functions. The latter means that $\operatorname{Im}\lambda\cdot\operatorname{Im}M(\lambda)\geqslant\alpha_\lambda I$ with some $\alpha_\lambda>0$ and $M^*(\lambda)=M(\overline{\lambda})$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

An important ingredient in the theory of boundary triplets is a self-adjoint extension $A_0:=\ker\Gamma_0$ of A associated with a boundary triplet $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ for A^*. Let $M(\cdot)$ and $M_r(\cdot)$ be the Weyl functions of boundary triplets Π and Π_r from Theorem 1.1. Then according to [11, 13] there exists a boundary triplet $\Pi_c=\{\mathscr{H}_c,\Gamma_0^c,\Gamma_1^c\}$ for $A^*\oplus A_r^*$ such that $\mathscr{H}_c=\mathscr{H}\oplus\mathscr{H}$, $A_0^c(=\ker\Gamma_0^c)$ coincides with the coupling $\widetilde{A}$ of A and A_r (see (1.2)) and the Weyl function $M_c(\lambda)(\in B(\mathscr{H}\oplus\mathscr{H}))$ of Π_c is

$$M_c(\lambda)=\begin{pmatrix}-(M(\lambda)+M_r(\lambda))^{-1} & I_{\mathscr{H}}-(M(\lambda)+M_r(\lambda))^{-1}M(\lambda)\\ I_{\mathscr{H}}-M(\lambda)(M(\lambda)+M_r(\lambda))^{-1} & (M^{-1}(\lambda)+M_r^{-1}(\lambda))^{-1}\end{pmatrix},\ \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{1.4}$$

The Weyl function $M_c(\cdot)$ turns out to be a very important and useful object in extension theory and its applications. To illustrate this assertion consider the Sturm -

Liouville expression $l[y]=-y''+q(x)y$ with the real potential $q(x)$ defined on $\mathbb{R}$. Let $T_{\min}$ and $T_{\max}$ be the corresponding minimal an maximal operator in $L^2(\mathbb{R})$ and let $T_{\min}^-$ and $T_{\max}^-$ ($T_{\min}^+$ and $T_{\max}^+$) be minimal and maximal operators in $L^2(\mathbb{R}_-)$ (resp. $L^2(\mathbb{R}_+)$) generated by the restriction of $l[y]$ onto $\mathbb{R}_-=(-\infty,0]$ (resp. $\mathbb{R}_+=[0,\infty)$). Assume that $l[y]$ is in the limit point case at $-\infty$ and ∞ (this means that $n_\pm(T_{\min}^-)=n_\pm(T_{\min}^+)=1$). Then there exist boundary triplets Π for $T_{\max}^-$ and Π_r for $T_{\max}^+$ such that the Weyl functions $M(\cdot)$ of Π and $M_r(\cdot)$ of Π_r coincide with the classical Titchmarsh -Weyl functions for $l[y]$ on $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively [16]. Moreover, the coupling (1.2) of $T_{\min}^-$ and $T_{\min}^+$ is $T_{\max}=T_{\max}^*(=T_{\min})$ and formula (1.4) for $M_c(\cdot)$ turns into the classical Titchmarsh formula for the characteristic function $\Omega(\lambda)=M_c(\lambda)$ of the Sturm - Liouville operator on $\mathbb{R}$ [44]. This fact enables one to consider (1.4) as an abstract Titchmarsh formula for the coupling of symmetric relations with equal deficiency indices.

Recall that a self-adjoint relation $\widetilde{A}\supset A$ in a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ is called an exit space extension of A. In the case $n_+(A)=n_-(A)$ the Krein formula for generalized resolvents

$$R_\tau(\lambda):=P_{\mathfrak{H}}(\widetilde{A}_\tau-\lambda)^{-1}\upharpoonright\mathfrak{H}=(A_0-\lambda)^{-1}-\gamma(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\overline{\lambda}),\quad \lambda\in\mathbb{C}_+ \tag{1.5}$$

gives a parametrization $\widetilde{A}=\widetilde{A}_\tau$ of all exit space extensions $\widetilde{A}=\widetilde{A}^*$ of A by means of all Nevanlinna functions $\tau(\cdot):\mathbb{C}_+\to\widetilde{\mathscr{C}}(\mathscr{H})$, which are holomorphic functions with values in the set of all maximal dissipative linear relations in $\mathscr{H}$ [26, 28]. In terms of a boundary triplet $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ for A^* elements of (1.5) are defined as follows [14, 30]: $A_0(=A_0^*)=\ker\Gamma_0$, $\gamma(\cdot):\mathbb{C}_+\to B(\mathscr{H},\mathfrak{H})$ is a unique operator-function such that $\Gamma_0(\gamma(\lambda),\lambda\gamma(\lambda))^\top=I_{\mathscr{H}}$ (the γ-field) and $M(\cdot)$ is the Weyl function of Π. Moreover, in [11, 13] the proof of the Krein formula (1.5) using the coupling construction (1.2) is presented. For extensions $\widetilde{A}$ satisfying (1.3) this proof is based on the following basic realization result: for every function $\tau\in R_u[\mathscr{H}]$ there exists a symmetric relation A_r in $\mathfrak{H}_r$ and a boundary triplet $\Pi_r=\{\mathscr{H},\Gamma_0^r,\Gamma_1^r\}$ for A_r^* such that the Weyl function of Π_r is $\tau(\cdot)$ [28, 15]. This result, Theorem 1.1 and Titchmarsh formula (1.4) with $M_r(\lambda)=\tau(\lambda)$ enabled to show in [11, 13] that an exit space extension $\widetilde{A}=\widetilde{A}_\tau$ satisfies (1.3) if and only if $\tau(\cdot)\in R_u[\mathscr{H}]$. Moreover, by using a certain modification of the above results the authors of [11, 13] parameterized all extensions $\widetilde{A}_\tau$ which are operators (when A is an operator) and all extensions $\widetilde{A}_\tau$ of the second kind in the sense of Naimark. Note also that the coupling construction (1.2) and formula for canonical resolvents related to it were used in the recent paper [10] for studying of the compressions of exit space extensions; moreover, a certain modification of such a construction was used in [4, 6, 39] for studying of multidimensional Schrodinger operators.

As is known [30] each boundary triplet $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ for A^* satisfies $n_+(A)=n_-(A)=\dim\mathscr{H}$. Therefore theory of boundary triplets is not applicable to couplings of relations with unequal deficiency indices. At the same time in applications couplings of relations (operators) A_1 and A_2 with unequal deficiency indices $n_+(A_j)\neq n_-(A_j)$ naturally appear. Consider for instance the differential expression $l[y]=-iy^{(3)}$ of the third order on $\mathbb{R}$. Let $T_{\max}$ be the corresponding maximal operator in $L^2(\mathbb{R})$ and let $T_{\min}^\pm$ and $T_{\max}^\pm$ be minimal and maximal operators in $L^2(\mathbb{R}_\pm)$ generated by restrictions of $l[y]$ on $\mathbb{R}_{\pm}$. Then $n_+(T^+_{\min}) = n_-(T^-_{\min}) = 1$, $n_-(T^+_{\min}) = n_+(T^-_{\min}) = 2$ and $T_{\max} = T^*_{\max}$ is the coupling of $T^+_{\min}$ and $T^-_{\min}$.

In [34] a new construction of a boundary triplet with two Hilbert space $\mathcal{H}_0$ and $\mathcal{H}_1$ was presented. Namely, assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ and P_j is the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1,2\}$. Then according to [34] a collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ with operators $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0,1\}$, is called a boundary triplet for A^* if the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective and the following Green's identity holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in A^*$:

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}_0} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}_0} + i(P_2 \Gamma_0 \widehat{f}, P_2 \Gamma_0 \widehat{g})_{\mathcal{H}_2} \tag{1.6}$$

If $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, then

$$\dim \mathcal{H}_1 = n_-(A) \leqslant n_+(A) = \dim \mathcal{H}_0 \tag{1.7}$$

and hence it is applicable to relations A with possibly unequal deficiency indices $n_{\pm}(A)$. By using this fact we extend in the present paper the above results from [42, 11, 13] to couplings of symmetric linear relations with arbitrary (possibly unequal and infinite) deficiency indices.

According to [34] with a boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* one associates two Weyl functions $M_+(\cdot) : \mathbb{C}_+ \to B(\mathcal{H}_0, \mathcal{H}_1)$ and $M_-(\cdot) : \mathbb{C}_- \to B(\mathcal{H}_1, \mathcal{H}_0)$ given by

$$\begin{gathered} \Gamma_1\{f_\lambda, \lambda f_\lambda\} = M_+(\lambda)\Gamma_0\{f_\lambda, \lambda f_\lambda\}, \quad f_\lambda \in \mathfrak{N}_\lambda(A), \ \lambda \in \mathbb{C}_+, \\ (\Gamma_1 + iP_2\Gamma_0)\{f_\lambda, \lambda f_\lambda\} = M_-(\lambda)P_1\Gamma_0\{f_\lambda, \lambda f_\lambda\}, \quad f_\lambda \in \mathfrak{N}_\lambda(A), \quad \lambda \in \mathbb{C}_-. \end{gathered}$$

Assume that the block representations of $M_{\pm}(\lambda)$ are

$$M_+(\lambda) = (M(\lambda), N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+, \tag{1.8}$$

$$M_-(\lambda) = (M(\lambda), N_-(\lambda))^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_-. \tag{1.9}$$

Then according to [34, 36]the equalities

$$\mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & N_+(\lambda) \\ 0 & \frac{i}{2} I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+ \tag{1.10}$$

$$\mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ N_-(\lambda) & -\frac{i}{2} I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_- \tag{1.11}$$

define the operator-function $\mathcal{M}(\cdot) \in R_u[\mathcal{H}_0]$ with $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. In the present paper we prove the following inverse theorem.

THEOREM 1.2. *Let (M_+, M_-) be a pair of holomorphic operator-functions $M_+(\cdot) : \mathbb{C}_+ \to B(\mathcal{H}_0, \mathcal{H}_1)$ and $M_-(\cdot) : \mathbb{C}_- \to B(\mathcal{H}_1, \mathcal{H}_0)$ with the block representations* (1.8), (1.9) *such that the operator function $\mathcal{M}(\cdot)$ defined by* (1.10) *and* (1.11) *belongs to $R_u[\mathcal{H}_0]$. Then there exist a symmetric relation A in $\mathfrak{H}$ and a boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* such that $M_+(\cdot)$ and $M_-(\cdot)$ are the Weyl functions of* Π.

By using Theorem 1.2 we show that a simple symmetric operator A with $n_+(A) = n_-(A)$ admits the representation $A = A_1 \oplus A_2$ with the maximal symmetric operator A_2 if and only if there is a characteristic function $C(\lambda)$ of A (in the sense of [43, 22, 30]) admitting the block representation $C(\lambda) = \begin{pmatrix} C_1(\lambda) & C_2(\lambda) \\ 0 & C_3 \end{pmatrix}$ with the constant entry C_3. This statement covers the following known result [2, 27]: a densely defined simple symmetric operator A with $n_+(A) = n_-(A)$ admits the representation $A = A_1 \oplus A_2$ with the maximal symmetric operators A_1 and A_2 if and only if some (and hence any) characteristic function of A is constant.

Next we prove the main coupling theorem concerning the coupling of symmetric relations A in $\mathfrak{H}$ and A_r in $\mathfrak{H}_r$ with possibly unequal deficiency indices satisfying $n_\pm(A) = n_\pm(-A_r)$. Namely, we show that in this case Theorem 1.1 remains valid with boundary triplets $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^*, $\Pi_r = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0^r, \Gamma_1^r\}$ for $(-A_r)^*$ and the coupling $\widetilde{A}$ given by

$$\widetilde{A} = \{\widehat{f} \oplus \widehat{f}_r \in A^* \oplus A_r^* : \Gamma_0 \widehat{f} - \Gamma_0^r J_r \widehat{f}_r = \Gamma_1 \widehat{f} - \Gamma_1^r J_r \widehat{f}_r = 0\} \tag{1.12}$$

instead of (1.2) (in (1.12) $J_r\{f, f'\} := \{f, -f'\}$, $\{f, f'\} \in \mathfrak{H}^2$, so that $J_r A_r^* = -A_r^*$). Moreover, we show that there exists a boundary triplet $\Pi_c = \{\mathcal{H}_c, \Gamma_0^c, \Gamma_1^c\}$ for $A^* \oplus A_r^*$ (the coupling of boundary triplets Π and Π_r) such that $\mathcal{H}_c = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_1$, $\ker \Gamma_0 = \widetilde{A}$ and the Weyl function $M_c(\lambda)$ $(\lambda \in \mathbb{C}_+)$ of Π_c is

$$M_c(\lambda) = \begin{pmatrix} \Phi(\lambda) & \Phi(\lambda) N_+(\lambda) & I_{\mathcal{H}_1} + \Phi(\lambda) M(\lambda) \\ N_{r-}(-\lambda)\Phi(\lambda) & \frac{i}{2}\mathcal{I}_{\mathcal{H}_2} + N_{r-}(-\lambda)\Phi(\lambda)N_+(\lambda) & N_{r-}(-\lambda)\Phi(\lambda)M(\lambda) \\ M_r(-\lambda)\Phi(\lambda) & M_r(-\lambda)\Phi(\lambda)N_+(\lambda) & M_r(-\lambda)\Phi(\lambda)M(\lambda) \end{pmatrix}, \tag{1.13}$$

where $\Phi(\lambda) = -(M(\lambda) - M_r(-\lambda) - iN_+(\lambda)N_{r-}(-\lambda))^{-1}$. In this equalities $M(\cdot)$ and $N_+(\cdot)$ are taken from the block representation (1.8) of the Weyl function $M_+(\lambda)$ corresponding to the boundary triplet Π for A^* and $M_r(\cdot)$ and $N_{r-}(\cdot)$ are taken from the block representation

$$M_{r-}(z) = (M_r(z), N_{r-}(z))^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad z \in \mathbb{C}_-$$

of the Weyl function $M_{r-}(\cdot)$ corresponding to the boundary triplet Π_r for $(-A_r)^*$.

Equality (1.13) is the abstract Titchmarsh formula for the coupling of symmetric relations (in particular operators) with arbitrary defects. Its role in the extension theory of such relations is similar to that of the formula (1.4) for relations with equal defects. For instance (1.13) enables us to describe all exit space extensions $\widetilde{A} = \widetilde{A}^*$ of a symmetric relation A with arbitrary defects satisfying (1.3). Namely, let A be a symmetric relation in $\mathfrak{H}$ with $n_-(A) \leqslant n_+(A)$ and let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then according to [34] the Krein type formula for generalized resolvents

$$R_\tau(\lambda) := P_{\mathfrak{H}}(\widetilde{A}_\tau - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} - \gamma_+(\lambda) K_0(\lambda)(K_1(\lambda) + M_+(\lambda) K_0(\lambda))^{-1} \gamma_-^*(\overline{\lambda})$$

with $\lambda \in \mathbb{C}_+$ gives a parametrization $\widetilde{A} = \widetilde{A}_\tau$ of all exit space extensions $\widetilde{A} = \widetilde{A}^*$ of A by means of pairs $\tau = \{K_0(\cdot), K_1(\cdot)\}$ of holomorphic operator-functions $K_j(\cdot) : \mathbb{C}_+ \to$

$B(\mathscr{H}_1,\mathscr{H}_j)$, $j\in\{0,1\}$, belonging to the Nevanlinna type class $\widetilde{R}(\mathscr{H}_0,\mathscr{H}_1)$ (see subsection 2.2). In this formula $A_0 := \ker\Gamma_0$ is a maximal symmetric extension of A with $n_-(A_0)=0$, $\gamma_\pm(\cdot)$ are γ-fields (see Proposition 2.7) and $M_+(\cdot)$ is the Weyl function of Π.

Using the realisation Theorem 1.2, the main coupling theorem 1.1 and Titchmarsh formula (1.13) we parametrize in terms of τ all extensions $\widetilde{A}_\tau$ of the second kind (in the sense of Naimark) of a densely defined symmetric operator A with finite possibly unequal deficiency indices $n_\pm(A)$. In the case $n_+(A)=n_-(A)$ such a parametrization follows from the results of [13].

In the final part of the paper we demonstrate the obtained results on a symmetric differential system [3, 18]

$$Jy'-B(t)y=\lambda H(t)y,\quad t\in\mathbb{R},\quad \lambda\in\mathbb{C} \tag{1.14}$$

on $\mathbb{R}$. Here

$$J=\begin{pmatrix}0&0&-I_\nu\\0&iI_{\widehat{\nu}}&0\\I_\nu&0&0\end{pmatrix}:\underbrace{\mathbb{C}^\nu\oplus\mathbb{C}^{\widehat{\nu}}\oplus\mathbb{C}^\nu}_{\mathbb{C}^n}\to\underbrace{\mathbb{C}^\nu\oplus\mathbb{C}^{\widehat{\nu}}\oplus\mathbb{C}^\nu}_{\mathbb{C}^n} \tag{1.15}$$

and $B(t)=B^*(t)$, $H(t)\geqslant 0$, $t\in\mathbb{R}$, are $n\times n$-matrix functions ($n=2\nu+\widehat{\nu}$). Let $T_{\min}$, $T_{\min}^-$ and $T_{\min}^+$ be minimal (symmetric) relations in $L^2(H,\mathbb{R})$, $L^2(H,\mathbb{R}_-)$ and $L^2(H,\mathbb{R}_+)$ generated by system (1.14) and its restrictions onto $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively [5, 29]. We show that in the case when $T_{\min}^-$ and $T_{\min}^+$ have minimal (unequal) deficiency indices

$$n_+(T_{\min}^-)=\nu+\widehat{\nu},\quad n_-(T_{\min}^-)=\nu,\quad n_+(T_{\min}^+)=\nu,\qquad n_-(T_{\min}^+)=\nu+\widehat{\nu}, \tag{1.16}$$

the relation $T_{\min}=T_{\min}^*$ is the coupling of $T_{\min}^-$ and $T_{\min}^+$. Moreover, we show that in this case the characteristic matrix $\Omega(\cdot)$ of the system in the sense of [9, 40] coincides (up to a self-adjoint constant) with the Weyl function $M_c(\cdot)$ of the coupling Π_c of certain boundary triplets Π for $(T_{\min}^-)^*$ and Π_r for $(-T_{\min}^+)^*$ (see Proposition 4.14, (2)). This implies that for system (1.14) abstract Titchmarsh formula (1.13) turns into the Titchmarsh formula from [35], which gives a representation of $\Omega(\lambda)$ in terms of Weyl functions $m^-(\cdot)$ and $m^+(\cdot)$ for restrictions of the system onto $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively (see Definition 4.13).

As is known [23] the equation $l[y]=\lambda y$, where $l[y]$ is a formally self-adjoint differential expression of an odd order can be reduced to system (1.14) with J of the form (1.15). Therefore the results of the paper concerning system (1.14) can be reformulated for differential operators on $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ generated by $l[y]$ (c.f. [37]).

2. Preliminaries

2.1. Notations

The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathscr{H}$ denote separable Hilbert spaces; $B(\mathscr{H}_1,\mathscr{H}_2)$ is the set of all bounded linear operators defined on $\mathscr{H}_1$ with values in $\mathscr{H}_2$; $\mathbf{C}_0(\mathscr{H}_1,\mathscr{H}_2):=\{N\in B(\mathscr{H}_0,\mathscr{H}_1):||N||<1\}$ is the set of all strict contractions from $\mathcal{H}_1$ to $\mathcal{H}_2$; $B(\mathcal{H}) := B(\mathcal{H},\mathcal{H})$; $\mathbf{C}_0(\mathcal{H}) := \mathbf{C}_0(\mathcal{H},\mathcal{H})$; $\mathcal{C}(\mathfrak{H})$ is the set of all closed (possibly non densely defined) operators in $\mathfrak{H}$; $A \upharpoonright \mathcal{L}$ is a restriction of the operator $A \in B(\mathcal{H}_1,\mathcal{H}_2)$ onto the linear manifold $\mathcal{L} \subset \mathcal{H}_1$; $\mathbb{C}_+$ $(\mathbb{C}_-)$ is the open upper (lower) half-plane of the complex plane; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk in $\mathbb{C}$.

If $\mathcal{H}$ is a subspace in $\widetilde{\mathcal{H}}$, then $P_{\mathcal{H}}(\in [\widetilde{\mathcal{H}}])$ denote the orthoprojector in $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$ and $P_{\widetilde{\mathcal{H}},\mathcal{H}}(\in [\widetilde{\mathcal{H}},\mathcal{H}])$ denote the same orthoprojector considered as an operator from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$.

Recall that a linear manifold T in the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$ ($\mathcal{H} \oplus \mathcal{H}$) is called a linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ (resp. in $\mathcal{H}$). The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ (resp. $\widetilde{\mathcal{C}}(\mathcal{H})$). In the following an operator $T \in \mathcal{C}(\mathfrak{H})$ is identified with its graph $\operatorname{gr} T$. This enables one to consider $\mathcal{C}(\mathfrak{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathfrak{H})$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we denote by $\operatorname{dom} T, \operatorname{ran} T$ and $\ker T$ the domain, range and kernel of T respectively. For $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we will denote by $T^{-1}(\in \widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0))$ and $T^*(\in \widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0))$ the inverse and adjoint linear relations of T respectively.

As is known a linear relation $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is called symmetric (self-adjoint) if $T \subset T^*$ (resp. $T = T^*$). Let $J_{\mathfrak{H}} \in B(\mathfrak{H}^2)$ be the operator given by

$$J_{\mathfrak{H}} = \begin{pmatrix} 0 & -I_{\mathfrak{H}} \\ I_{\mathfrak{H}} & 0 \end{pmatrix} : \mathfrak{H} \oplus \mathfrak{H} \to \mathfrak{H} \oplus \mathfrak{H}. \tag{2.1}$$

Then for $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ the following equivalence holds:

$$T \subset T^* \iff (J_{\mathfrak{H}}\widehat{f},\widehat{g}) = 0, \;\; \widehat{f},\widehat{g} \in T. \tag{2.2}$$

For an operator $T = T^* \in B(\mathfrak{H})$ we write $T \geqslant 0$ if $(Tf,f) \geqslant 0$, $f \in \mathfrak{H}$, and $T > 0$ if $T - \alpha I \geqslant 0$ with some $\alpha > 0$.

Recall that a holomorphic operator function $\Phi(\cdot) : \mathbb{C}\setminus\mathbb{R} \to B(\mathcal{H})$ is called a Nevanlinna function if $\operatorname{Im}\lambda \cdot \operatorname{Im}\Phi(\lambda) \geqslant 0$ and $\Phi^*(\lambda) = \Phi(\overline{\lambda})$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. The class of all Nevanlinna $B(\mathcal{H})$-valued functions will be denoted by $R[\mathcal{H}]$. Moreover, we denote by $R_u[\mathcal{H}]$ the set of all functions $\Phi(\cdot) \in R[\mathcal{H}]$ such that $\operatorname{Im}\lambda \cdot \operatorname{Im}\Phi(\lambda) > 0$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

2.2. The classes $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ and $\widetilde{R}(\mathcal{H})$

In the following $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, $P_1 := P_{\mathcal{H}_0,\mathcal{H}_1}$ and $P_2 = P_{\mathcal{H}_2}$.

DEFINITION 2.1. [33, 36] A function $\tau(\cdot) : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ is referred to the class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ if:

(i) $2\operatorname{Im}(h_1,h_0) - ||P_2 h_0||^2 \geqslant 0$, $\{h_0,h_1\} \in \tau(\lambda)$, $\lambda \in \mathbb{C}_+$;

(ii) $(\tau(\lambda)+iP_1)^{-1}\in\mathscr{B}(\mathscr{H}_1,\mathscr{H}_0)$, $\lambda\in\mathbb{C}_+$, and the operator-function $(\tau(\lambda)+iP_1)^{-1}$ is holomorphic on $\mathbb{C}_+$.

According to [33, 36] the equality

$$\tau(\lambda)=\{\{K_0(\lambda)h,K_1(\lambda)h\}:h\in\mathscr{H}_1\},\ \ \lambda\in\mathbb{C}_+$$

establishes a bijective correspondence between all functions $\tau=\tau(\cdot)\in\widetilde{R}(\mathscr{H}_0,\mathscr{H}_1)$ and all pairs $\{K_0(\cdot),K_1(\cdot)\}$ of holomorphic operator-functions $K_j(\cdot):\mathbb{C}_+\to B(\mathscr{H}_1,\mathscr{H}_j)$, $j\in\{0,1\}$, with the block representation

$$K_0(\lambda)=(K_{01}(\lambda),K_{02}(\lambda))^\top:\mathscr{H}_1\to\mathscr{H}_1\oplus\mathscr{H}_2 \tag{2.3}$$

satisfying for all $\lambda\in\mathbb{C}_+$ the following relations:

$$2\,\mathrm{Im}(K_{01}^*(\lambda)K_1(\lambda))-K_{02}^*(\lambda)K_{02}(\lambda)\geqslant 0,\qquad (K_1(\lambda)+iK_{01}(\lambda))^{-1}\in B(\mathscr{H}_1). \tag{2.4}$$

In the following we write $\tau=\{K_0(\cdot),K_1(\cdot)\}$ identifying a function $\tau\in\widetilde{R}(\mathscr{H}_0,\mathscr{H}_1)$ and the corresponding pair $\{K_0(\cdot),K_1(\cdot)\}$ of holomorphic operator functions satisfying (2.4)(more precisely the equivalence class of such pairs [33]).

DEFINITION 2.2. A pair $\tau=\{K_0(\cdot),K_1(\cdot)\}$ of holomorphic operator-functions $K_j(\cdot):\mathbb{C}_+\to B(\mathscr{H}_1,\mathscr{H}_j)$, $j\in\{0,1\}$, is referred to the class $\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$ if

$$K_0(\lambda)=(I_{\mathscr{H}_1},K_{02}(\lambda))^\top:\mathscr{H}_1\to\mathscr{H}_1\oplus\mathscr{H}_2,\quad \lambda\in\mathbb{C}_+ \tag{2.5}$$

(that is, $K_0(\lambda)$ has the block representation (2.3) with $K_{01}(\lambda)=I_{\mathscr{H}_1}$) and

$$2\mathrm{Im}K_1(\lambda)-K_{02}^*(\lambda)K_{02}(\lambda)>0,\quad \lambda\in\mathbb{C}_+. \tag{2.6}$$

REMARK 2.3. (1) If $\tau=\{K_0(\cdot),K_1(\cdot)\}\in\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$, then by (2.6) $\mathrm{Im}K_1(\lambda)>0$ and, consequently, the second relation in (2.4) is satisfied. Therefore $\tau\in\widetilde{R}(\mathscr{H}_0,\mathscr{H}_1)$ and hence $\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)\subset\widetilde{R}(\mathscr{H}_0,\mathscr{H}_1)$.

(2) In the case $\mathscr{H}_1=\mathscr{H}_0=:\mathscr{H}$ the class $\widetilde{R}(\mathscr{H},\mathscr{H})$ coincides with the well-known class $\widetilde{R}(\mathscr{H})$ of Nevanlinna $\widetilde{\mathscr{C}}(\mathscr{H})$-valued functions (Nevanlinna operator pairs) $\tau=\{K_0(\lambda),K_1(\lambda)\}$, $\lambda\in\mathbb{C}_+$ (see e.g [11]).Moreover, identifying a function $\Phi(\cdot)\in R_u[\mathscr{H}]$ with a pair $\tau=\{I_{\mathscr{H}},\Phi(\lambda)\}\in\widetilde{R}_u(\mathscr{H},\mathscr{H})$, $\lambda\in\mathbb{C}_+$, one gets $\widetilde{R}_u(\mathscr{H},\mathscr{H})=R_u[\mathscr{H}]$.

2.3. Boundary triplets and Weyl functions

In the following we denote by A a closed symmetric linear relation (in particular closed not necessarily densely defined symmetric operator) in a Hilbert space $\mathfrak{H}$. Let $\mathfrak{N}_\lambda(A)=\ker(A^*-\lambda)$ ($\lambda\in\mathbb{C}\setminus\mathbb{R}$) be a defect subspace of A, let $\widehat{\mathfrak{N}}_\lambda(A)=\{\{f,\lambda f\}:f\in\mathfrak{N}_\lambda(A)\}$ and let $n_\pm(A):=\dim\mathfrak{N}_\lambda(A)\leqslant\infty$, $\lambda\in\mathbb{C}_\pm$, be deficiency indices of A.

As before we assume that $\mathscr{H}_0$ is a Hilbert space, $\mathscr{H}_1$ is a subspace in $\mathscr{H}_0$ and $\mathscr{H}_2:=\mathscr{H}_0\ominus\mathscr{H}_1$, so that $\mathscr{H}_0=\mathscr{H}_1\oplus\mathscr{H}_2$. Moreover, we let $P_1=P_{\mathscr{H}_0,\mathscr{H}_1}$ and $P_2=P_{\mathscr{H}_2}$.

Below within this subsection we specify some definitions and results from [34, 36].

DEFINITION 2.4. A collection $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$, where $\Gamma_j: A^*\to\mathcal{H}_j$, $j\in\{0,1\}$ are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma:\widehat{f}\to\{\Gamma_0\widehat{f},\Gamma_1\widehat{f}\},\widehat{f}\in A^*$, from A^* into $\mathcal{H}_0\oplus\mathcal{H}_1$ is surjective and the Green's identity (1.6) holds for all $\widehat{f}=\{f,f'\}$, $\widehat{g}=\{g,g'\}\in A^*$.

By using the operator $J_{\mathfrak{H}}$ (see (2.1)) one may rewrite (1.6) as

$$-(J_{\mathfrak{H}}\widehat{f},\widehat{g})=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})_{\mathcal{H}_0}-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})_{\mathcal{H}_0}+i(P_2\Gamma_0\widehat{f},P_2\Gamma_0\widehat{g})_{\mathcal{H}_2},\quad \widehat{f},\widehat{g}\in A^*. \tag{2.7}$$

In the following propositions some properties of boundary triplets are specified.

PROPOSITION 2.5. *If $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ is a boundary triplet for A^*, then* (1.7) *holds. Conversely, let A be a symmetric relation with $n_-(A)\leqslant n_+(A)$. Then for any Hilbert space $\mathcal{H}_0$ and a subspace $\mathcal{H}_1\subset\mathcal{H}_0$ satisfying* (1.7) *there exists a boundary triplet $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for A^*.*

PROPOSITION 2.6. *Let $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*. Then:*

(1) $\ker\Gamma_0\cap\ker\Gamma_1=A$ *and Γ_j is a bounded operator from A^* onto $\mathcal{H}_j$, $j\in\{0,1\}$.*

(2) *The equality $A_0:=\ker\Gamma_0=\{\hat{f}\in A^*:\Gamma_0\hat{f}=0\}$ define a maximal symmetric extension A_0 of A such that $n_-(A_0)=0$.*

PROPOSITION 2.7. *Let $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*. Then there exists a unique pair of operator-functions $\gamma_+(\cdot):\mathbb{C}_+\to B(\mathcal{H}_0,\mathfrak{H})$ and $\gamma_-(\cdot):\mathbb{C}_-\to B(\mathcal{H}_1,\mathfrak{H})$ (γ-fields of the triplet Π) such that $\gamma_+(\lambda)\mathcal{H}_0\subset\mathfrak{N}_\lambda(A)$, $\gamma_-(\lambda)\mathcal{H}_1\subset\mathfrak{N}_\lambda(A)$ and*

$$\Gamma_0\widehat{\gamma}_+(\lambda)=I_{\mathcal{H}_0},\ \lambda\in\mathbb{C}_+;\qquad P_1\Gamma_0\widehat{\gamma}_-(\lambda)=I_{\mathcal{H}_1},\ \lambda\in\mathbb{C}_-, \tag{2.8}$$

where $\widehat{\gamma}_+(\lambda)=(\gamma_+(\lambda),\lambda\gamma_+(\lambda))^\top(\in B(\mathcal{H}_0,\mathfrak{H}^2))$, $\lambda\in\mathbb{C}_+$, and $\widehat{\gamma}_-(\lambda)=(\gamma_-(\lambda),\lambda\gamma_-(\lambda))^\top(\in B(\mathcal{H}_1,\mathfrak{H}^2))$, $\lambda\in\mathbb{C}_-$.

DEFINITION 2.8. The operator functions $M_+(\cdot):\mathbb{C}_+\to B(\mathcal{H}_0,\mathcal{H}_1)$ and $M_-(\cdot):\mathbb{C}_-\to B(\mathcal{H}_1,\mathcal{H}_0)$ defined by

$$M_+(\lambda)=\Gamma_1\widehat{\gamma}_+(\lambda),\ \lambda\in\mathbb{C}_+;\qquad M_-(\lambda)=(\Gamma_1+iP_2\Gamma_0)\widehat{\gamma}_-(\lambda),\ \lambda\in\mathbb{C}_- \tag{2.9}$$

are called the (abstract) Weyl functions of the triplet $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for A^*.

It was shown in [34] that the operator-functions $\gamma_\pm(\cdot)$ and $M_\pm(\cdot)$ are holomorphic on their domains and $M_+^*(\lambda)=M_-(\overline{\lambda})$, $\lambda\in\mathbb{C}_+$. Moreover, the following theorem was proved in [36].

THEOREM 2.9. *Let A be a symmetric operator in $\mathfrak{H}$, let $\Pi=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^* and let $M_\pm$ be the Weyl functions of Π with the block representations* (1.8), (1.9). *Then :* (i) *the equalities* (1.10) *and* (1.11) *define the operator-function $\mathscr{M}(\cdot)\in R_u[\mathscr{H}_0]$;* (ii) *the operator A is densely defined if and only if the following two conditions are satisfied:*

$$s-\lim_{y\to\infty}\mathscr{M}(iy)=0 \quad \text{and} \quad \lim_{y\to\infty} y\mathrm{Im}(\mathscr{M}(iy)h,h)=\infty,\ \ h\in\mathscr{H}_0\setminus\{0\}. \tag{2.10}$$

2.4. Exit space extensions and generalized resolvents

In the following with each unitary operator $U\in B(\mathfrak{H}_1,\mathfrak{H}_2)$ we associate a unitary operator $\widetilde{U}:=U\oplus U\in B(\mathfrak{H}_1^2,\mathfrak{H}_2^2)$.

DEFINITION 2.10. Linear relations $T_j\in\widetilde{\mathscr{C}}(\mathfrak{H}_j)$, $j\in\{1,2\}$, are said to be unitarily equivalent if there exists a unitary operator $U\in B(\mathfrak{H}_1,\mathfrak{H}_2)$ such that $T_2=\widetilde{U}T_1$.

DEFINITION 2.11. Let $\mathfrak{H}$ be a subspace in a Hilbert space $\widetilde{\mathfrak{H}}$. The relation $\widetilde{A}=\widetilde{A}^*\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ is called $\mathfrak{H}$-minimal if $\overline{\mathrm{span}}\{\mathfrak{H},(\widetilde{A}-\lambda)^{-1}\mathfrak{H}:\lambda\in\mathbb{C}\setminus\mathbb{R}\}=\widetilde{\mathfrak{H}}$.

Recall further the following definition.

DEFINITION 2.12. The operator function $R(\cdot):\mathbb{C}_+\to B(\mathfrak{H})$ is called the generalized resolvent of a symmetric relation $A\in\widetilde{\mathscr{C}}(\mathfrak{H})$ if there exist a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ and a self-adjoint relation $\widetilde{A}\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ such that $A\subset\widetilde{A}$ and the following equality holds:

$$R(\lambda)=P_{\widetilde{\mathfrak{H}},\mathfrak{H}}(\widetilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H},\quad \lambda\in\mathbb{C}_+. \tag{2.11}$$

The relation $\widetilde{A}\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ in (2.11) is called an exit space self-adjoint extension of A. Such an extension exists for any symmetric relation A.

According to [28] each generalized resolvent of A is generated by some $\mathfrak{H}$-minimal exit space extension $\widetilde{A}$ of A. Moreover, if the $\mathfrak{H}$-minimal exit space extensions $\widetilde{A}_1\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}}_1)$ and $\widetilde{A}_2\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}}_2)$ of A induce the same generalized resolvent $R(\lambda)$, then there exists a unitary operator $V\in[\widetilde{\mathfrak{H}}_1\ominus\mathfrak{H},\widetilde{\mathfrak{H}}_2\ominus\mathfrak{H}]$ such that $\widetilde{A}_1$ and $\widetilde{A}_2$ are unitarily equivalent by means of the unitary operator $U=I_{\mathfrak{H}}\oplus V$. By using this fact we suppose in the following that an exit space extension $\widetilde{A}$ is $\mathfrak{H}$-minimal, so that it is defined by (2.11) uniquely up to the unitary equivalence.

A description of all generalized resolvents of a symmetric relation A with possibly unequal deficiency indices $n_\pm(A)$ is given by the following theorem obtained in [34].

THEOREM 2.13. *Assume that $n_-(A)\leqslant n_+(A)$, $\Pi=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0,\Gamma_1\}$ is a boundary triplet for A^*, $A_0=\ker\Gamma_0$ and $\gamma_\pm(\cdot)$ and $M_+(\cdot)$ are the γ-fields and the Weyl function of Π respectively. Then the equality (the Krein formula for generalized resolvents)*

$$R_\tau(\lambda)=(A_0-\lambda)^{-1}-\gamma_+(\lambda)K_0(\lambda)(K_1(\lambda)+M_+(\lambda)K_0(\lambda))^{-1}\gamma_-^*(\overline{\lambda}),\ \ \lambda\in\mathbb{C}_+ \tag{2.12}$$

establishes a bijective correspondence $R(\lambda) = R_\tau(\lambda)$ *between all pairs* $\tau = \{K_0(\cdot), K_1(\cdot)\} \in \widetilde{R}(\mathscr{H}_0, \mathscr{H}_1)$ *and all generalized resolvents* $R(\lambda)$ *of* A.

REMARK 2.14. It follows from Theorem 2.13 that formula for resolvents (2.12) together with the equality

$$R_\tau(\lambda) = P_{\widetilde{\mathfrak{H}},\mathfrak{H}}(\widetilde{A}_\tau - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C}_+ \tag{2.13}$$

gives a parametrization $\widetilde{A} = \widetilde{A}_\tau$ of all ($\mathfrak{H}$-minimal) exit space self-adjoint extensions $\widetilde{A}$ of A by means of all pairs $\tau = \{K_0(\cdot), K_1(\cdot)\} \in \widetilde{R}(\mathscr{H}_0, \mathscr{H}_1)$.

REMARK 2.15. If $\mathscr{H}_0 = \mathscr{H}_1 := \mathscr{H}$, then the triplet $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ in the sense of Definition 2.4 turns into the boundary triplet (boundary value space) $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ for A^* in the sense of [19, 8]. In this case:

(i) $n_+(A) = n_-(A) = \dim \mathscr{H}$ and the Green identity (1.6) takes the form (1.1).

(ii) $A_0 = A_0^*$ and the γ-fields $\gamma_\pm(\cdot)$ of Π turn into the γ-field $\gamma(\cdot)$ defined in the papers [14, 15, 30] as a unique operator-function $\gamma(\cdot) : \mathbb{C} \setminus \mathbb{R} \to B(\mathscr{H}, \mathfrak{H})$ such that $\gamma(\lambda)\mathscr{H} \subset \mathfrak{N}_\lambda(A)$ and

$$\Gamma_0 \widehat{\gamma}(\lambda) = I_{\mathscr{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.14}$$

with $\widehat{\gamma}(\lambda) = (\gamma(\lambda), \lambda\gamma(\lambda))^\top (\in B(\mathscr{H}, \mathfrak{H}^2))$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, the Weyl function $M_+(\cdot)$ of Π turns into the Weyl function $M(\cdot)$ defined in the same papers by

$$M(\lambda) = \Gamma_1 \widehat{\gamma}(\lambda), \ \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.15}$$

(iii) $M(\cdot)$ is a Q-function of the pair (A, A_0) and formula (2.12) turns into the classical Krein formula for generalized resolvents of a symmetric relation A with equal deficiency indices [26, 28, 14, 30].

PROPOSITION 2.16. [14, 30] *Let* $n_+(A) = n_-(A)$ *and let* $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*. *Then for each operator* $B = B^* \in B(\mathscr{H})$ *the equality* $\widetilde{A}_B = \{\hat{f} \in A^* : \Gamma_1 \hat{f} = B\Gamma_0 \hat{f}\}$ *defines a self-adjoint extension* $\widetilde{A}_B \in \widetilde{\mathscr{C}}(\mathfrak{H})$ *of* A *and the following Krein formula for canonical resolvents holds:*

$$(\widetilde{A}_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(B - M(\lambda))^{-1}\gamma^*(\overline{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.16}$$

3. Inner characterization of the Weyl functions

Recall that a symmetric relation $A \in \widetilde{\mathscr{C}}(\mathfrak{H})$ is called simple if there is not a decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that $\mathfrak{H}_2 \neq \{0\}$ and $A = A_1 \oplus A_2$ with symmetric $A_1 \in \widetilde{\mathscr{C}}(\mathfrak{H}_1)$ and self-adjoint $A_2 \in \widetilde{\mathscr{C}}(\mathfrak{H}_2)$. The simplicity of A is equivalent to the equality $\mathfrak{H} = \overline{\operatorname{span}}\{\mathfrak{N}_\lambda(A) : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$. If $A \in \widetilde{\mathscr{C}}(\mathfrak{H})$ is simple, then $A \in \mathscr{C}(\mathfrak{H})$, that is A is an operator. Moreover, for each symmetric $A\in\widetilde{\mathscr{C}}(\mathfrak{H})$ there exists a unique pair if decompositions

$$\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2,\qquad A=A_1\oplus A_2 \tag{3.1}$$

with a simple operator $A_1\in\mathscr{C}(\mathfrak{H}_1)$ (the simple part of A) and a self-adjoint relation $A_2\in\widetilde{\mathscr{C}}(\mathfrak{H}_2)$.

DEFINITION 3.1. Let $A_j\in\widetilde{\mathscr{C}}(\mathfrak{H}_j)$ and let $\Pi_j=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0^{(j)},\Gamma_1^{(j)}\}$ be a boundary triplet for A_j^*, $j\in\{1,2\}$. The boundary triplets Π_1 and Π_2 are called unitarily equivalent if there exists a unitary operator $U\in B(\mathfrak{H}_1,\mathfrak{H}_2)$ such that

$$\widetilde{U}A_1^*=A_2^*\quad\text{and}\quad\Gamma_k^{(1)}=\Gamma_k^{(2)}\widetilde{U}\upharpoonright A_1^*,\quad k\in\{0,1\}. \tag{3.2}$$

If boundary triplets $\Pi_j=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0^{(j)},\Gamma_1^{(j)}\}$ for A_j^* are unitarily equivalent and $U\in B(\mathfrak{H}_1,\mathfrak{H}_2)$ is a unitary operator such that (3.2) holds, then $\widetilde{U}A_1=A_2$, i.e., relations A_1 and A_2 are unitarily equivalent.

The inner characterization of abstract Weyl functions for symmetric relations A with equal deficiency indices is given by the following two theorems [14, 15, 16].

THEOREM 3.2. *Let $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^* and let $M(\cdot)$ be the Weyl function of Π. Then $M(\cdot)\in R_u[\mathscr{H}]$.*

Conversely, let $\mathscr{H}$ be a Hilbert space and let $M(\cdot)\in R_u[\mathscr{H}]$. Then there exist a Hilbert space $\mathfrak{H}$, a simple symmetric operator $A\in\mathscr{C}(\mathfrak{H})$ with $n_+(A)=n_-(A)$ and a boundary triplet $\Pi=\{\mathscr{H},\Gamma_0,\Gamma_1\}$ for A^* such that $M(\cdot)$ is the Weyl function of Π.

THEOREM 3.3. *Assume that $A_j\in\mathscr{C}(\mathfrak{H}_j)$ is a simple symmetric operator with $n_+(A_j)=n_-(A_j)$, $\Pi_j=\{\mathscr{H},\Gamma_0^{(j)},\Gamma_1^{(j)}\}$ is a boundary triplet for A_j^* and $M_j(\cdot)$ is the Weyl function of Π_j, $j\in\{1,2\}$. Then the triplets Π_1 and Π_2 are unitarily equivalent if and only if $M_1(\lambda)=M_2(\lambda)$, $\lambda\in\mathbb{C}_+$.*

Our next goal is to extend the above theorems onto symmetric relations A with unequal deficiency indices $n_\pm(A)$. To this end we first introduce a new class of holomorphic operator-functions. Namely, let $\mathscr{H}_0$ be a Hilbert space, let $\mathscr{H}_1$ be a subspace in $\mathscr{H}_0$ and let $\mathscr{H}_2=\mathscr{H}_0\ominus\mathscr{H}_1$, so that $\mathscr{H}_0=\mathscr{H}_1\oplus\mathscr{H}_2$.

DEFINITION 3.4. A pair (M_+,M_-) of holomorphic operator-functions $M_+(\cdot):\mathbb{C}_+\to B(\mathscr{H}_0,\mathscr{H}_1)$ and $M_-(\cdot):\mathbb{C}_-\to B(\mathscr{H}_1,\mathscr{H}_0)$ with the block representations

$$M_+(\lambda)=(M(\lambda),N_+(\lambda)):\mathscr{H}_1\oplus\mathscr{H}_2\to\mathscr{H}_1,\quad\lambda\in\mathbb{C}_+ \tag{3.3}$$

$$M_-(\lambda)=(M(\lambda),N_-(\lambda))^\top:\mathscr{H}_1\to\mathscr{H}_1\oplus\mathscr{H}_2,\quad\lambda\in\mathbb{C}_- \tag{3.4}$$

will be referred to the class $R_u[\mathscr{H}_0,\mathscr{H}_1]$ if $M_+^*(\lambda)=M_-(\overline{\lambda})$, $\lambda\in\mathbb{C}_+$, and

$$2\mathrm{Im}M(\lambda)-N_+(\lambda)N_+^*(\lambda)>0,\quad\lambda\in\mathbb{C}_+ \tag{3.5}$$

$$2\mathrm{Im}M(\lambda)+N_-^*(\lambda)N_-(\lambda)<0,\quad\lambda\in\mathbb{C}_- \tag{3.6}$$

REMARK 3.5. (1) Clearly each function $M_+(\cdot)$ of the form (3.3) ($M_-(\cdot)$ of the form (3.4)) satisfying (3.5) (resp, (3.6)) generates a pair $(M_+,M_-)\in R_u[\mathscr{H}_0,\mathscr{H}_1]$ (with $\mathscr{H}_0=\mathscr{H}_1\oplus\mathscr{H}_2$) by means of the equality $M_-(\lambda)=M_+^*(\overline{\lambda})$, $\lambda\in\mathbb{C}_-$ (resp. $M_+(\lambda)=M_-^*(\overline{\lambda})$, $\lambda\in\mathbb{C}_+$).

(2) Let $M_+(\cdot)$ and $M_-(\cdot)$ be operator-functions (3.3), (3.4), let $\mathscr{M}(\lambda)$ be given by (1.10), (1.11) and let $\mathscr{H}_0=\mathscr{H}_1\oplus\mathscr{H}_2$. One can easily verify that $(M_+,M_-)\in R_u[\mathscr{H}_0,\mathscr{H}_1]$ if and only if $\mathscr{M}(\cdot)\in R_u[\mathscr{H}_0]$. Therefore in the case $\mathscr{H}_0=\mathscr{H}_1=:\mathscr{H}$ on has $R_u[\mathscr{H}_0,\mathscr{H}_1]=R_u[\mathscr{H}]$.

The following lemma directly follows from Definitions 2.2, 3.4 and Remark 3.5, (1).

LEMMA 3.6. *The equalities*

$$K_0(\lambda)=(I_{\mathscr{H}_1},-iN_-(-\lambda))^\top:\mathscr{H}_1\to\mathscr{H}_1\oplus\mathscr{H}_2,\quad K_1(\lambda)=-M(-\lambda),\quad \lambda\in\mathbb{C}_+,$$

establish a bijective correspondence between all pairs $(M_+,M_-)\in R_u[\mathscr{H}_0,\mathscr{H}_1]$ with the block representation (3.4) of $M_-(\lambda)$ and all pairs $\tau=\{K_0(\cdot),K_1(\cdot)\}\in\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$.

In the following proposition we specify a connection between pairs $(M_+,M_-)\in R_u[\mathscr{H}_0,\mathscr{H}_1]$ and strictly contractive operator-functions.

PROPOSITION 3.7. *Let φ be a conformal mapping of $\mathbb{C}_+$ onto $\mathbb{D}$ given by*

$$\varphi(\lambda)=(\lambda-i)(\lambda+i)^{-1},\quad \lambda\in\mathbb{C}_+ \tag{3.7}$$

and let

$$Y_1=\begin{pmatrix} iI_{\mathscr{H}_1} & 0\\ 0 & \sqrt{2}I_{\mathscr{H}_2}\end{pmatrix}:\underbrace{\mathscr{H}_1\oplus\mathscr{H}_2}_{\mathscr{H}_0}\to\underbrace{\mathscr{H}_1\oplus\mathscr{H}_2}_{\mathscr{H}_0},\quad Y_2=\begin{pmatrix} I_{\mathscr{H}_1}\\ 0\end{pmatrix}:\mathscr{H}_1\to\underbrace{\mathscr{H}_1\oplus\mathscr{H}_2}_{\mathscr{H}_0}, \tag{3.8}$$

$$Y_3=(-iI_{\mathscr{H}_1},0):\underbrace{\mathscr{H}_1\oplus\mathscr{H}_2}_{\mathscr{H}_0}\to\mathscr{H}_1. \tag{3.9}$$

Then the equality

$$\widehat{M}(z)=(Y_3+M_+(\varphi^{-1}(z)))(Y_1+Y_2M_+(\varphi^{-1}(z)))^{-1},\quad z\in\mathbb{D} \tag{3.10}$$

establishes a bijective correspondence between all holomorphic operator-functions $M_+(\cdot):\mathbb{C}_+\to B(\mathscr{H}_0,\mathscr{H}_1)$ with block representation (3.3) satisfying (3.5) and all holomorphic operator-functions $\widehat{M}(\cdot):\mathbb{D}\to\mathbf{C}_0(\mathscr{H}_0,\mathscr{H}_1)$.

Proof. Since

$$Y_1+Y_2M_+(\varphi^{-1}(z))=\begin{pmatrix} M(\varphi^{-1}(z))+iI_{\mathscr{H}_1} & N_+(\varphi^{-1}(z))\\ 0 & \sqrt{2}I_{\mathscr{H}_2}\end{pmatrix},\quad z\in\mathbb{D}$$

and by (3.5) $\mathrm{Im} M(\varphi^{-1}(z) \geqslant 0$, the operator $Y_1+Y_2M_+(\varphi^{-1}(z))$ is invertible. Therefore the equality (3.10) gives a bijective correspondence between holomorphic functions $M_+(\cdot):\mathbb{C}_+\to B(\mathscr{H}_0,\mathscr{H}_1)$ and $\widehat{M}(\cdot):\mathbb{D}\to B(\mathscr{H}_0,\mathscr{H}_1)$.

Let $z\in\mathbb{D}$ and let $\lambda=\varphi^{-1}(z)\in\mathbb{C}_+$. Then the operator $\widehat{M}(z)$ admits the representation by means of the following two equalities:

$$k_0=(Y_1+Y_2M_+(\lambda))h_0=(M_+(\lambda)h_0+ih_{01})\oplus\sqrt{2}h_{02}, \tag{3.11}$$
$$\widehat{M}(z)k_0=(Y_3+M_+(\lambda))h_0=M_+(\lambda)h_0-ih_{01},\quad h_0=h_{01}\oplus h_{02}\in\mathscr{H}_1\oplus\mathscr{H}_2=\mathscr{H}_0$$

and the immediate calculations give

$$||k_0||^2-||\widehat{M}(z)k_0||^2=4(B(\lambda)h_0,h_0), \tag{3.12}$$

where $B(\lambda)=\begin{pmatrix}\mathrm{Im}M(\lambda) & \frac{1}{2i}N_+(\lambda)\\ -\frac{1}{2i}N_+^*(\lambda) & \frac{1}{2}I_{\mathscr{H}_2}\end{pmatrix}$. Since the operator $Y_1+Y_2M_+(\lambda)$ is invertible, it follows from (3.11) and (3.12) that $||\widehat{M}(z)||<1$ if and only if $B(\lambda)>0$ or, equivalently, (3.5) holds. Hence $\widehat{M}(z)\in\mathbf{C}_0(\mathscr{H}_0,\mathscr{H}_1)$ for any $z\in\mathbb{D}$ if and only if $M_+(\lambda)$ satisfies (3.5) for any $\lambda\in\mathbb{C}_+$.

Recall that an operator $V\in B(\mathrm{dom}V,\mathfrak{H})$ with the closed domain $\mathrm{dom}V\subset\mathfrak{H}$ is called an isometry in $\mathfrak{H}$ if $||Vf||=||f||$, $f\in\mathrm{dom}V$. Clearly a linear relation $V\in\widetilde{\mathscr{C}}(\mathfrak{H})$ is (a graph of) an isometry if and only if $V\subset V^{-1*}$.

DEFINITION 3.8. [31, 32] Let V be an isometry in $\mathfrak{H}$. Then a collection $\widehat{\Pi}=\{\widehat{\mathscr{H}_0}\oplus\widehat{\mathscr{H}_1},\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$ consisting of Hilbert spaces $\widehat{\mathscr{H}_j}$ and linear mappings $\widehat{\Gamma}_j:V^{-1*}\to\widehat{\mathscr{H}_j}$, $j\in\{0,1\}$, is called a boundary triplet for V if the mapping $\widehat{\Gamma}=(\widehat{\Gamma}_0,\widehat{\Gamma}_1)^\top: V^{-1*}\to\widehat{\mathscr{H}_0}\oplus\widehat{\mathscr{H}_1}$ is surjective and

$$(f',g')-(f,g)=(\widehat{\Gamma}_0\widehat{f},\widehat{\Gamma}_0\widehat{g})-(\widehat{\Gamma}_1\widehat{f},\widehat{\Gamma}_1\widehat{g}),\quad \widehat{f}=\{f,f'\},\widehat{g}=\{g,g'\}\in V^{-1*}. \tag{3.13}$$

PROPOSITION 3.9. [31, 32] *Let $\widehat{\Pi}=\{\widehat{\mathscr{H}_0}\oplus\widehat{\mathscr{H}_1},\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$ be a boundary triplet for V. Then for each $z\in\mathbb{D}$ the operator $\widehat{\Gamma}_1\upharpoonright\widehat{\mathfrak{N}}_z(V^{-1})$ isomorphically maps $\widehat{\mathfrak{N}}_z(V^{-1})$ onto $\widehat{\mathscr{H}_1}$ and the equality*

$$\widehat{\Gamma}_0\upharpoonright\widehat{\mathfrak{N}}_z(V^{-1})=\widehat{M}(z)\widehat{\Gamma}_1\upharpoonright\widehat{\mathfrak{N}}_z(V^{-1}),\quad z\in\mathbb{D} \tag{3.14}$$

correctly defines the holomorphic operator function $\widehat{M}(\cdot):\mathbb{D}\to\mathbf{C}_0(\widehat{\mathscr{H}_1},\widehat{\mathscr{H}_0})$.

The operator-function $\widehat{M}(\cdot)$ is called the Weyl function of the triplet $\widehat{\Pi}$. Actually the Weyl function $\widehat{M}(\cdot)$ is the characteristic function of some contraction $\widetilde{A}\supset V$ in the sense of [7, 38].

The following theorem directly follows from the results of [31, 32].

THEOREM 3.10. *Let $\widehat{\mathscr{H}_0}$ and $\widehat{\mathscr{H}_1}$ be Hilbert spaces and let $\widehat{M}(\cdot):\mathbb{D}\to\mathbf{C}_0(\widehat{\mathscr{H}_1},\widehat{\mathscr{H}_0})$ be a holomorphic operator-function. Then there exist a Hilbert space $\mathfrak{H}$, an isometry V in $\mathfrak{H}$ and a boundary triplet $\widehat{\Pi}=\{\widehat{\mathscr{H}_0}\oplus\widehat{\mathscr{H}_1},\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$ for V such that $\widehat{M}(\cdot)$ is the Weyl function of $\widehat{\Pi}$.*

LEMMA 3.11. *Let V be an isometry in $\mathfrak{H}$, let $\widehat{\Pi}=\{\widehat{\mathcal{H}_0}\oplus\widehat{\mathcal{H}_1},\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$ be a boundary triplet for V with $\widehat{\mathcal{H}_0}\subset\widehat{\mathcal{H}_1}$, let $\widehat{\mathcal{H}_2}=\widehat{\mathcal{H}_1}\ominus\widehat{\mathcal{H}_0}$ and let $\widehat{M}(\cdot)$ be the Weyl function of $\widehat{\Pi}$. Moreover, let*

$$X=\frac{1}{\sqrt{2}}\begin{pmatrix} iI_{\mathfrak{H}} & I_{\mathfrak{H}}\\ -iI_{\mathfrak{H}} & I_{\mathfrak{H}}\end{pmatrix}:\mathfrak{H}\oplus\mathfrak{H}\to\mathfrak{H}\oplus\mathfrak{H}. \tag{3.15}$$

Then:

(1) *The operator X is unitary and the equalities*

$$A=X^*\,\mathrm{gr}V,\qquad A^*=X^*V^{-1*} \tag{3.16}$$

define a symmetric relation $A\in\widetilde{\mathscr{C}}(\mathfrak{H})$ and its adjoint A^*.

(2) *The equalities*

$$\mathcal{H}_0=\widehat{\mathcal{H}_1},\qquad \mathcal{H}_1=\widehat{\mathcal{H}_0} \tag{3.17}$$

$$\Gamma_0\widehat{f}=\tfrac{i}{\sqrt{2}}(\widehat{\Gamma}_0\widehat{x}-P_{\widehat{\mathcal{H}_1},\widehat{\mathcal{H}_0}}\widehat{\Gamma}_1\widehat{x})\oplus P_{\widehat{\mathcal{H}_1},\widehat{\mathcal{H}_2}}\widehat{\Gamma}_1\widehat{x},\qquad \Gamma_1\widehat{f}=\tfrac{1}{\sqrt{2}}(\widehat{\Gamma}_0\widehat{x}+P_{\widehat{\mathcal{H}_1},\widehat{\mathcal{H}_0}}\widehat{\Gamma}_1\widehat{x}), \tag{3.18}$$

where $\widehat{f}\in A^*$ and $\widehat{x}=X\widehat{f}\in V^{-1*}$ define a boundary triplet $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for A^*.

(3) *The Weyl function $M_+(\cdot)$ of the triplet Π is connected with $\widehat{M}(\cdot)$ via (3.10).*

Proof.

(1) Since $X^*=\frac{1}{\sqrt{2}}\begin{pmatrix}-iI_{\mathfrak{H}} & iI_{\mathfrak{H}}\\ I_{\mathfrak{H}} & I_{\mathfrak{H}}\end{pmatrix}$, it follows that $X^*X=XX^*=I$ and hence the operator X is unitary. Let $\widehat{f}=\{f,f'\}$, $\widehat{g}=\{g,g'\}$ and $\widehat{x}=\{x,x'\}$, $\widehat{y}=\{y,y'\}$ be elements of $\mathfrak{H}^2$ connected via $\widehat{x}=X\widehat{f}$ and $\widehat{y}=X\widehat{g}$ or, equivalently, $\widehat{f}=X^*\widehat{x}$ and $\widehat{g}=X^*\widehat{y}$. This means that

$$x=\tfrac{1}{\sqrt{2}}(f'+if),\quad x'=\tfrac{1}{\sqrt{2}}(f'-if),\quad y=\tfrac{1}{\sqrt{2}}(g'+ig),\quad y'=\tfrac{1}{\sqrt{2}}(g'-ig)$$

and, consequently,

$$(x',y')-(x,y)=i[(f',g)-(f,g')]. \tag{3.19}$$

Let $A:=X^*\mathrm{grV}\in\widetilde{\mathscr{C}}(\mathfrak{H})$. Then by (3.19)

$$\{f,f'\}\in A^*\iff\{x',x\}\in V^*\iff\{x,x'\}\in V^{-1*}$$

end hence $A^*=X^*V^{-1*}$. Moreover, since $\mathrm{gr}V\subset V^{-1*}$, it follows that $A\subset A^*$, i.e., the relation A is symmetric.

(2) Let $\widehat{f}=\{f,f'\}$, $\widehat{g}=\{g,g'\}\in A^*$ and let $\widehat{x},\widehat{y}\in V^{-1*}$ be given by $\widehat{x}=\{x,x'\}:=X\widehat{f}$, $\widehat{y}=\{y,y'\}:=X\widehat{g}$. Then in view of (3.18) and (3.19) one has

$$(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})+i(P_2\Gamma_0\widehat{f},P_2\Gamma_0\widehat{g})=i((\widehat{\Gamma}_1\widehat{x},\widehat{\Gamma}_1\widehat{y})-(\widehat{\Gamma}_0\widehat{x},\widehat{\Gamma}_0\widehat{y}))$$
$$=-i((x',y')-(x,y))=(f',g)-(f,g')$$

(here we made use of the equality $\mathscr{H}_2(=\mathscr{H}_0\ominus\mathscr{H}_1)=\widehat{\mathscr{H}_2}$, which follows from (3.17)). Hence the operators Γ_0 and Γ_1 satisfy the Green's identity (1.6). Surjectivity of the operator $(\Gamma_0,\Gamma_1)^\top$ directly follows from (3.18) and surjectivity of $(\widehat{\Gamma}_0,\widehat{\Gamma}_1)^\top$.

(3) It follows from (3.18) that

$$\sqrt{2}\widehat{\Gamma}_1\widehat{x}=Y_1\Gamma_0\widehat{f}+Y_2\Gamma_1\widehat{f},\qquad \sqrt{2}\widehat{\Gamma}_0\widehat{x}=Y_3\Gamma_0\widehat{f}+\Gamma_1\widehat{f},$$

where $\widehat{f}\in A^*$, $\widehat{x}=X\widehat{f}$ and Y_j are operators (3.8), (3.9). Let $\lambda\in\mathbb{C}_+$ and let $z=\varphi(\lambda)\in\mathbb{D}$, where φ is the mapping (3.7). Assume that $h\in\widehat{\mathscr{H}_1}(=\mathscr{H}_0)$. Since the operator $Y_1+Y_2M_+(\lambda)$ is invertible, there exists $\widehat{f}=\{f,\lambda f\}\in\widehat{\mathfrak{N}}_\lambda(A)$ such that

$$h=(Y_1+Y_2M_+(\lambda))\Gamma_0\widehat{f}=Y_1\Gamma_0\widehat{f}+Y_2\Gamma_1\widehat{f}=\sqrt{2}\widehat{\Gamma}_1\widehat{x},$$

where

$$\widehat{x}=X\{f,\lambda f\}=\tfrac{1}{\sqrt{2}}\{(\lambda+i)f,(\lambda-i)f\}=\tfrac{\lambda+i}{\sqrt{2}}\{f,zf\}\in V^{-1*}.$$

Hence $\widehat{x}\in\widehat{\mathfrak{N}}_z(V^{-1})$ and

$$\widehat{M}(z)h=\sqrt{2}\widehat{M}(z)\widehat{\Gamma}_1\widehat{x}=\sqrt{2}\widehat{\Gamma}_0\widehat{x}=(Y_3+M_+(\lambda))\Gamma_0\widehat{f}$$
$$=(Y_3+M_+(\lambda))(Y_1+Y_2M_+(\lambda))^{-1}h.$$

This yields statement (3).

The following lemma directly follows from [36, Proposition 4.2].

LEMMA 3.12. *Let $\Pi=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^* and let $M_+(\cdot)$ be the Weyl function of Π represented as in (3.3). Moreover, let A_r be a simple maximal symmetric operator in a Hilbert space $\mathfrak{H}_r$ with $n_+(A_r)=0$, $n_-(A_r)=\dim\mathscr{H}_2$ and let $\mathfrak{H}_e:=\mathfrak{H}\oplus\mathfrak{H}_r$. Then $A_e:=A\oplus A_r$ is a symmetric relation in $\mathfrak{H}_e$, $A_e^*:=A^*\oplus A_r^*$ and there exists a linear mapping $\Gamma_r:A_r^*\to\mathscr{H}_2$ such that the operators*

$$\Gamma_0^e\widehat{f}_e=P_1\Gamma_0\widehat{f}\oplus(P_2\Gamma_0\widehat{f}+\Gamma_r\widehat{f}_r)(\in\mathscr{H}_1\oplus\mathscr{H}_2),\tag{3.20}$$
$$\Gamma_1^e\widehat{f}_e=\Gamma_1\widehat{f}\oplus\tfrac{i}{2}(P_2\Gamma_0\widehat{f}-\Gamma_r\widehat{f}_r)(\in\mathscr{H}_1\oplus\mathscr{H}_2),\quad \widehat{f}_e=\widehat{f}\oplus\widehat{f}_r\in A^*\oplus A_r^*\tag{3.21}$$

form a boundary triplet $\Pi_e=\{\mathscr{H}_0,\Gamma_0^e,\Gamma_1^e\}$ for A_e^* and the Weyl function $\mathscr{M}(\cdot)$ of Π_e admits the representation (1.10).

Now we are ready to prove the two main theorems of this section, which provides an inner characterisation of abstract Weyl functions for symmetric relations with arbitrary (possibly unequal) deficiency indices (cf. Theorems 3.2 and 3.3).

THEOREM 3.13. *Let $A \in \widetilde{\mathscr{C}}(\mathfrak{H})$ be a symmetric linear relation with deficiency indices $n_-(A) \leqslant n_+(A)$, let $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let $M_\pm(\cdot)$ be the Weyl functions of this triplet. Then $(M_+, M_-) \in R_u[\mathscr{H}_0, \mathscr{H}_1]$.*

Conversely, let $\mathscr{H}_0$ be a Hilbert space, let $\mathscr{H}_1$ be a subspace in $\mathscr{H}_0$ and let $(M_+, M_-) \in R_u[\mathscr{H}_0, \mathscr{H}_1]$. Then there exist a Hilbert space $\mathfrak{H}$, a simple symmetric operator $A \in \mathscr{C}(\mathfrak{H})$ with $n_-(A) \leqslant n_+(A)$ and a boundary triplet $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ for A^* such that $M_+(\cdot)$ and $M_-(\cdot)$ are the Weyl functions of Π.

Proof. The direct statement of the theorem follows from [34, Corollary 3.18].

Let the assumptions of the inverse statement be satisfied. Then by Proposition 3.7 equality (3.10) defines the holomorphic operator-function $\widehat{M}(\cdot) : \mathbb{D} \to \mathbf{C}_0(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_0})$ with $\widehat{\mathscr{H}_1} = \mathscr{H}_0$, $\widehat{\mathscr{H}_0} = \mathscr{H}_1$ and according to Theorem 3.10 there exist a Hilbert space $\mathfrak{H}$, an isometry V in $\mathfrak{H}$ and a boundary triplet $\widehat{\Pi} = \{\widehat{\mathscr{H}_0} \oplus \widehat{\mathscr{H}_1}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ for V such that $\widehat{M}(\cdot)$ is the Weyl function of $\widehat{\Pi}$. Let $A = X^* \mathrm{gr} V$, where X is given by (3.15). Then in view of Lemma 3.11 A is a symmetric relation in $\mathfrak{H}$ and the equalities (3.17), (3.18) define a boundary triplet $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ for A^*. Denote by $M_{\Pi+}(\cdot)$ and $M_{\Pi-}(\cdot)$ the Weyl functions of the triplet Π. In accordance with lemma 3.11 $M_{\Pi+}(\cdot)$ is connected with $\widehat{M}(\cdot)$ via (3.10) (with $M_{\Pi+}$ instead of M_+) and since the correspondence (3.10) is one-to-one, one has $M_+(\lambda) = M_{\Pi+}(\lambda)$, $\lambda \in \mathbb{C}_+$. Moreover, $M_-(\lambda) = M_+^*(\overline{\lambda}) = M_{\Pi+}^*(\overline{\lambda}) = M_{\Pi-}(\lambda)$, $\lambda \in \mathbb{C}_-$.

Next assume that A_1 is a simple part of A. Then in view of decompositions (3.1) $A^* = A_1^* \oplus A_2$ and by Proposition 2.6, (1) $\Gamma_0 \upharpoonright A_2 = \Gamma_1 \upharpoonright A_2 = 0$. Hence the equalities $\Gamma_j' = \Gamma_j \upharpoonright A_1^*$, $j \in \{0,1\}$, define a boundary triplet $\Pi' = \{\mathscr{H}_0, \oplus \mathscr{H}_1, \Gamma_0', \Gamma_1'\}$ for A_1^* with the same Weyl functions $M_\pm(\cdot)$.

THEOREM 3.14. *Assume that $A_j \in \mathscr{C}(\mathfrak{H}_j)$ is a simple symmetric operator with $n_-(A_j) \leqslant n_+(A_j)$, $\Pi_j = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0^{(j)}, \Gamma_1^{(j)}\}$ is a boundary triplet for A_j^* and $M_+^{(j)}(\cdot)$ is the Weyl function of Π_j, $j \in \{1,2\}$. Then the triplets Π_1 and Π_2 are unitarily equivalent if and only if $M_+^{(1)}(\lambda) = M_+^{(2)}(\lambda)$, $\lambda \in \mathbb{C}_+$.*

Proof. If triplets Π_1 and Π_2 are unitarily equivalent, then obviously $M_+^{(1)}(\lambda) = M_+^{(2)}(\lambda)$. Conversely, let $M_+^{(j)}(\lambda) = (M^{(j)}(\lambda), N_+^{(j)}(\lambda))$ be the block representations of $M_+^{(j)}(\lambda)$, $j \in \{1,2\}$, and let $M_+^{(1)}(\lambda) = M_+^{(2)}(\lambda)$, $\lambda \in \mathbb{C}_+$. Moreover, let A_r be a simple maximal symmetric operator in a Hilbert space $\mathfrak{H}_r$ with $n_+(A_r) = 0$, $n_-(A_r) = \dim \mathscr{H}_2$, let $\mathfrak{H}_e^{(j)} := \mathfrak{H}_j \oplus \mathfrak{H}_r$ and let $A_e^{(j)} = A_j \oplus A_r$, $j \in \{1,2\}$. Then by Lemma 3.12 the equalities (3.20) and (3.21) with $\Gamma_0 = \Gamma_0^{(j)}$ and $\Gamma_1 = \Gamma_1^{(j)}$ define a boundary triplet $\Pi_e^{(j)} = \{\mathscr{H}_0, \Gamma_0^{e(j)}, \Gamma_1^{e(j)}\}$ for $(A_e^{(j)})^* (= A_j^* \oplus A_r^*)$ and the Weyl function $\mathscr{M}_j(\cdot)$

of $\Pi_e^{(j)}$ is $\mathscr{M}_j(\lambda) = \begin{pmatrix} M^{(j)}(\lambda) & N_+^{(j)}(\lambda) \\ 0 & \frac{i}{2} I_{\mathscr{H}_2} \end{pmatrix}$, $\lambda \in \mathbb{C}_+$. Clearly, $\mathscr{M}_1(\lambda) = \mathscr{M}_2(\lambda)$ and by Theorem 3.3 there is a unitary operator $U_e \in B(\mathfrak{H}_e^{(1)}, \mathfrak{H}_e^{(2)})$ such that

$$\widetilde{U}_e (A_e^{(1)})^* = (A_e^{(2)})^* \quad \text{and} \quad \Gamma_k^{e(1)} = \Gamma_k^{e(2)} \widetilde{U}_e \upharpoonright (A_e^{(1)})^*, \quad k \in \{0,1\}. \tag{3.22}$$

Let $\gamma_e^{(j)}(\cdot)$ be the γ-field of $\Pi_e^{(j)}$. Since $\widehat{\mathfrak{N}}_\lambda (A_e^{(j)}) = \widehat{\mathfrak{N}}_\lambda (A_j) \oplus \widehat{\mathfrak{N}}_\lambda (A_r)$ and for each $\lambda \in \mathbb{C}_-$ the operators $\Gamma_0^{e(j)} \upharpoonright \widehat{\mathfrak{N}}_\lambda (A_e^{(j)})$ and $P_1 \Gamma_0^{(j)} \upharpoonright \widehat{\mathfrak{N}}_\lambda (A_j))$ are invertible, it follows from (3.20) that the operator $\Gamma_0^{e(j)} \upharpoonright \widehat{\mathfrak{N}}_\lambda (A_r)$ isomorphically maps $\widehat{\mathfrak{N}}_\lambda (A_r)$ onto $\mathscr{H}_2$ and hence $\gamma_e^{(j)}(\lambda)\mathscr{H}_2 = \mathfrak{N}_\lambda (A_r)$, $\lambda \in \mathbb{C}_-$, $j \in \{1,2\}$. Moreover, in view of (3.22) $\gamma_e^{(2)}(\lambda) = U_e \gamma_e^{(1)}(\lambda)$ and, therefore, $U_e \mathfrak{N}_\lambda (A_r) = \mathfrak{N}_\lambda (A_r)$, $\lambda \in \mathbb{C}_-$. This and the equality $\mathfrak{H}_r = \overline{\text{span}} \{\mathfrak{N}_\lambda (A_r) : \lambda \in \mathbb{C}_-\}$ imply that $U_e \mathfrak{H}_r = \mathfrak{H}_r$ and, consequently, $U_e = U \oplus U_r$ with unitary operators $U \in B(\mathfrak{H})$ and $U_r \in B(\mathfrak{H}_r)$. Combining this fact with (3.20), (3.21) and (3.22) one obtains the equalities (3.2). Thus triplets Π_1 and Π_2 are unitarily equivalent.

DEFINITION 3.15. [22, 30] Let $A \in \widetilde{\mathscr{C}}(\mathfrak{H})$, $A \subset A^*$, $n_+(A) = n_-(A)$ and let $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. An operator function $C(\cdot) : \mathbb{C}_+ \to B(\mathscr{H})$ defined by

$$C(\lambda)(\Gamma_1 + i\Gamma_0) \upharpoonright \widehat{\mathfrak{N}}_\lambda (A) = (\Gamma_1 - i\Gamma_0) \upharpoonright \widehat{\mathfrak{N}}_\lambda (A), \quad \lambda \in \mathbb{C}_+ \tag{3.23}$$

is called the characteristic function of A.

For a special triplet Π for A^* the function $C(\cdot)$ coincides with the characteristic function of A in the sense of A. Shtraus [43].

LEMMA 3.16. *Let $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, let $M(\cdot)$ be the Weyl function of Π and let $C(\cdot)$ be the characteristic function of A. Assume also that $\mathscr{H}$ is decomposed as*

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2. \tag{3.24}$$

Then $M(\lambda)$ has the block representation

$$M(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ 0 & M_3 \end{pmatrix} : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}_1 \oplus \mathscr{H}_2, \quad \lambda \in \mathbb{C}_+ \tag{3.25}$$

with the constant entry M_3 if and only if $C(\lambda)$ has the block representation

$$C(\lambda) = \begin{pmatrix} C_1(\lambda) & C_2(\lambda) \\ 0 & C_3 \end{pmatrix} : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}_1 \oplus \mathscr{H}_2, \quad \lambda \in \mathbb{C}_+ \tag{3.26}$$

with the constant entry C_3.

Proof. It follows from (3.23) that $M(\lambda)$ and $C(\lambda)$ are connected via

$$C(\lambda) = (M(\lambda) - i)(M(\lambda) + i)^{-1}, \quad \lambda \in \mathbb{C}_+.$$

Now the immediate checking gives the result.
In the following theorem we characterise in terms of the Weyl functions and characteristic functions symmetric operators A admitting the representation $A = A_1 \oplus A_2$ with the maximal symmetric operator A_2.

THEOREM 3.17. *Let $A \in \mathscr{C}(\mathfrak{H})$ be a simple symmetric operator with equal deficiency indices $n_+(A) = n_-(A)$. Then the following statements are equivalent:*

(1) *There exist decompositions*

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \qquad A = A_1 \oplus A_2, \tag{3.27}$$

where A_1 and A_2 are symmetric operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively such that $n_+(A_2) = 0$, $n_-(A_2) \neq 0$ (this implies that A_2 is maximal symmetric).

(2) *There exist a boundary triplet $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ for A^* and decomposition* (3.24) *with $\mathscr{H}_2 \neq \{0\}$ such that the Weyl function $M(\cdot)$ of Π has the block representation* (3.25) *with the constant entry M_3 or, equivalently, the characteristic function $C(\cdot)$ of A has the block representation* (3.26) *with the constant entry C_3 (see Lemma* 3.16*).*

Moreover, $n_+(A_1) = \dim \mathscr{H}$, $n_-(A_1) = \dim \mathscr{H}_1$ and $n_-(A_2) = \dim \mathscr{H}_2$.

Proof. (1)$\Rightarrow$ (2). Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces with $\dim \mathscr{H}_1 = n_-(A_1)$, $\dim \mathscr{H}_2 = n_-(A_2)$ and let $\mathscr{H}_0 = \mathscr{H}_1 \oplus \mathscr{H}_2$. Since

$$\dim \mathscr{H}_0 = n_-(A_1) + n_-(A_2) = n_-(A) = n_+(A) = n_+(A_1)$$

and hence $n_-(A_1) \leqslant n_+(A_1)$, it follows from Proposition 2.5 that there exists a boundary triplet $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ for A_1^*. Applying to this triplet Lemma 3.12 we obtain a boundary triplet Π_e for A^* with the Weyl function $\mathscr{M}(\cdot)$ of the form (1.10).

(2)$\Rightarrow$ (1). Let $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that for some decomposition (3.24) of $\mathscr{H}$ the Weyl function $M(\cdot)$ of Π has the block representation (3.25). Assume that $K := \operatorname{Re} M_3$, $N := \operatorname{Im} M_3$. Since $M(\cdot) \in R_u[\mathscr{H}]$, it follows that $N > 0$. Let

$$X = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} N^{-\frac{1}{2}} K N^{-\frac{1}{2}} \end{pmatrix}, \qquad Y = \begin{pmatrix} I_{\mathscr{H}_1} & 0 \\ 0 & \frac{1}{\sqrt{2}} N^{-\frac{1}{2}} \end{pmatrix}$$

and let $\mathscr{M}(\lambda) = X + Y^* M(\lambda) Y$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Clearly, $X^* = X$ and the operator Y is invertible. Therefore according to [14, 30] $\mathscr{M}(\cdot)$ is the Weyl function of some boundary triplet $\widetilde{\Pi} = \{\mathscr{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^*. Moreover, the direct calculations show that in the upper half-plane $\mathscr{M}(\lambda)$ is of the form (1.10) with some $M(\lambda)$ and $N_+(\lambda)$. Since

$\mathscr{M}(\cdot) \in R_u[\mathscr{H}]$, it follows from Remark 3.5 that a par (M_+, M_-) with $M_+(\lambda) = (M(\lambda), N_+(\lambda))$, $\lambda \in \mathbb{C}_+$, and $M_-(\lambda) = M_+^*(\overline{\lambda})$, $\lambda \in \mathbb{C}_-$, belongs to $R_u[\mathscr{H}, \mathscr{H}_1]$. Therefore by Theorem 3.13 there exist a Hilbert space $\mathfrak{H}'$, a simple symmetric operator A' in $\mathfrak{H}'$ and a boundary triplet $\Pi' = \{\mathscr{H} \oplus \mathscr{H}_1, \Gamma_0', \Gamma_1'\}$ for $(A')^*$ such that $M_+(\cdot)$ is the Weyl function of Π'. Let $\mathfrak{H}_r$ be a Hilbert space and let A_r be a simple maximal symmetric operator in $\mathfrak{H}_r$ with $n_+(A_r) = 0$ and $n_-(A_r) = \dim \mathscr{H}_2$. Moreover, let

$$\mathfrak{H}_e := \mathfrak{H}' \oplus \mathfrak{H}_r, \qquad A_e = A' \oplus A_r. \tag{3.28}$$

Clearly A_e is a simple symmetric operator in $\mathfrak{H}_e$ and according to Lemma 3.12 there exists a boundary triplet $\Pi_e = \{\mathscr{H}, \Gamma_0^e, \Gamma_1^e\}$ for A_e^* such that the Weyl function of Π_e coincides with $\mathscr{M}(\cdot)$. Thus, triplets $\widetilde{\Pi}$ for A^* and Π_e for A_e^* have the same Weyl function $\mathscr{M}(\cdot)$ and by Theorem 3.3 there exists a unitary operator $U \in B(\mathfrak{H}_e, \mathfrak{H})$ such that $\mathrm{gr} A = \widetilde{U} \mathrm{gr} A_e$. This and (3.28) imply that (3.27) holds with $\mathfrak{H}_1 = U\mathfrak{H}'$, $\mathfrak{H}_2 = U\mathfrak{H}_r$ and symmetric operators $A_1 \in \mathscr{C}(\mathfrak{H}_1)$ and $A_2 \in \mathscr{C}(\mathfrak{H}_2)$ given by $\mathrm{gr} A_1 = \widetilde{U} \mathrm{gr} A'$ and $\mathrm{gr} A_2 = \widetilde{U} \mathrm{gr} A_r$. Moreover, $n_+(A_2) = n_+(A_r) = 0$ and $n_-(A_2) = n_-(A_r) = \dim \mathscr{H}_2 \neq 0$.

COROLLARY 3.18. *Under the assumptions of Theorem 3.17 the following statements are equivalent:*

(1) *There exist decompositions (3.27) with maximal symmetric operators A_1 and A_2 satisfying $n_-(A_1) = n_+(A_2) = 0$.*

(2) *For some (and hence for all) boundary triplet Π for A^* the Weyl function $M(\cdot)$ of Π is constant or, equivalently, the characteristic function $C(\cdot)$ of A is constant.*

Proof. According to [30] the Weyl functions $M(\cdot)$ and $\widetilde{M}(\cdot)$ of boundary triplets $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ and $\widetilde{\Pi} = \{\mathscr{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* are connected by

$$\widetilde{M}(\lambda) = (X_3 + X_4 M(\lambda))(X_1 + X_2 M(\lambda))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

with some operators $X_j \in B(\mathscr{H})$. Therefore if $M(\lambda)$ is constant for some boundary triplet, then the same holds for any triplet. Observe also that in Theorem 3.17 $n_-(A_1) = 0$ if and only if $\mathscr{H}_1 = \{0\}$. This and Theorem 3.17 yield the result.

REMARK 3.19. For a densely defined operator A Corollary 3.18 by another method was proved in [2, 27].

4. Couplings and their Weyl functions

4.1. Couplings of symmetric relations

In the sequel we use the following assumptions:

(A1) $\mathfrak{H}$ and $\mathfrak{H}_r$ are Hilbert spaces, $A \in \widetilde{\mathscr{C}}(\mathfrak{H})$ and $A_r \in \widetilde{\mathscr{C}}(\mathfrak{H}_r)$ are symmetric linear relations with $n_-(A) = n_-(-A_r) \leqslant n_+(A) = n_+(-A_r)$, $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ and $\Pi_r = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0^r, \Gamma_1^r\}$ are boundary triplets for A^* and $(-A_r)^*$ respectively and $\gamma_\pm(\cdot)$ and $M_\pm(\cdot)$ ($\gamma_{r\pm}(\cdot)$ and $M_{r\pm}(\cdot)$) are the γ-fields and the Weyl functions of Π (resp. Π_r) with the block representations

$$\gamma_+(\lambda) = (\gamma(\lambda), \delta_+(\lambda)) : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathfrak{H}, \quad \lambda \in \mathbb{C}_+, \tag{4.1}$$

$$M_+(\lambda) = (M(\lambda), N_+(\lambda)) : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}_1, \quad \lambda \in \mathbb{C}_+, \tag{4.2}$$

$$M_-(\lambda) = (M(\lambda), N_-(\lambda))^\top : \mathscr{H}_1 \to \mathscr{H}_1 \oplus \mathscr{H}_2, \quad \lambda \in \mathbb{C}_-, \tag{4.3}$$

$$\gamma_{r+}(\lambda) = (\gamma_r(\lambda), \delta_{r+}(\lambda)) : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathfrak{H}_r, \quad \lambda \in \mathbb{C}_+, \tag{4.4}$$

$$M_{r+}(\lambda) = (M_r(\lambda), N_{r+}(\lambda)) : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}_1, \quad \lambda \in \mathbb{C}_+, \tag{4.5}$$

$$M_{r-}(\lambda) = (M_r(\lambda), N_{r-}(\lambda))^\top : \mathscr{H}_1 \to \mathscr{H}_1 \oplus \mathscr{H}_2, \quad \lambda \in \mathbb{C}_-. \tag{4.6}$$

(A2) $\widetilde{\mathfrak{H}}$ is a Hilbert space given by

$$\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_r. \tag{4.7}$$

In the following $J_r \in B(\mathfrak{H}_r^2)$ is the operator given by

$$J_r = \begin{pmatrix} I_{\mathfrak{H}_r} & 0 \\ 0 & -I_{\mathfrak{H}_r} \end{pmatrix} : \mathfrak{H}_r \oplus \mathfrak{H}_r \to \mathfrak{H}_r \oplus \mathfrak{H}_r. \tag{4.8}$$

Clearly, the equality $J_r A_r^* = (-A_r)^*$ is valid.

LEMMA 4.1. *Let the assumptions* (A1) *and* (A2) *be fulfilled. Then:*

(1) *The equalities*

$$A_e = A \oplus A_r, \qquad A_e^* = A^* \oplus A_r^* \tag{4.9}$$

define a symmetric relation $A_e \in \widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ and its adjoint A_e^*.

(2) *The Hilbert space $\mathscr{H}_e$ and the operators $\widetilde{\Gamma}_j^e : A_e^* \to \mathscr{H}_e$ given by*

$$\mathscr{H}_e = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1, \tag{4.10}$$

$$\widetilde{\Gamma}_0^e(\widehat{f} \oplus \widehat{f}_r) = P_1\Gamma_0\widehat{f} \oplus \tfrac{i}{2}P_2(\Gamma_0^r J_r \widehat{f}_r + \Gamma_0 \widehat{f}) \oplus P_1\Gamma_0^r J_r \widehat{f}_r (\in \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1), \tag{4.11}$$

$$\widetilde{\Gamma}_1^e(\widehat{f} \oplus \widehat{f}_r) = \Gamma_1\widehat{f} \oplus P_2(\Gamma_0^r J_r \widehat{f}_r - \Gamma_0\widehat{f}) \oplus (-\Gamma_1^r J_r \widehat{f}_r) (\in \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1) \tag{4.12}$$

form a boundary triplet $\widetilde{\Pi}_e = \{\mathscr{H}_e, \widetilde{\Gamma}_0^e, \widetilde{\Gamma}_1^e\}$ for A_e^* (here $\widehat{f} \in A^*$ and $\widehat{f}_r \in A_r^*$).

(3) *For each $\lambda \in \mathbb{C}_-$ the γ-field $\widetilde{\gamma}_e(\lambda)$ of $\widetilde{\Pi}_e$ admits the representation*

$$\widetilde{\gamma}_e(\lambda) = \begin{pmatrix} \gamma_-(\lambda) & 0 & 0 \\ i\delta_{r+}(-\lambda)N_-(\lambda) & -2i\delta_{r+}(-\lambda) & \gamma_r(-\lambda) \end{pmatrix} : \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1 \to \mathfrak{H} \oplus \mathfrak{H}_r. \tag{4.13}$$

Proof.

(1) Statement (1) is obvious.

(2) Let $\widehat{\varphi}=\widehat{f}\oplus\widehat{f}_r$, $\widehat{\psi}=\widehat{g}\oplus\widehat{g}_r\in A_e^*$, where $\widehat{f}=\{f,f'\}$, $\widehat{g}=\{g,g'\}\in A^*$ and $\widehat{f}_r=\{f_r,f_r'\}$, $\widehat{g}_r=\{g_r,g_r'\}\in A_r^*$. Then

$$\widehat{\varphi}=\{\varphi,\varphi'\}=\{f\oplus f_r,f'\oplus f_r'\},\qquad \widehat{\psi}=\{\psi,\psi'\}=\{g\oplus g_r,g'\oplus g_r'\}$$

and hence

$$\begin{aligned}(\varphi',\psi)-(\varphi,\psi')=&(f',g)-(f,g')+(f_r',g_r)-(f_r,g_r')\\ =&(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})+i(P_2\Gamma_0\widehat{f},P_2\Gamma_0\widehat{g})\\ &-(\Gamma_1^rJ_r\widehat{f}_r,\Gamma_0^rJ_r\widehat{g}_r)+(\Gamma_0^rJ_r\widehat{f}_r,\Gamma_1^rJ_r\widehat{g}_r)-i(P_2\Gamma_0^rJ_r\widehat{f}_r,P_2\Gamma_0^rJ_r\widehat{g}_r).\end{aligned}$$

On the other hand

$$\begin{aligned}&(\widetilde{\Gamma}_1^e\widehat{\varphi},\widetilde{\Gamma}_0^e\widehat{\psi})-(\widetilde{\Gamma}_0^e\widehat{\varphi},\widetilde{\Gamma}_1^e\widehat{\psi})\\ =&(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})-\tfrac{i}{2}(P_2\Gamma_0^rJ_r\widehat{f}_r-P_2\Gamma_0\widehat{f},P_2\Gamma_0^rJ_r\widehat{g}_r+P_2\Gamma_0\widehat{g})\\ &-\tfrac{i}{2}(P_2\Gamma_0^rJ_r\widehat{f}_r+P_2\Gamma_0\widehat{f},P_2\Gamma_0^rJ_r\widehat{g}_r-P_2\Gamma_0\widehat{g})-(\Gamma_1^rJ_r\widehat{f}_r,\Gamma_0^rJ_r\widehat{g}_r)+(\Gamma_0^rJ_r\widehat{f}_r,\Gamma_1^rJ_r\widehat{g}_r)\\ =&(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})+i(P_2\Gamma_0\widehat{f},P_2\Gamma_0\widehat{g})-i(P_2\Gamma_0^rJ_r\widehat{f}_r,P_2\Gamma_0^rJ_r\widehat{g}_r)\\ &-(\Gamma_1^rJ_r\widehat{f}_r,\Gamma_0^rJ_r\widehat{g}_r)+(\Gamma_0^rJ_r\widehat{f}_r,\Gamma_1^rJ_r\widehat{g}_r).\end{aligned}$$

This yields the Green identity (1.1) for mappings $\widetilde{\Gamma}_0^e$ and $\widetilde{\Gamma}_1^e$. Surjectivity of the operator $(\widetilde{\Gamma}_0^e,\widetilde{\Gamma}_1^e)^\top$ directly follows from surjectivity of $(\Gamma_0,\Gamma_1)^\top$, $(\Gamma_0^r,\Gamma_1^r)^\top$ and (4.11), (4.12).

(3) Let $\widehat{\gamma}_\pm(\lambda)=(\gamma_\pm(\lambda),\lambda\gamma_\pm(\lambda))^\top$ and $\widehat{\gamma}_{r\pm}(\lambda)=(\gamma_{r\pm}(\lambda),\lambda\gamma_{r\pm}(\lambda))^\top$, $\lambda\in\mathbb{C}_\pm$. Then by (4.1) and (4.4)

$$\begin{aligned}\widehat{\gamma}_+(\lambda)=(\widehat{\gamma}(\lambda),\widehat{\delta}_+(\lambda)):\mathscr{H}_1\oplus\mathscr{H}_2\to\mathfrak{H}^2,\quad \lambda\in\mathbb{C}_+,\\ \widehat{\gamma}_{r+}(\lambda)=(\widehat{\gamma}_r(\lambda),\widehat{\delta}_{r+}(\lambda)):\mathscr{H}_1\oplus\mathscr{H}_2\to\mathfrak{H}_r^2,\quad \lambda\in\mathbb{C}_+,\end{aligned}$$

where $\widehat{\gamma}(\lambda)=(\gamma(\lambda),\lambda\gamma(\lambda))^\top$, $\widehat{\gamma}_r(\lambda)=(\gamma_r(\lambda),\lambda\gamma_r(\lambda))^\top$ and $\widehat{\delta}_+(\lambda)=(\delta_+(\lambda),\lambda\delta_+(\lambda))^\top$, $\widehat{\delta}_{r+}(\lambda)=(\delta_{r+}(\lambda),\lambda\delta_{r+}(\lambda))^\top$. This and (2.8), (2.9) yield

$$P_1\Gamma_0\widehat{\gamma}(\lambda)=P_1\Gamma_0^r\widehat{\gamma}_r(\lambda)=I_{\mathscr{H}_1},\qquad P_1\Gamma_0\widehat{\delta}_+(\lambda)=P_1\Gamma_0^r\widehat{\delta}_{r+}(\lambda)=0,\quad \lambda\in\mathbb{C}_+,\tag{4.14}$$

$$P_2\Gamma_0\widehat{\gamma}(\lambda)=P_2\Gamma_0^r\widehat{\gamma}_r(\lambda)=0,\qquad P_2\Gamma_0\widehat{\delta}_+(\lambda)=P_2\Gamma_0^r\widehat{\delta}_{r+}(\lambda)=I_{\mathscr{H}_2},\quad \lambda\in\mathbb{C}_+,\tag{4.15}$$

$$P_1\Gamma_0\widehat{\gamma}_-(\lambda)=P_1\Gamma_0^r\widehat{\gamma}_{r-}(\lambda)=I_{\mathscr{H}_1},\ \ \lambda\in\mathbb{C}_-;\qquad \Gamma_1\widehat{\gamma}(\lambda)=M(\lambda),\ \ \lambda\in\mathbb{C}_+,\tag{4.16}$$

$$\Gamma_1\widehat{\delta}_+(\lambda)=N_+(\lambda);,\quad \Gamma_1^r\widehat{\gamma}_r(\lambda)=M_r(\lambda),\quad \Gamma_1^r\widehat{\delta}_{r+}(\lambda)=N_{r+}(\lambda),\quad \lambda\in\mathbb{C}_+,\tag{4.17}$$

$$\Gamma_1\widehat{\gamma}_-(\lambda)=M(\lambda), \quad P_2\Gamma_0\widehat{\gamma}_-(\lambda)=-iN_-(\lambda), \quad \lambda\in\mathbb{C}_-, \tag{4.18}$$
$$\Gamma_1^r\widehat{\gamma}_{r-}(\lambda)=M_r(\lambda), \quad P_2\Gamma_0^r\widehat{\gamma}_{r-}(\lambda)=-iN_{r-}(\lambda), \quad \lambda\in\mathbb{C}_-. \tag{4.19}$$

Let $\lambda\in\mathbb{C}_-$, let $\widetilde{\gamma}_e(\lambda)$ be given by (4.13) and let $\widehat{\widetilde{\gamma}}_e(\lambda)=(\widetilde{\gamma}_e(\lambda),\lambda\widetilde{\gamma}_e(\lambda))(\in B(\mathcal{H}_e,\widetilde{\mathfrak{H}}^2))$. Since $\gamma_-(\lambda)\mathcal{H}_1=\mathfrak{N}_\lambda(A)$, $\delta_{r+}(-\lambda)\mathcal{H}_2\subset\mathfrak{N}_{-\lambda}(-A_r)$, $\gamma_r(-\lambda)\mathcal{H}_1\subset\mathfrak{N}_{-\lambda}(-A_r)$ and $\mathfrak{N}_{-\lambda}(-A_r)=\mathfrak{N}_\lambda(A_r)$, $\lambda\in\mathbb{C}_-$, it follows from the equality $\mathfrak{N}_\lambda(A_e)=\mathfrak{N}_\lambda(A)\oplus\mathfrak{N}_\lambda(A_r)$ that $\widetilde{\gamma}_e(\lambda)\mathcal{H}_e\subset\mathfrak{N}_\lambda(A_e)$. Next, the operators $\widehat{\widetilde{\gamma}}_e(\lambda)$ and $\widetilde{\Gamma}_0^e$ have the block representations

$$\widehat{\widetilde{\gamma}}_e(\lambda)=\begin{pmatrix}\widehat{\gamma}_-(\lambda) & 0 & 0\\ iJ_r\widehat{\delta}_{r+}(-\lambda)N_-(\lambda) & -2iJ_r\widehat{\delta}_{r+}(-\lambda) & J_r\widehat{\gamma}_r(-\lambda)\end{pmatrix}:\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_1 \to\mathfrak{H}^2\oplus\mathfrak{H}_r^2$$

$$\widetilde{\Gamma}_0^e=\begin{pmatrix}P_1\Gamma_0 & 0\\ \frac{i}{2}P_2\Gamma_0 & \frac{i}{2}P_2\Gamma_0^rJ_r\upharpoonright A_r^*\\ 0 & P_1\Gamma_0^rJ_r\upharpoonright A_r^*\end{pmatrix}:A^*\oplus A_r^*\to\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_1. \tag{4.20}$$

Therefore for $\lambda\in\mathbb{C}_-$ one has

$$\widetilde{\Gamma}_0^e\widehat{\widetilde{\gamma}}_e(\lambda)=\begin{pmatrix}P_1\Gamma_0 & 0\\ \frac{i}{2}P_2\Gamma_0 & \frac{i}{2}P_2\Gamma_0^rJ_r\upharpoonright A_r^*\\ 0 & P_1\Gamma_0^rJ_r\upharpoonright A_r^*\end{pmatrix}\begin{pmatrix}\widehat{\gamma}_-(\lambda) & 0 & 0\\ iJ_r\widehat{\delta}_{r+}(-\lambda)N_-(\lambda) & -2iJ_r\widehat{\delta}_{r+}(-\lambda) & J_r\widehat{\gamma}_r(-\lambda)\end{pmatrix}$$
$$=\begin{pmatrix}I_{\mathcal{H}_1} & 0 & 0\\ a(\lambda) & I_{\mathcal{H}_2} & 0\\ 0 & 0 & I_{\mathcal{H}_1}\end{pmatrix},$$

where

$$a(\lambda)=\tfrac{i}{2}P_2\Gamma_0\widehat{\gamma}_-(\lambda)-\tfrac{1}{2}P_2\Gamma_0^r\widehat{\delta}_{r+}(-\lambda)N_-(\lambda)=\tfrac{1}{2}N_-(\lambda)-\tfrac{1}{2}N_-(\lambda)=0$$

(here we made use of the relations (4.14) - (4.19)). Thus $\widetilde{\Gamma}_0^e\widehat{\widetilde{\gamma}}_e(\lambda)=I_{\mathcal{H}_e}$ and by Remark 2.15, (ii) $\widetilde{\gamma}_e(\lambda)$ is the γ-field of $\widetilde{\Pi}_e$.

PROPOSITION 4.2. *Let the assumptions* (A1) *and* (A2) *be satisfied and let A_e and A_e^* be the same as in Lemma 4.1. Then:*

(1) *The Hilbert space $\mathcal{H}_e$ of the form* (4.10) *and the operators $\Gamma_j^e:A_e^*\to\mathcal{H}_e$ given by*

$$\Gamma_0^e(\widehat{f}\oplus\widehat{f}_r)=P_1\Gamma_0\widehat{f}\oplus\tfrac{i}{2}P_2(\Gamma_0^rJ_r\widehat{f}_r+\Gamma_0\widehat{f})\oplus\Gamma_1^rJ_r\widehat{f}_r(\in\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_1) \tag{4.21}$$
$$\Gamma_1^e(\widehat{f}\oplus\widehat{f}_r)=\Gamma_1\widehat{f}\oplus P_2(\Gamma_0^rJ_r\widehat{f}_r-\Gamma_0\widehat{f})\oplus P_1\Gamma_0^rJ_r\widehat{f}_r(\in\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_1) \tag{4.22}$$

form a boundary triplet $\Pi_e=\{\mathcal{H}_e,\Gamma_0^e,\Gamma_1^e\}$ for A_e^* (here $\widehat{f}\in A^*$ and $\widehat{f}_r\in A_r^*$).

(2) *The γ-field $\gamma_e(\cdot)$ and the Weyl function $M_e(\cdot)$ of the triplet Π_e are*

$$\gamma_e(\lambda)=\begin{pmatrix}\gamma(\lambda) & -2i\delta_+(\lambda) & i\delta_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 0 & \gamma_{r-}(-\lambda)M_r^{-1}(-\lambda)\end{pmatrix}:\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_1 \tag{4.23}$$
$$\to\mathfrak{H}\oplus\mathfrak{H}_r,\quad \lambda\in\mathbb{C}_+$$

$$\gamma_e(\lambda)=\begin{pmatrix}\gamma_-(\lambda) & 0 & 0\\ * & * & *\end{pmatrix}:\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_1\to\mathfrak{H}\oplus\mathfrak{H}_r,\quad \lambda\in\mathbb{C}_- \tag{4.24}$$

$$M_e(\lambda)=\begin{pmatrix}M(\lambda) & -2iN_+(\lambda) & iN_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 2iI_{\mathscr{H}_2} & -2iN_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 0 & M_r^{-1}(-\lambda)\end{pmatrix}:\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_1 \tag{4.25}$$
$$\to\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_1,\quad \lambda\in\mathbb{C}_+.$$

Proof.

(1) One can easily prove that a collection $\dot{\Pi}_r=\{\mathscr{H}_0\oplus\mathscr{H}_1,\dot{\Gamma}_0^r,\dot{\Gamma}_1^r\}$ with

$$\dot{\Gamma}_0^r=(\Gamma_1^r,P_2\Gamma_0^r)^\top:A_r^*\to\mathscr{H}_1\oplus\mathscr{H}_2,\qquad \dot{\Gamma}_1^r=-P_1\Gamma_0^r \tag{4.26}$$

is a boundary triplet for A_r^*. Substituting $\dot{\Gamma}_j^r$ (instead of Γ_j^r) into (4.11) and (4.12) and taking Lemma 4.1, (2) into account one gets statement (1).

(2) Let $\gamma_e(\lambda)$ be given by (4.23) and let $\widehat{\gamma}_e(\lambda)=(\gamma_e(\lambda),\lambda\gamma_e(\lambda))(\in B(\mathscr{H}_e,\widetilde{\mathfrak{H}}^2))$. Then the same arguments as in Lemma 4.1 show that $\gamma_e(\lambda)\mathscr{H}_e\subset\mathfrak{N}_\lambda(A_e)$. Let $\widehat{\gamma}(\cdot)$, $\widehat{\delta}_+(\cdot)$ and $\widehat{\gamma}_{r-}(\cdot)$ be the same as in Lemma 4.1. Since $\widehat{\gamma}_e(\lambda)$ and Γ_j^e has the block representations

$$\widehat{\gamma}_e(\lambda)=\begin{pmatrix}\widehat{\gamma}(\lambda) & -2i\widehat{\delta}_+(\lambda) & i\widehat{\delta}_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 0 & J_r\widehat{\gamma}_{r-}(-\lambda)M_r^{-1}(-\lambda)\end{pmatrix},\quad \lambda\in\mathbb{C}_+,$$

$$\Gamma_0^e=\begin{pmatrix}P_1\Gamma_0 & 0\\ \frac{i}{2}P_2\Gamma_0 & \frac{i}{2}P_2\Gamma_0^rJ_r\upharpoonright A_r^*\\ 0 & \Gamma_1^rJ_r\upharpoonright A_r^*\end{pmatrix},\qquad \Gamma_1^e=\begin{pmatrix}\Gamma_1 & 0\\ -P_2\Gamma_0 & P_2\Gamma_0^rJ_r\upharpoonright A_r^*\\ 0 & P_1\Gamma_0^rJ_r\upharpoonright A_r^*\end{pmatrix},$$

it follows that

$$\Gamma_0^e\widehat{\gamma}_e(\lambda)=\begin{pmatrix}P_1\Gamma_0 & 0\\ \frac{i}{2}P_2\Gamma_0 & \frac{i}{2}P_2\Gamma_0^rJ_r\upharpoonright A_r^*\\ 0 & \Gamma_1^rJ_r\upharpoonright A_r^*\end{pmatrix}\begin{pmatrix}\widehat{\gamma}(\lambda) & -2i\widehat{\delta}_+(\lambda) & i\widehat{\delta}_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 0 & J_r\widehat{\gamma}_{r-}(-\lambda)M_r^{-1}(-\lambda)\end{pmatrix}$$
$$=\begin{pmatrix}I_{\mathscr{H}_1} & 0 & 0\\ 0 & I_{\mathscr{H}_2} & c(\lambda)\\ 0 & 0 & I_{\mathscr{H}_1}\end{pmatrix},$$

where

$$\begin{aligned} c(\lambda) &= -\tfrac{1}{2}P_2\Gamma_0\widehat{\delta}_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)+\tfrac{i}{2}P_2\Gamma_0^r\widehat{\gamma}_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ &= -\tfrac{1}{2}N_{r-}(-\lambda)M_r^{-1}(-\lambda)+\tfrac{i}{2}(-iN_{r-}(-\lambda))M_r^{-1}(-\lambda)=0 \end{aligned}$$

(here we made use of the relations (4.14) - (4.19)). Thus $\Gamma_0^e\widehat{\gamma}_e(\lambda)=I_{\mathscr{H}_e}$ and by Remark 2.15, (ii) $\gamma_e(\lambda)$ $(\lambda\in\mathbb{C}_+)$ is the γ-field of Π_e. Moreover, applying (2.15) to the triplet Π_e and taking (4.14) - (4.19) into account we obtain that the Weyl function $M_e(\cdot)$ of the triplet Π_e is

$$\begin{aligned} M_e(\lambda) &= \Gamma_1^e\widehat{\gamma}_e(\lambda)\\ &= \begin{pmatrix}\Gamma_1 & 0\\ -P_2\Gamma_0 & P_2\Gamma_0^rJ_r\restriction A_r^*\\ 0 & P_1\Gamma_0^rJ_r\restriction A_r^*\end{pmatrix}\begin{pmatrix}\widehat{\gamma}(\lambda) & -2i\widehat{\delta}_+(\lambda) & i\widehat{\delta}_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 0 & J_r\widehat{\gamma}_{r-}(-\lambda)M_r^{-1}(-\lambda)\end{pmatrix}\\ &= \begin{pmatrix}M(\lambda) & -2iN_+(\lambda) & iN_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & 2iI_{\mathscr{H}_2} & c(\lambda)\\ 0 & 0 & M_r^{-1}(-\lambda)\end{pmatrix}, \quad \lambda\in\mathbb{C}_+, \end{aligned}$$

where

$$\begin{aligned} c(\lambda) &= -iP_2\Gamma_0\widehat{\delta}_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)+P_2\Gamma_0^r\widehat{\gamma}_{r-}(-\lambda)M_r^{-1}(-\lambda)=\\ &= -iN_{r-}(-\lambda)M_r^{-1}(-\lambda)-iN_{r-}(-\lambda)M_r^{-1}(-\lambda)=-2iN_{r-}(-\lambda)M_r^{-1}(-\lambda). \end{aligned}$$

Hence $M_e(\lambda)$ is of the form (4.25). Finally, (4.24) directly follows from Lemma 4.1 applied to the triplet Π_e.

Assume that $\mathfrak{H}$ and $\mathfrak{H}_r$ are Hilbert spaces, $\widetilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}_r$, P (P_r) is the orthoprojector in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ (resp. $\mathfrak{H}_r$) and $\widehat{P}$ $(\widehat{P}_r)$ is the orthoprojector in $\widetilde{\mathfrak{H}}^2$ onto $\mathfrak{H}^2$ (resp. $\mathfrak{H}_r^2$):

$$\widehat{P}\{h,h'\}=\{Ph,Ph'\},\qquad \widehat{P}_r\{h,h'\}=\{P_rh,P_rh'\},\quad \{h,h'\}\in\widetilde{\mathfrak{H}}^2.$$

For a self-adjoint relation $\widetilde{A}\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ we let

$$A=\widetilde{A}\cap\mathfrak{H}^2,\qquad A_r=\widetilde{A}\cap\mathfrak{H}_r^2, \tag{4.27}$$

$$T=\widehat{P}\widetilde{A}=\{\{Ph,Ph'\}:\{h,h'\}\in\widetilde{A}\},\quad T_r=\widehat{P}_r\widetilde{A}=\{\{P_rh,P_rh'\}:\{h,h'\}\in\widetilde{A}\}. \tag{4.28}$$

Clearly, $A\in\widetilde{\mathscr{C}}(\mathfrak{H})$ and $A_r\in\widetilde{\mathscr{C}}(\mathfrak{H}_r)$ are closed symmetric relations and T and T_r are not necessarily closed linear relations in $\mathfrak{H}$ and $\mathfrak{H}_r$ respectively. If T and T_r are operators, then $\widetilde{A}$ is an operator, which is a coupling of T and T_r in the sense of [42].

The following relations are valid [12, 42]:

$$n_+(A)=n_+(-A_r)(=n_-(A_r)),\qquad n_-(A)=n_-(-A_r)(=n_+(A_r)) \tag{4.29}$$

$$A^*=\operatorname{clos}T,\qquad A_r^*=\operatorname{clos}T_r \tag{4.30}$$

$$\operatorname{clos}T=T\iff\operatorname{clos}T_r=T_r \tag{4.31}$$

It follows from (4.30) that $T\subset A^*$ and T is closed if and only if $A^*=T$.

THEOREM 4.3. *Assume that $\mathfrak{H}$ is a Hilbert space, $A \in \widetilde{\mathscr{C}}(\mathfrak{H})$ is a symmetric relation in $\mathfrak{H}$ with $n_-(A) \leqslant n_+(A)$, $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* and $\mathscr{H}_2 = \mathscr{H}_0 \ominus \mathscr{H}_1$, so that $\mathscr{H}_0 = \mathscr{H}_1 \oplus \mathscr{H}_2$. Moreover, let $\mathfrak{H}_r$ be a Hilbert space, let $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_r$ and let $J_r \in B(\mathfrak{H}_r^2)$ be operator (4.8). Then:*

(1) *If $A_r \in \widetilde{\mathscr{C}}(\mathfrak{H}_r)$ is a symmetric relation in $\mathfrak{H}_r$ with $n_\pm(-A_r) = n_\pm(A)$ and $\Pi_r = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0^r, \Gamma_1^r\}$ is a boundary triplet for $(-A_r)^*$, then the equality*

$$\widetilde{A} = \{\widehat{f} \oplus \widehat{f}_r \in A^* \oplus A_r^* : \Gamma_0 \widehat{f} = \Gamma_0^r J_r \widehat{f}_r,\ \Gamma_1 \widehat{f} = \Gamma_1^r J_r \widehat{f}_r\} \tag{4.32}$$

defines a self-adjoint relation $\widetilde{A} \in \widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ such that $A \subset \widetilde{A}$ and

$$\widetilde{A} \cap \mathfrak{H}^2 = A, \qquad A^* = T (= \widehat{P}\widetilde{A}). \tag{4.33}$$

Moreover, $A_r = \widetilde{A} \cap \mathfrak{H}_r^2$.

(2) *If a relation $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ satisfies (4.33) and $A_r = \widetilde{A} \cap \mathfrak{H}_r^2$, then there exists a boundary triplet $\Pi_r = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0^r, \Gamma_1^r\}$ for $(-A_r)^*$ such (4.32) holds.*

Proof.

(1) Let $A_e = A \oplus A_r$. Then according to Proposition 4.2 the equalities (4.10) and (4.21), (4.22) define a boundary triplet $\Pi_e = \{\mathscr{H}_e, \Gamma_0^e, \Gamma_1^e\}$ for A_e^*. Let

$$B = B^* := \begin{pmatrix} 0 & 0 & I_{\mathscr{H}_1} \\ 0 & 0 & 0 \\ I_{\mathscr{H}_1} & 0 & 0 \end{pmatrix} : \underbrace{\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1}_{\mathscr{H}_e} \to \underbrace{\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1}_{\mathscr{H}_e}. \tag{4.34}$$

Then according to Proposition 2.16 the equality

$$\widetilde{A}_B = \{\widehat{f} \oplus \widehat{f}_r \in A^* \oplus A_r^* : \Gamma_1^e(\widehat{f} \oplus \widehat{f}_r) = B\Gamma_0^e(\widehat{f} \oplus \widehat{f}_r)\} \tag{4.35}$$

defines a self-adjoint extension $\widetilde{A}_B$ of A_e and the immediate checking shows that $\widetilde{A} = \widetilde{A}_B$. Hence $\widetilde{A} = \widetilde{A}^*$.

Let us show that $\widetilde{A}$ satisfies (4.33). Since $A \subset A_e \subset \widetilde{A}$, it follows that $A \subset (\widetilde{A} \cap \mathfrak{H}^2)$. Conversely, let $\widehat{f} \in \widetilde{A} \cap \mathfrak{H}^2$. Then $\widehat{f} = \widehat{f} \oplus \widehat{f}_r \in A^* \oplus A_r^*$ with $\widehat{f}_r = 0$ and hence

$$\Gamma_0 \widehat{f} = \Gamma_0^r J_r \widehat{f}_r = 0, \qquad \Gamma_1 \widehat{f} = \Gamma_1^r J_r \widehat{f}_r = 0.$$

Therefore by Proposition 2.6, (1) $\widehat{f} \in A$ and, consequently, $(\widetilde{A} \cap \mathfrak{H}^2) \subset A$. This yields the first equality in (4.33).

Next, in view of (4.30) one has $T \subset A^*$. Conversely, let $\widehat{f} \in A^*$. Since the mapping $(\Gamma_0^r, \Gamma_1^r)^\top$ is surjective, there exist $\widehat{f}_r \in A_r^*$ such that $\Gamma_0^r J_r \widehat{f}_r = \Gamma_0 \widehat{f}$ and $\Gamma_1^r J_r \widehat{f}_r = \Gamma_1 \widehat{f}$. Therefore $\widehat{f} \oplus \widehat{f}_r \in \widetilde{A}$ and hence $\widehat{f} \in T$. This implies that $A^* \subset T$, which gives the second equality in (4.33).

(2) Let $\widetilde{A}=\widetilde{A}^*\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ satisfies (4.33) and let $A_r:=\widetilde{A}\cap\mathfrak{H}_r^2$. Then $A_r\in\widetilde{\mathscr{C}}(\mathfrak{H}_r)$, $A_r\subset A_r^*$ and by (4.30), (4.31) $A_r^*=T_r(=\widehat{P}_r\widetilde{A})$. Hence $J_r\widehat{P}_r\widetilde{A}=J_rA_r^*=(-A_r)^*$. Let $\widehat{f}_e\in\widetilde{A}$ and $J_r\widehat{P}_r\widehat{f}_e=0$. Then $\widehat{f}_e\in\widetilde{A}\cap\mathfrak{H}^2=A$ and hence $\Gamma_j\widehat{P}\widehat{f}_e=\Gamma_j\widehat{f}_e=0$, $j\in\{0,1\}$. Therefore the equalities

$$\widehat{f}_r=J_r\widehat{P}_r\widehat{f}_e,\qquad \Gamma_j^r\widehat{f}_r=\Gamma_j\widehat{P}\widehat{f}_e,\quad \widehat{f}_e\in\widetilde{A} \tag{4.36}$$

correctly define linear operators $\Gamma_j^r:(-A_r)^*\to\mathscr{H}_j$, $j\in\{0,1\}$. Let us show that $\Pi_r=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0^r,\Gamma_1^r\}$ is a boundary triplet for $(-A_r)^*$. Since obviously $J_{\widetilde{\mathfrak{H}}}=\operatorname{diag}(J_{\mathfrak{H}},J_{\mathfrak{H}_r})$ (see (2.1)), it follows from (2.2) that

$$0=(J_{\widetilde{\mathfrak{H}}}\widehat{f}_e,\widehat{g}_e)=(J_{\mathfrak{H}}\widehat{P}\widehat{f}_e,\widehat{P}\widehat{g}_e)+(J_{\mathfrak{H}_r}\widehat{P}_r\widehat{f}_e,\widehat{P}_r\widehat{g}_e),\quad \widehat{f}_e,\widehat{g}_e\in\widetilde{A}. \tag{4.37}$$

Let $\widehat{f}_r,\widehat{g}_r\in(-A_r)^*$. Then there exist $\widehat{f}_e,\widehat{g}_e\in\widetilde{A}$ such that (4.36) holds and $\widehat{g}_r=J_r\widehat{P}_r\widehat{g}_e$, $\Gamma_j^r\widehat{g}_r=\Gamma_j\widehat{P}\widehat{g}_e$, $j\in\{0,1\}$. By using (4.37), (2.7) and the equality $J_r^*J_{\mathfrak{H}_r}J_r=-J_{\mathfrak{H}_r}$ one gets

$$\begin{aligned}-(J_{\mathfrak{H}_r}\widehat{f}_r,\widehat{g}_r)&=-(J_{\mathfrak{H}_r}J_r\widehat{P}_r\widehat{f}_e,J_r\widehat{P}_r\widehat{g}_e)=(J_{\mathfrak{H}_r}\widehat{P}_r\widehat{f}_e,\widehat{P}_r\widehat{g}_e)=-(J_{\mathfrak{H}}\widehat{P}\widehat{f}_e,\widehat{P}\widehat{g}_e)\\&=(\Gamma_1\widehat{P}\widehat{f}_e,\Gamma_0\widehat{P}\widehat{g}_e)-(\Gamma_0\widehat{P}\widehat{f}_e,\Gamma_1\widehat{P}\widehat{g}_e)+i(P_2\Gamma_0\widehat{P}\widehat{f}_e,P_2\Gamma_0\widehat{P}\widehat{g}_e)\\&=(\Gamma_1^r\widehat{f}_r,\Gamma_0^r\widehat{g}_r)-(\Gamma_0^r\widehat{f}_r,\Gamma_1^r\widehat{g}_r)+i(P_2\Gamma_0^r\widehat{f}_r,P_2\Gamma_0^r\widehat{g}_r).\end{aligned}$$

Thus the operators Γ_j^r satisfy the Green identity (2.7).

Next assume that $h_0\in\mathscr{H}_0$ and $h_1\in\mathscr{H}_1$. Then there is $\widehat{f}\in A^*$ such that $\Gamma_0\widehat{f}=h_0$, $\Gamma_1\widehat{f}=h_1$ and by the second equality in (4.33) there is $\widehat{f}_e\in\widetilde{A}$ such that $\widehat{f}=\widehat{P}\widehat{f}_e$. Let $\widehat{f}_r:=J_r\widehat{P}_r\widehat{f}_e$. Then $\widehat{f}_r\in(-A_r)^*$ and by (4.36)

$$\Gamma_j^r\widehat{f}_r=\Gamma_j\widehat{P}\widehat{f}_e=\Gamma_j\widehat{f}=h_j,\quad j\in\{0,1\},$$

which proves surjectivity of the operator $(\Gamma_0^r,\Gamma_1^r)^\top$. Thus Π_r is a boundary triplet for A_r^* and according to statement (1) the equality

$$\widetilde{A}'=\{\widehat{f}\oplus\widehat{f}_r\in A^*\oplus A_r^*:\Gamma_0\widehat{f}=\Gamma_0^rJ_r\widehat{f}_r,\ \Gamma_1\widehat{f}=\Gamma_1^rJ_r\widehat{f}_r\}$$

defines a self-adjoint relation $\widetilde{A}'\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$. Let $\widehat{f}_e\in\widetilde{A}$. Then $\widehat{f}_e=\widehat{f}\oplus\widehat{f}_r$ with $\widehat{f}=\widehat{P}\widehat{f}_e\in A^*$ and $\widehat{f}_r=\widehat{P}_r\widehat{f}_e\in A_r^*$. Moreover, by (4.36) $\Gamma_j\widehat{f}=\Gamma_j\widehat{P}\widehat{f}_e=\Gamma_j^rJ_r\widehat{P}_r\widehat{f}_e=\Gamma_j^rJ_r\widehat{f}_r$, $j\in\{0,1\}$, and hence $\widehat{f}_e\in\widetilde{A}'$. Thus $\widetilde{A}\subset\widetilde{A}'$ and the equality $\widetilde{A}^*=\widetilde{A}$ yields $\widetilde{A}'=\widetilde{A}$. This implies that (4.32) is valid.

DEFINITION 4.4. A self-adjoint relation $\widetilde{A}\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ defined by (4.32) is called the coupling of linear relations A and A_r corresponding to boundary triplets Π for A^* and Π_r for $(-A_r)^*$.

REMARK 4.5. (1) Let A be a symmetric relation in $\mathfrak{H}$ with not necessarily equal deficiency indices $n_\pm(A)$ and let $\Pi=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*. Then according to Theorem 4.3 there is a bijective correspondence between all exit space self-adjoint extensions $\widetilde{A}$ of A satisfying (4.33) and all couplings of A with symmetric relations $A_r \in \widetilde{\mathscr{C}}(\mathfrak{H}_r)$ corresponding to Π and a boundary triplet Π_r for A_r^*.

(2) Assume that A has equal deficiency indices $n_+(A) = n_-(A)$ and $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* with a single Hilbert space $\mathscr{H}$ (see Remark 2.15). Then the coupling in the sense of Definition 4.4 turns into the coupling of A and $A_r \in \widetilde{\mathscr{C}}(\mathfrak{H}_r)$ $(n_+(A_r) = n_-(A_r) = n_\pm(A))$ corresponding to boundary triplets Π for A^* and $\Pi_r = \{\mathscr{H}, \Gamma_0^r, \Gamma_1^r\}$ for A_r^* (see [11, 13]). For this case Theorem 4.3 was proved in [11, 13].

(3) Suppose that A is a densely defined symmetric operator in $\mathfrak{H}$. Then in Definition 2.4 of a boundary triplet $\Pi = \{\mathscr{H}_0 \oplus \mathscr{H}_1, \Gamma_0, \Gamma_1\}$ for A^* one may assume that the operators Γ_0 and Γ_1 are defined on $\operatorname{dom} A^*$. Letting

$$\widetilde{J} = i \begin{pmatrix} 0 & 0 & -I_{\mathscr{H}_1} \\ 0 & -iI_{\mathscr{H}_2} & 0 \\ I_{\mathscr{H}_1} & 0 & 0 \end{pmatrix} : \underbrace{\mathscr{H}_1 \oplus \mathscr{H}_2}_{\mathscr{H}_0} \oplus \mathscr{H}_1 \to \underbrace{\mathscr{H}_1 \oplus \mathscr{H}_2}_{\mathscr{H}_0} \oplus \mathscr{H}_1$$

one rewrites (1.6) as

$$\tfrac{1}{i}[(A^* f, g) - (f, A^* g)] = (\widetilde{J}\Gamma_{A^*} f, \Gamma_{A^*} g), \quad f, g \in \operatorname{dom} A^*,$$

where $\Gamma_{A^*} = (\Gamma_0, \Gamma_1)^\top$. This implies that a linear space $\mathscr{L} = \mathscr{H}_0 \oplus \mathscr{H}_1$ with an indefinite inner product $[h, h'] := (\widetilde{J}h, h')$, $h, h' \in \mathscr{L}$, is a boundary space and $\Gamma_{A^*} : \operatorname{dom} A^* \to \mathscr{L}$ is a boundary operator of A^* in the sense of A.V. Shtraus [41]. Moreover, the coupling of densely defined operators A and A_r in the sense of Definition 4.4 is a coupling of A^* and A_r^* with respect to boundary operators Γ_{A^*} and $\Gamma_{-A_r^*}$ (see [42]). Therefore in the particular case of a densely defined operator A with finite deficiency indices $n_\pm(A)$ Theorem 4.3 follows from [42, Theorems 1 and 2].

4.2. Weyl function of the coupling

THEOREM 4.6. *Let the assumptions* (A1) *and* (A2) *be satisfied and let* $\widetilde{A}$ *be the coupling* (4.32) *of* A *and* A_r. *Moreover, let* $A_e \in \widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ *be symmetric linear relation* (4.9) *and let* J_r *be operator* (4.8)*. Then:*

(i) *The Hilbert space* $\mathscr{H}_e$ *(see* (4.10)*) and the operators* $\Gamma_j^c : A_e^* \to \mathscr{H}_e$ *defined by*

$$\Gamma_0^c(\widehat{f} \oplus \widehat{f}_r) = (\Gamma_1^r J_r \widehat{f}_r - \Gamma_1 \widehat{f}) \oplus P_2(\Gamma_0 \widehat{f} - \Gamma_0^r J_r \widehat{f}_r) \oplus P_1(\Gamma_0 \widehat{f} - \Gamma_0^r J_r \widehat{f}_r) \quad (4.38)$$

$$\Gamma_1^c(\widehat{f} \oplus \widehat{f}_r) = P_1 \Gamma_0 \widehat{f} \oplus \tfrac{i}{2} P_2(\Gamma_0 \widehat{f} + \Gamma_0^r J_r \widehat{f}_r) \oplus \Gamma_1^r J_r \widehat{f}_r \quad (4.39)$$

form a boundary triplet $\Pi_c = \{\mathscr{H}_e, \Gamma_0^c, \Gamma_1^c\}$ *for* A_e^* *(here* $\widehat{f} \in A^*$ *and* $\widehat{f}_r \in A_r^*$*). Moreover, for this triplet* $\widetilde{A} = \ker \Gamma_0^c$.

(ii) $\Phi(\lambda):=-(M(\lambda)-M_r(-\lambda)-iN_+(\lambda)N_{r-}(-\lambda))^{-1}\in B(\mathcal{H}_1)$ $(\lambda\in\mathbb{C}_+)$ *and the Weyl function* $M_c(\cdot)$ *of the triplet* Π_c *has the block representation*

$$M_c(\lambda)=\begin{pmatrix}\Phi(\lambda) & \Phi(\lambda)N_+(\lambda) & I_{\mathcal{H}_1}+\Phi(\lambda)M(\lambda)\\ N_{r-}(-\lambda)\Phi(\lambda) & \frac{i}{2}\mathcal{I}_{\mathcal{H}_2}+N_{r-}(-\lambda)\Phi(\lambda)N_+(\lambda) & N_{r-}(-\lambda)\Phi(\lambda)M(\lambda)\\ M_r(-\lambda)\Phi(\lambda) & M_r(-\lambda)\Phi(\lambda)N_+(\lambda) & M_r(-\lambda)\Phi(\lambda)M(\lambda)\end{pmatrix}, \tag{4.40}$$

$\lambda\in\mathbb{C}_+$, *with respect to decomposition* (4.10) *of* $\mathcal{H}_e$.

Proof. Let $\Pi_e=\{\mathcal{H}_e,\Gamma_0^e,\Gamma_1^e\}$ be boundary triplet (4.21), (4.22) for A_e^*, let $M_e(\cdot)$ be the Weyl function (4.25) of Π_e and let $B=B^*\in B(\mathcal{H}_e)$ be operator (4.34). Then according to [15] the equalities $\Gamma_0^c=B\Gamma_0^e-\Gamma_1^e$ and $\Gamma_1^c=\Gamma_0^e$ define a boundary triplet $\Pi_c=\{\mathcal{H}_e,\Gamma_0^c,\Gamma_1^c\}$ for A_e^* and the Weyl function $M_c(\cdot)$ of this triplet is

$$M_c(\lambda)=(B-M_e(\lambda))^{-1},\quad \lambda\in\mathbb{C}_+. \tag{4.41}$$

The immediate calculation shows that Γ_0^c and Γ_1^c are of the form (4.38) and (4.39). Moreover, the equality $\widetilde{A}=\ker\Gamma_0^c$ directly follows from (4.38).

Assume that

$$\mathcal{M}(\lambda)=\begin{pmatrix}M(\lambda) & N_+(\lambda)\\ 0 & \frac{i}{2}I_{\mathcal{H}_2}\end{pmatrix},\quad \mathcal{M}_r(\lambda)=\begin{pmatrix}M_r(-\lambda) & 0\\ N_{r-}(-\lambda) & -\frac{i}{2}I_{\mathcal{H}_2}\end{pmatrix},\quad \lambda\in\mathbb{C}_+.$$

Since by Theorem 3.13 $(M_+,M_-)\in R_u[\mathcal{H}_0,\mathcal{H}_1]$ and $(M_{r+},M_{r-})\in R_u[\mathcal{H}_0,\mathcal{H}_1]$, it follows from Remark 3.5, (2) that $\operatorname{Im}\mathcal{M}(\lambda)>0$ and $\operatorname{Im}\mathcal{M}_r(\lambda)<0$. Hence the operator

$$\widetilde{\mathcal{M}}(\lambda):=\mathcal{M}(\lambda)-\mathcal{M}_r(\lambda)=\begin{pmatrix}M(\lambda)-M_r(-\lambda) & N_+(\lambda)\\ -N_{r-}(-\lambda) & iI_{\mathcal{H}_2}\end{pmatrix}$$

satisfies $\operatorname{Im}\widetilde{\mathcal{M}}(\lambda)>0$, $\lambda\in\mathbb{C}_+$, and, consequently, the operator $\widetilde{\mathcal{M}}(\lambda)$ is invertible. Therefore the operator

$$M(\lambda)-M_r(-\lambda)-N_+(\lambda)(-iI_{\mathcal{H}_2})(-N_{r-}(-\lambda))=M(\lambda)-M_r(-\lambda)-iN_+(\lambda)N_{r-}(-\lambda)$$

is invertible, which implies that $\Phi(\lambda)\in B(\mathcal{H}_1)$. Moreover,

$$B-M_e(\lambda)=\begin{pmatrix}-M(\lambda) & 2iN_+(\lambda) & I_{\mathcal{H}_1}-iN_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ 0 & -2iI_{\mathcal{H}_2} & 2iN_{r-}(-\lambda)M_r^{-1}(-\lambda)\\ I_{\mathcal{H}_1} & 0 & -M_r^{-1}(-\lambda)\end{pmatrix}$$

and the immediate checking shows that the operator-function $M_c(\cdot)$ of the form (4.40) satisfies

$$(B-M_e(\lambda))M_c(\lambda)=I_{\mathcal{H}_e},\quad \lambda\in\mathbb{C}_+.$$

Therefore by (4.41) $M_c(\cdot)$ is the Weyl function of Π_c.

DEFINITION 4.7. The operator function $M_c(\cdot)$ of the form (4.40) is called the Weyl function of the coupling of A and A_r.

REMARK 4.8. Let in Theorem 4.6 $n_+(A) = n_+(A_r) = n_-(A) = n_-(A_r)$, let $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\}$ ($\widetilde{\Pi}^r = \{\mathscr{H}, \widetilde{\Gamma}_0^r, \widetilde{\Gamma}_1^r\}$) be a boundary triplet for A^* (resp. A_r^*) and let $M(\cdot)$ ($\widetilde{M}_r(\cdot)$) be the Weyl function of Π (resp. $\widetilde{\Pi}^r$). Then the equalities $\Gamma_0^r = \widetilde{\Gamma}_0^r J_r \upharpoonright (-A_r)^*$, $\Gamma_1^r = -\widetilde{\Gamma}_1^r J_r \upharpoonright (-A_r)^*$ define a boundary triplet $\Pi_r = \{\mathscr{H}, \Gamma_0^r, \Gamma_1^r\}$ for $(-A_r)^*$ with the Weyl function $M_r(\lambda) = -\widetilde{M}_r(-\lambda)$ and, therefore, the Weyl function $M_c(\cdot)$ of the coupling of A and A_r is

$$M_c(\lambda) = \begin{pmatrix} -(M(\lambda) + \widetilde{M}_r(\lambda))^{-1} & I_{\mathscr{H}} - (M(\lambda) + \widetilde{M}_r(\lambda))^{-1} M(\lambda) \\ I_{\mathscr{H}} - M(\lambda)(M(\lambda) + \widetilde{M}_r(\lambda))^{-1} & \widetilde{M}_r(\lambda)(M(\lambda) + \widetilde{M}_r(\lambda))^{-1} M(\lambda) \end{pmatrix}, \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Note, that this equality was obtained in [11, 13].

4.3. Parametrization of special exit space extensions

PROPOSITION 4.9. *Let the assumptions* (A1) *and* (A2) *be satisfied, let* $\widetilde{A} \in \widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ *be the self-adjoint extension* (4.32) *of* A *and let* $R(\lambda)$ *be the corresponding generalized resolvent* (2.11) *of* A. *Assume also that* $A_0 = \ker \Gamma_0$ *and*

$$K_0(\lambda) = (I_{\mathscr{H}_1}, -iN_{r-}(-\lambda))^\top : \mathscr{H}_1 \to \mathscr{H}_1 \oplus \mathscr{H}_2, \quad K_1(\lambda) = -M_r(-\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.42}$$

Then

$$R(\lambda) = (A_0 - \lambda)^{-1} - \gamma_+(\lambda) K_0(\lambda)(K_1(\lambda) + M_+(\lambda) K_0(\lambda))^{-1} \gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+. \tag{4.43}$$

Proof. Let $A_e = A \oplus A_r$. Then according to Proposition 4.2 the equalities (4.10) and (4.21), (4.22) define a boundary triplet $\Pi_e = \{\mathscr{H}_e, \Gamma_0^e, \Gamma_1^e\}$ for $A_e^* (= A^* \oplus A_r^*)$ such that the γ-field $\gamma_e(\cdot)$ and the Weyl function $M_e(\cdot)$ of Π_e are of the form (4.23) - (4.25). It follows from (4.32) that $\widetilde{A} = \widetilde{A}_B$, where $\widetilde{A}_B$ is the extension (4.35) of A_e with $B = B^*$ defined by (4.34). Now applying Proposition 2.16 to the triplet Π_e and the extension $\widetilde{A} = \widetilde{A}_B$ one gets

$$(\widetilde{A} - \lambda)^{-1} = (A_0^e - \lambda)^{-1} + \gamma_e(\lambda)(B - M_e(\lambda))^{-1} \gamma_e^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+ \tag{4.44}$$

where $A_0^e = \ker \Gamma_0^e$ is a self-adjoint extension of A_e. It follows from (4.44) that

$$R(\lambda) = P_{\widetilde{\mathfrak{H}}, \mathfrak{H}} (A_0^e - \lambda)^{-1} \upharpoonright \mathfrak{H} + P_{\widetilde{\mathfrak{H}}, \mathfrak{H}} \gamma_e(\lambda)(B - M_e(\lambda))^{-1} \gamma_e^*(\overline{\lambda}) \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C}_+. \tag{4.45}$$

Let $\widehat{f} \in A_0$. Then $\widehat{f} = \widehat{f} \oplus \widehat{f}_r$ with $\widehat{f}_r = 0$ and hence $\widehat{f} \in A_e^* (= A^* \oplus A_r^*)$. Moreover, by (4.21) $\Gamma_0^e \widehat{f} = 0$ and, consequently, $\widehat{f} \in A_0^e$. This implies that A_0^e is an exit space self-adjoint extension of A_0. Since by Proposition 2.6, (2) $(A_0 - \lambda)^{-1} \in B(\mathfrak{H})(\lambda \in \mathbb{C}_+)$, it follows that

$$P_{\widetilde{\mathfrak{H}}, \mathfrak{H}} (A_0^e - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{4.46}$$

Next, in view of (4.23) and (4.24) one has

$$P_{\widetilde{\mathfrak{H}},\mathfrak{H}}\gamma_e(\lambda)=(\gamma(\lambda),\,-2i\delta_+(\lambda),\,i\delta_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)):\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_1\to\mathfrak{H},$$
$$\gamma_e^*(\overline{\lambda})\upharpoonright\mathfrak{H}=(\gamma_-^*(\overline{\lambda}),0,0)^\top:\mathfrak{H}\to\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_1$$

and by (4.41) the operator $(B-M_e(\lambda))^{-1}$ is equal to the right-hand side of (4.40). This implies that

$$\begin{aligned}&P_{\widetilde{\mathfrak{H}},\mathfrak{H}}\gamma_e(\lambda)(B-M_e(\lambda))^{-1}\gamma_e^*(\overline{\lambda})\upharpoonright\mathfrak{H}\\&=\big(\gamma(\lambda),\,-2i\delta_+(\lambda),\,i\delta_+(\lambda)N_{r-}(-\lambda)M_r^{-1}(-\lambda)\big)\begin{pmatrix}I\\N_{r-}(-\lambda)\\M_r(-\lambda)\end{pmatrix}\Phi(\lambda)\gamma_-^*(\overline{\lambda})\\&=-(\gamma(\lambda)-i\delta_+(\lambda)N_{r-}(-\lambda))(M(\lambda)-M_r(-\lambda)-iN_+(\lambda)N_{r-}(-\lambda))^{-1}\gamma_-^*(\overline{\lambda})\\&=-\gamma_+(\lambda)K_0(\lambda)(K_1(\lambda)+M_+(\lambda)K_0(\lambda))^{-1}\gamma_-^*(\overline{\lambda}),\quad\lambda\in\mathbb{C}_+.\end{aligned}$$

Combining of this equality with (4.45) and (4.46) yields (4.43).
In the following theorem we parameterize all exit space extensions $\widetilde{A}$ of a symmetric relation A satisfying (4.33).

THEOREM 4.10. *Assume that $A\in\widetilde{\mathscr{C}}(\mathfrak{H})$ is a symmetric relation in $\mathfrak{H}$ with deficiency indices $n_-(A)\leqslant n_+(A)$, $\Pi=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0,\Gamma_1\}$ is a boundary triplet for A^*, $A_0=\ker\Gamma_0$ and $\gamma_\pm(\cdot)$ and $M_+(\cdot)$ are the γ-fields and the Weyl function of Π respectively. Moreover, let $\tau=\{K_0(\cdot),K_1(\cdot)\}\in\widetilde{R}(\mathscr{H}_0,\mathscr{H}_1)$ and let $\widetilde{A}_\tau$ be the corresponding exit space self-adjoint extension of A defined by (2.12) and (2.13) (see Remark 2.14). Then $\widetilde{A}=\widetilde{A}_\tau$ satisfies (4.33) if and only if $\tau\in\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$.*

Proof. Let $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ be a Hilbert space and let $\widetilde{A}=\widetilde{A}_\tau\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ satisfies (4.33). Moreover, let $\mathfrak{H}_r=\widetilde{\mathfrak{H}}\ominus\mathfrak{H}$ and let $A_r=\widetilde{A}\cap\mathfrak{H}_r^2$. Then according to Theorem 4.3, (2) there exists a boundary triplet $\Pi_r=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0^r,\Gamma_1^r\}$ for $(-A_r)^*$ such that (4.32) holds. Assume that $M_{r\pm}$ are the Weyl functions of Π_r and let $M_{r-}(\lambda)$ has the block representation (4.6). Then according to Proposition 4.9 the generalized resolvent $R(\lambda)=R_\tau(\lambda)$ generated by $\widetilde{A}=\widetilde{A}_\tau$ is of the form (4.43) with operator functions $K_0(\cdot)$ and $K_1(\cdot)$ defined by (4.42). Moreover, since by Theorem 3.13 $(M_{r+},M_{r-})\in R_u[\mathscr{H}_0,\mathscr{H}_1]$, it follows from Lemma 3.6 that $K_0(\cdot)$ and $K_1(\cdot)$ form a pair $\tau'=\{K_0(\cdot),K_1(\cdot)\}\in\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$. Since by Theorem 2.13 $\tau=\tau'$, it follows that $\tau\in\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$.

Conversely, let $\tau\in\widetilde{R}_u(\mathscr{H}_0,\mathscr{H}_1)$. Then according to Lemma 3.6 there exists a pair $(M_{r+},M_{r-})\in R_u[\mathscr{H}_0,\mathscr{H}_1]$ with the block representation (4.6) of $M_{r-}(\lambda)$ such that the operator functions $K_0(\cdot)$ and $K_1(\cdot)$ admit the representation (4.42). Moreover, by Theorem 3.13 there exist a Hilbert space $\mathfrak{H}_r$, a symmetric operator $A_r\in\mathscr{C}(\mathfrak{H}_r)$ and a boundary triplet $\Pi_r=\{\mathscr{H}_0\oplus\mathscr{H}_1,\Gamma_0^r,\Gamma_1^r\}$ for $(-A_r)^*$ such that $M_{r-}(\cdot)$ is the Weyl functions of Π_r. Let $\widetilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}_r$. Then according to Theorem 4.3, (1) the equality (4.32) defines an exit space self-adjoint extension $\widetilde{A}\in\widetilde{\mathscr{C}}(\widetilde{\mathfrak{H}})$ of A such that (4.33) holds and by Proposition 4.9 the corresponding generalized resolvent $R(\lambda)$ of A admits the representation (4.43). Therefore by Theorem 2.13 $\widetilde{A}$ coincides with $\widetilde{A}_\tau$ and hence $\widetilde{A}_\tau$ satisfies (4.33).

REMARK 4.11. It follows from Theorem 4.10 and Remark 2.3, (2) that in the case $n_+(A)=n_-(A)$ and a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* an exit space extension $\widetilde{A}=\widetilde{A}_\tau$ of A satisfies (4.33) if and only if $\tau\in R_u[\mathcal{H}]$. This fact was proved in [13].

Next assume that A is a closed densely defined symmetric operator in $\mathfrak{H}$, so that each ($\mathfrak{H}$-minimal) exit space extension $\widetilde{A}=\widetilde{A}^*$ of A is a densely defined operator as well (see e.g. [13]). Recall that according to M.A. Naimark (see e.g. [1]) an extension $\widetilde{A}=\widetilde{A}^*$ of A on a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ is said to be: (i) of the first kind, if $\operatorname{dom}\widetilde{A}\cap\mathfrak{H}=\operatorname{dom}\widetilde{A}$ (i.e., if $\widetilde{A}$ acts in $\mathfrak{H}$); (ii) of the second kind, if $\operatorname{dom}\widetilde{A}\cap\mathfrak{H}=\operatorname{dom}A$; (iii) of the third kind, if $\operatorname{dom}\widetilde{A}\cap\mathfrak{H}\neq\operatorname{dom}A$ and $\operatorname{dom}\widetilde{A}\cap\mathfrak{H}\neq\operatorname{dom}\widetilde{A}$. The set of all extensions of A of the second kind will be denoted by $\mathrm{Nai}_2(A)$.

A description of the set $\mathrm{Nai}_2(A)$ for an operator A with finite possibly unequal deficiency indices is given in the following theorem.

THEOREM 4.12. *Let in addition to the assumptions of Theorem 4.10 A is a densely defined operator in $\mathfrak{H}$ with finite deficiency indices $n_-(A)\leqslant n_+(A)$. Then $\widetilde{A}_\tau\in\mathrm{Nai}_2(A)$ if and only if $\tau\in\widetilde{R}_u(\mathcal{H}_0,\mathcal{H}_1)$ (so that $K_0(\lambda)$ has the block representation (2.5)) and the operator-function*

$$\mathcal{K}(\lambda)=\begin{pmatrix}K_1(\lambda) & 0\\ -iK_{02}(\lambda) & \frac{i}{2}I_{\mathcal{H}_2}\end{pmatrix};\underbrace{\mathcal{H}_1\oplus\mathcal{H}_2}_{\mathcal{H}_0}\to\underbrace{\mathcal{H}_1\oplus\mathcal{H}_2}_{\mathcal{H}_0},\quad\lambda\in\mathbb{C}_+,$$

satisfies the conditions

$$\lim_{y\to+\infty}\frac{1}{y}\mathcal{K}(iy)=0\quad\text{and}\quad\lim_{y\to+\infty}y\operatorname{Im}(\mathcal{K}(iy)h,h)=\infty,\ \ h\in\mathcal{H}_0\setminus\{0\}.\tag{4.47}$$

Proof. Assume that $\widetilde{A}_\tau$ acts in a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ and let $\mathfrak{H}_r=\widetilde{\mathfrak{H}}\ominus\mathfrak{H}$. Moreover, let T and A_r be operators in $\mathfrak{H}$ and $\mathfrak{H}_r$ given by $\operatorname{gr}T=\widehat{P}\operatorname{gr}\widetilde{A}_\tau$ and $\operatorname{gr}A_r=\operatorname{gr}\widetilde{A}_\tau\cap\mathfrak{H}_r^2$ (see (4.27) and (4.28)). Clearly, A_r is a closed symmetric operator. Moreover, since $n_\pm(A)<\infty$, the operator T is closed, that is $A^*=T$. Therefore by[13, Proposition 7.5]

$$\widetilde{A}_\tau\in\mathrm{Nai}_2(A)\iff\operatorname{gr}A=\operatorname{gr}\widetilde{A}_\tau\cap\mathfrak{H}^2\quad\text{and}\quad\overline{\operatorname{dom}A_r}=\mathfrak{H}_r.\tag{4.48}$$

Since $A^*=T$, it follows from Theorem 4.10 that $\operatorname{gr}A=\operatorname{gr}\widetilde{A}_\tau\cap\mathfrak{H}^2$ if and only if $\tau\in\widetilde{R}_u(\mathcal{H}_0,\mathcal{H}_1)$ (so that $K_0(\lambda)$ has the block representation (2.5)). Next, according to Theorem 4.3 there is a boundary triplet $\Pi_r=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0^r,\Gamma_1^r\}$ for $(-A_r)^*$ such that $\widetilde{A}_\tau$ is a coupling of A and A_r corresponding to triplets Π and Π_r. Let $M_{r-}(\cdot)$ be the Weyl function of Π_r with the block representation (4.6) and let $\mathcal{M}_r(\lambda)$, $\lambda\in\mathbb{C}_-$, be the corresponding operator-function (1.11). Then by Proposition 4.9 $M_{r-}(\cdot)$ and $\tau=\{K_0(\cdot),K_1(\cdot)\}$ are connected via (4.42) and hence $M_r(-\lambda)=-K_1(\lambda)$, $N_{r-}(-\lambda)=iK_{02}(\lambda)$, $\lambda\in\mathbb{C}_+$. Therefore $\mathcal{K}(\lambda)=-\mathcal{M}_r(-\lambda)$, $\lambda\in\mathbb{C}_+$, and by Theorem 2.9 $\overline{\operatorname{dom}A_r}=\mathfrak{H}_r$ if and only if conditions (4.47) are satisfied. This and (4.48) yield the statement of the theorem.

4.4. Example: coupling of symmetric systems

Let $\mathscr{I} = \langle a,b\rangle$, $-\infty \leqslant a < b \leqslant \infty$, be an interval in $\mathbb{R}$. As is known (see e.g [3, 18]) a symmetric differential system on $\mathscr{I}$ is of the form

$$Jy' - B(t)y = \lambda H(t)y, \quad t \in \mathscr{I}, \quad \lambda \in \mathbb{C}, \tag{4.49}$$

where $J \in B(\mathbb{C}^n)$ is an operator ($n\times n$-matrix) satisfying $J^* = J^{-1} = -J$ and $B(t) = B^*(t)$, $H(t) \geqslant 0$ (a.e. on $\mathscr{I}$) are $B(\mathbb{C}^n)$-valued operator functions ($n\times n$-matrix functions) on $\mathscr{I}$ integrable on each compact subinterval $[\alpha,\perp\subset \mathscr{I}$. We assume that system (4.49) is definite. The latter means that for some (and hence all) $\lambda \in \mathbb{C}$ there is only a trivial solution $y(t) \equiv 0$ of (4.49) such that $H(t)y(t) = 0$ (a.e. on $\mathscr{I}$).

Denote by $L^2(H,\mathscr{I})$ the Hilbert space of all (equivalence classes of) vector-functions $f(\cdot) : \mathscr{I} \to \mathbb{C}^n$ such that $\int\limits_{\mathscr{I}} (H(t)f(t), f(t))\, dt < \infty$. With system (4.49) one associates the maximal relation

$$T_{\max} = \{\{y,f\} \in (L^2(H,\mathscr{I}))^2 : y \in AC(\mathscr{I}) \text{ and } Jy'(t) - B(t)y = H(t)f(t) \text{ a.e. on } \mathscr{I}\}$$

and the minimal relation $T_{\min}$ defined as the closure of

$$T'_{\min} := \{\{y,f\} \in T_{\max} : y \text{ has the compact support}\}.$$

It turns out that $T_{\min}$ is a closed symmetric relation in $L^2(H,\mathscr{I})$ with finite deficiency indices $n_\pm(T_{\min}) \leqslant n$, which coincide with the number of linearly independent solutions $y \in L^2(H,\mathscr{I})$ of (4.1) for $\lambda \in \mathbb{C}_\pm$. Moreover, the equality $T_{\max} = T^*_{\min}$ is valid (see e.g. [5, 29]).

In the following we consider system (4.49) on $\mathscr{I} = \mathbb{R}$ and systems on the semiaxes $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0,\infty)$ obtained by restriction of (4.49) onto $\mathbb{R}_-$ and $\mathbb{R}_+$. We assume that all these systems are definite. Moreover, without loss of generality we assume that $n = 2\nu + \widehat{\nu}$ (hence $\mathbb{C}^n = \mathbb{C}^\nu \oplus \mathbb{C}^{\widehat{\nu}} \oplus \mathbb{C}^\nu$) and the operator J is given by (1.15).

Put $\widetilde{\mathfrak{H}} = L^2(H,\mathbb{R})$, $\mathfrak{H} = L^2(H,\mathbb{R}_-)$, $\mathfrak{H}_r = L^2(H,\mathbb{R}_+)$ and denote by $T^-_{\max}$ ($T^+_{\max}$) and $T^-_{\min}$ ($T^+_{\min}$) the maximal and minimal relations in $\mathfrak{H}$ (resp. $\mathfrak{H}_r$) generated by restriction of the system (4.49) onto $\mathbb{R}_-$ (resp. $\mathbb{R}_+$). In the following we assume that the relations $T^-_{\min}$ and $T^+_{\min}$ have minimal deficiency indices (1.16), i.e., system is in the limit point case at ∞ and $-\infty$. Then

$$T^-_{\min} = \{\{y^-,f^-\} \in T^-_{\max} : y^-(0) = 0\}, \qquad T^+_{\min} = \{\{y^+,f^+\} \in T^+_{\max} : y^+(0) = 0\}$$

and $T_{\min} (= T_{\max})$ is a self-adjoint linear relation in $\widetilde{\mathfrak{H}}$ (see e.g. [5, 29]).Moreover, according to [35, 37] the equalities

$$\Gamma_0\{y^-,f^-\} = \{y_3^-(0), -iy_2^-(0)\}, \qquad \Gamma_1\{y^-,f^-\} = y_1^-(0), \quad \{y^-,f^-\} \in T^-_{\max}, \tag{4.50}$$

$$\Gamma_0^r\{y^+,f^+\} = \{y_3^+(0), -iy_2^+(0)\}, \qquad \Gamma_1^r\{y^+,f^+\} = y_1^+(0), \quad \{y^+,f^+\} \in -T^+_{\max} \tag{4.51}$$

define boundary triplets $\Pi=\{\mathbb{C}^{\nu+\widehat{\nu}}\oplus\mathbb{C}^{\nu},\Gamma_0,\Gamma_1\}$ for $T_{\max}^-$ and $\Pi_r=\{\mathbb{C}^{\nu+\widehat{\nu}}\oplus\mathbb{C}^{\nu},\Gamma_0^r,\Gamma_1^r\}$ for $-T_{\max}^+$. In (4.50) and (4.51) $y_j^{\pm}(0)$ are taken from the representations

$$y^- = y^-(t) = y_1^-(t)\oplus y_2^-(t)\oplus y_3^-(t)(\in \mathbb{C}^{\nu}\oplus\mathbb{C}^{\widehat{\nu}}\oplus\mathbb{C}^{\nu}), \quad t\in\mathbb{R}_-,$$
$$y^+ = y^+(t) = y_1^+(t)\oplus y_2^+(t)\oplus y_3^+(t)(\in \mathbb{C}^{\nu}\oplus\mathbb{C}^{\widehat{\nu}}\oplus\mathbb{C}^{\nu}), \quad t\in\mathbb{R}_+$$

of functions $y^-\in\operatorname{dom}T_{\max}^-$ and $y^+\in\operatorname{dom}(-T_{\max}^+)(=\operatorname{dom}T_{\max}^+)$.

Let

$$M_+(\lambda) = (M(\lambda),N_+(\lambda)) : \mathbb{C}^{\nu}\oplus\mathbb{C}^{\widehat{\nu}}\to\mathbb{C}^{\nu}, \quad \lambda\in\mathbb{C}_+,$$
$$M_{r-}(\lambda) = (M_r(\lambda),N_{r-}(\lambda))^{\top} : \mathbb{C}^{\nu}\to\mathbb{C}^{\nu}\oplus\mathbb{C}^{\widehat{\nu}}, \quad \lambda\in\mathbb{C}_-$$

be the Weyl functions of triplets Π and Π_r respectively.

DEFINITION 4.13. [35, 37] The operator-functions

$$m^-(\lambda) = \begin{pmatrix} m_1^-(\lambda) & m_2^-(\lambda)\\ 0 & \frac{i}{2}I_{\widehat{\nu}}\end{pmatrix} := \begin{pmatrix} M(\lambda) & N_+(\lambda)\\ 0 & \frac{i}{2}I_{\widehat{\nu}}\end{pmatrix}, \quad \lambda\in\mathbb{C}_+, \tag{4.52}$$
$$m^+(\lambda) = \begin{pmatrix} m_1^+(\lambda) & 0\\ m_2^+(\lambda) & \frac{i}{2}I_{\widehat{\nu}}\end{pmatrix} := \begin{pmatrix} -M_r(-\lambda) & 0\\ -N_{r-}(-\lambda) & \frac{i}{2}I_{\widehat{\nu}}\end{pmatrix}, \quad \lambda\in\mathbb{C}_+ \tag{4.53}$$

are called the m-functions (Titchmarsh - Weyl functions) for restrictions of the system (4.49) onto $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively.

The Weyl functions $M_{\pm}(\cdot)$ and hence m-functions $m^{\pm}(\cdot)$ can be expressed in terms of values at 0 of certain matrix solutions of the system with entries belonging to $L^2(H,\mathbb{R}_{\pm})$. Moreover, $m^{\pm}(\cdot)\in R[\mathbb{C}^{\nu}\oplus\mathbb{C}^{\widehat{\nu}}]$ and spectral functions of $m_-(\cdot)$ and $m_+(\cdot)$ are matrix pseudo-spectral functions of the dimension $\nu+\widehat{\nu}$ for generalized Fourier transforms generated by restrictions of the system onto $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively (for more details see [35, 37]). Observe also that $m^-(\lambda)=\mathscr{M}(\lambda)$ and $m^+(\lambda)=-\mathscr{M}_r(-\lambda)$, $\lambda\in\mathbb{C}_+$, where $\mathscr{M}(\cdot)$ and $\mathscr{M}_r(\cdot)$ are operator functions (1.10), (1.11) for $M_+(\cdot)$ and $M_{r-}(\cdot)$ respectively.

Let $Y(\cdot,\lambda)(\in B(\mathbb{C}^n))$ be the fundamental solution of the system (4.49) with $Y(0,\lambda)=I_n$. Then according to [9, 40] there exists a unique operator (matrix) function $\Omega(\cdot)\in R[\mathbb{C}^n]$ (the characteristic matrix of the system) such that the resolvent $(T_{\min}-\lambda)^{-1}$ of $T_{\min}(=T_{\min}^*)$ is

$$\left((T_{\min}-\lambda)^{-1}f\right)(x) = \int_{\mathscr{I}} Y(x,\lambda)(\Omega(\lambda)+\tfrac{1}{2}\operatorname{sgn}(t-x)J)Y^*(t,\overline{\lambda})H(t)f(t)\,dt, \quad f\in\mathfrak{H}.$$

This fact enables one to describe spectral properties of $T_{\min}$ in terms of $\Omega(\cdot)$.

It follows from [35, Theorem 5.9] that the following Titchmarsh formula holds

$$\Omega(\lambda) = \begin{pmatrix} \Phi(\lambda) & \Phi(\lambda)m_2^-(\lambda) & \frac{1}{2}I_{\nu}+\Phi(\lambda)m_1^-(\lambda)\\ -m_2^+(\lambda)\Phi(\lambda) & \frac{i}{2}\mathscr{I}_{\widehat{\nu}}-m_2^+(\lambda)\Phi(\lambda)m_2^-(\lambda) & -m_2^+(\lambda)\Phi(\lambda)m_1^-(\lambda)\\ -\frac{1}{2}I_{\nu}-m_1^+(\lambda)\Phi(\lambda) & -m_1^+(\lambda)\Phi(\lambda)m_2^-(\lambda) & -m_1^+(\lambda)\Phi(\lambda)m_1^-\lambda) \end{pmatrix}, \tag{4.54}$$

where $\Phi(\lambda) = -(m_1^-(\lambda)+m_1^+(\lambda)+im_2^-(\lambda)m_2^+(\lambda))^{-1}$, $\lambda \in \mathbb{C}_+$, and $m_j^\pm(\lambda)$ are taken from (4.52), (4.53). If system (4.49) is Hamiltonian (this means that $\widehat{\nu}=0$), then m-functions $m^-(\cdot)$ and $m^+(\cdot)$ in the sense of Definition 4.13 turn into m-functions (Titchmarsh-Weyl functions) for Hamiltonian systems on $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively [17, 20, 24] and (4.54) takes the well known form [21, 25]

$$\Omega(\lambda) = \begin{pmatrix} -(m^-(\lambda)+m^+(\lambda))^{-1} & \frac{1}{2}I_\nu - (m^-(\lambda)+m^+(\lambda))^{-1}m^-(\lambda) \\ -\frac{1}{2}I_\nu + m^+(\lambda)(m^-(\lambda)+m^+(\lambda))^{-1} & m^+(\lambda)(m^-(\lambda)+m^+(\lambda))^{-1}m^-(\lambda) \end{pmatrix}.$$

A connection of abstract objects from this section with symmetric systems is given by the following proposition.

PROPOSITION 4.14. *Let symmetric system (4.49) on $\mathbb{R}$ satisfies the above assumptions and let $\Pi = \{\mathbb{C}^{\nu+\widehat{\nu}} \oplus \mathbb{C}^\nu, \Gamma_0, \Gamma_1\}$ and $\Pi_r = \{\mathbb{C}^{\nu+\widehat{\nu}} \oplus \mathbb{C}^\nu, \Gamma_0^r, \Gamma_1^r\}$ be boundary triplets (4.50), (4.51) for $T_{\max}^-$ and $-T_{\max}^+$ respectively. Then:*

(1) *$T_{\min}$ is a coupling of $A = T_{\min}^-$ and $A_r = T_{\min}^+$ corresponding to boundary triplets Π and Π_r (see Definition 4.4).*

(2) *The characteristic matrix $\Omega(\cdot)$ of the system is associated with the Weyl function $M_c(\cdot)$ of the coupling $T_{\min}$ (see Definition 4.7) via*

$$\Omega(\lambda) = M_c(\lambda) + C,\ \lambda \in \mathbb{C}_+, \tag{4.55}$$

where

$$C = C^* = \begin{pmatrix} 0 & 0 & -\frac{1}{2}I_\nu \\ 0 & 0 & 0 \\ -\frac{1}{2}I_\nu & 0 & 0 \end{pmatrix} : \mathbb{C}^\nu \oplus \mathbb{C}^{\widehat{\nu}} \oplus \mathbb{C}^\nu \to \mathbb{C}^\nu \oplus \mathbb{C}^{\widehat{\nu}} \oplus \mathbb{C}^\nu.$$

Proof. Statement (1) is immediate from (4.50), (4.51) and definition (4.32) of the coupling. Moreover, combining (4.54) with (4.52), (4.53) and (4.40) we arrive at statement (2).

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(Received August 15, 2018)

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