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OPERATORS WITH MINIMAL PSEUDOSPECTRA AND CONNECTIONS TO NORMALITY

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Abstract. This paper mainly studies the class of bounded linear operators A with minimal pseudospectra $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for some $\varepsilon > 0$, where $\sigma(A)$ denotes the spectrum of A, and $\mathbb{D}_\varepsilon$ denotes the open disk of radius ε centered at the origin. Some characterizations of the normality of operators with minimal pseudospectra are provided in terms of only one $\varepsilon-$pseudospectrum. Furthermore, a characterization of the normality of arbitrary $N \times N$ complex matrices is given for $N \leqslant 4$. Some applications to numerical ranges are also presented.

1. Introduction

Let $\mathscr{B}(H)$ denote the Banach algebra of all bounded linear operators on a complex Hilbert space H with the identity I. If H is finite-dimensional, $\dim(H) = N$, we identify $\mathscr{B}(H)$ as the algebra $\mathbb{C}^{N\times N}$ of $N \times N$ complex matrices. Let $\|\cdot\|$ denote the operator norm induced by the inner product associated with H. For $\varepsilon > 0$, the $\varepsilon-$*pseudospectrum* of an operator $A \in \mathscr{B}(H)$ is defined by

$$\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} : \left\|(zI - A)^{-1}\right\| > \frac{1}{\varepsilon} \right\}. \tag{1}$$

By convention we write $\left\|(zI - A)^{-1}\right\| = \infty$ if $z \in \sigma(A)$, the spectrum of A. The pseudospectra $\{\sigma_\varepsilon(A)\}_{\varepsilon>0}$ of A are a family of strictly nested open sets, which grow to fill the whole complex plane as $\varepsilon \to \infty$ and shrink to $\sigma(A)$ as $\varepsilon \to 0$. Any connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$. Furthermore, $\bigcap_{\varepsilon>0} \sigma_\varepsilon(A) = \sigma(A)$, and $\sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A)$ for $0 < \varepsilon_1 < \varepsilon_2$. The pseudospectra can also be defined in terms of perturbations of the spectrum. We have

$$\sigma_\varepsilon(A) := \{ z \in \mathbb{C} : z \in \sigma(A + E) \text{ for some } E \text{ with } \|E\| < \varepsilon \}. \tag{2}$$

Let $\mathbb{D}_\varepsilon(\lambda)$ denote the open disk of radius ε centered at $\lambda \in \mathbb{C}$. For simplicity, we write $\mathbb{D}_\varepsilon$ to represent $\mathbb{D}_\varepsilon(0)$. We always have the inclusion $\sigma(A) + \mathbb{D}_\varepsilon \subseteq \sigma_\varepsilon(A)$ for all $\varepsilon > 0$, where

$$\begin{aligned} \sigma(A) + \mathbb{D}_\varepsilon &:= \{ \lambda + z : \lambda \in \sigma(A) \text{ and } z \in \mathbb{D}_\varepsilon \} \\ &= \bigcup_{\lambda \in \sigma(A)} \mathbb{D}_\varepsilon(\lambda). \end{aligned}$$

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For more details about the pseudospectra and the study of the resolvent norm $\left\|(zI-A)^{-1}\right\|$, we refer the reader to [17, 18, 22] and the references therein.

An operator $A \in \mathscr{B}(H)$ is called *normal* if $AA^* = A^*A$, where A^* is the operator adjoint to A. It is called *self-adjoint* if $A^* = A$. An interesting problem in operator theory is to investigate some conditions under which certain operators are normal. Several mathematicians have paid attention to this problem, see for example [3, 11, 12, 13, 15, 20]. It is well-known that the pseudospectra of a normal operator A are given by $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$. The converse is also valid if H is finite-dimensional. Jianlian Cui et al. [10] showed interesting characterizations of some special operators in terms of the pseudospectrum. In particular, they proved that an operator A is self-adjoint if and only if $\sigma_\varepsilon(A) \subseteq \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \varepsilon\}$ for some $\varepsilon > 0$. Let $\rho(A) := \sup\limits_{z\in\sigma(A)} |z|$ and $\rho_\varepsilon(A) := \sup\limits_{z\in\sigma_\varepsilon(A)} |z|$ denote the spectral radius and the $\varepsilon-$pseudospectral radius of A respectively. Recently, Boling Jia and Youling Feng [14] showed that $A \in \mathscr{B}(H)$ satisfies $\rho_\varepsilon(A) = \rho(A) + \varepsilon$ for all $\varepsilon > 0$ if and only if A is "approximately unitarily similar" to an operator of the form $N \oplus M$, where N is normal and $\rho_\varepsilon(M) \leqslant \rho_\varepsilon(N) = \rho_\varepsilon(A)$. In particular, if $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$, we have $\rho_\varepsilon(A) = \rho(A) + \varepsilon$. The main contribution of this paper is to derive new characterizations of the normality of certain operators in terms of only one $\varepsilon-$pseudospectrum. We also present some applications to the numerical range.

The remainder of this paper is organized as follows. In Section 2, we present new conditions in terms of one $\varepsilon-$pseudospectrum implying the normality of some operators. As an application, we present a simple proof of the non-stability of the spectrum of a quasi-nilpotent operator under perturbations. We also prove that an arbitrary complex matrix $A \in \mathbb{C}^{N\times N}$, $N \leqslant 4$, is normal if and only if there exists $\varepsilon_0 > 0$ such that $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$. Moreover, we construct a counter-example for the case $N \geqslant 5$. If the $\varepsilon-$pseudospectrum of $A \in \mathscr{B}(H)$ consists of disjoint disks of radii ε for some $\varepsilon > 0$, we show that A must be normal. In Section 3, we prove that the numerical ranges of the class of operators with pseudospectra $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ are the closure of the convex hull of their spectra. In Section 4, we use some known results and the main result of Section 3 to show the existence of an invertible operator S such that $\|S\|\left\|S^{-1}\right\| \leqslant 1+\sqrt{2}$, SAS^{-1} has a normal dilation, and the numerical ranges of A and SAS^{-1} have the same closure.

2. Some conditions on an operator implying normality

For a subset X of the complex plane, let $\mathrm{co}(X)$ denote the convex hull of X, $\mathrm{int}(X)$ denote the interior of X, $\overline{X}$ denote the closure of X, ∂X denote the boundary of X, and $\mathrm{d}(z,X)$ denote the Euclidean distance from $z \in \mathbb{C}$ to X. We consider the following conditions that an operator $A \in \mathscr{B}(H)$ may satisfy:

- (G_1) $\left\|(zI-A)^{-1}\right\| = \frac{1}{\mathrm{d}(z,\sigma(A))}$ for all $z \notin \sigma(A)$,

- (β) each point of $\sigma(A)$ is a bare point of $\sigma(A)$ (That is, it lies on the circumference of some closed disk that contains $\sigma(A)$).

Other conditions on a bounded linear operator that have been used to study the normality can be found in [2, 4]. We now add a new condition to the above.

- (β_1) $\sigma(A)$ has empty intersection with the interior of its convex hull, i.e. $\sigma(A)\cap \operatorname{int}(\operatorname{co}(\sigma(A)))=\emptyset$.

The following lemmas play a key role in establishing our results.

LEMMA 2.1. *Let $A\in\mathscr{B}(H)$. Then A satisfies the condition (G_1) if and only if $\sigma_\varepsilon(A)=\sigma(A)+\mathbb{D}_\varepsilon$ for all $\varepsilon>0$.*

Proof. Let $z\in\sigma(A)+\mathbb{D}_\varepsilon$. Then $z=\lambda+\alpha$ for some $\lambda\in\sigma(A)$ and $0\leqslant|\alpha|<\varepsilon$. This leads to

$$z\in\sigma(A)+\alpha I=\sigma(A+\alpha I)\subset\sigma_\varepsilon(A),\quad \text{by}\quad (2).$$

Thus we always have the inclusion $\sigma(A)+\mathbb{D}_\varepsilon\subseteq\sigma_\varepsilon(A)$. By (1), the condition (G_1) implies that $\sigma_\varepsilon(A)\subseteq\sigma(A)+\mathbb{D}_\varepsilon$, and then $\sigma_\varepsilon(A)=\sigma(A)+\mathbb{D}_\varepsilon$ for all $\varepsilon>0$.

Assume now that $\sigma_\varepsilon(A)=\sigma(A)+\mathbb{D}_\varepsilon$ for all $\varepsilon>0$. Let $z\notin\sigma(A)$ and set $\varepsilon=\mathrm{d}(z,\sigma(A))$. Thus z is in the boundary of the set $\sigma(A)+\mathbb{D}_\varepsilon$, and so it is in the boundary of $\sigma_\varepsilon(A)$. Therefore $\left\|(zI-A)^{-1}\right\|=\frac{1}{\varepsilon}=\frac{1}{\mathrm{d}(z,\sigma(A))}$, and then A satisfies (G_1). $\square$

Let $A\in\mathscr{B}(H)$ and M be a closed linear subspace of H. If for every $x\in M$, $Ax\in M$, we say that M is an *invariant subspace of* A. Let $M^\perp$ denote the orthogonal complement of M. If both M and $M^\perp$ are invariant subspaces of A, we say that M is a *reduced subspace of* A. We shall denote the set of all eigenvalues of A by $\sigma_p(A)$, and the eigenspace associated with $\lambda\in\sigma_p(A)$ by $\ker(\lambda I-A)$. We note that an eigenspace of A is an invariant subspace of A, but it is not a reduced subspace in general.

LEMMA 2.2. [10, Lemma 2.1] *Let $\varepsilon>0$ and $A\in\mathscr{B}(H)$. Assume that $\lambda\in\sigma_p(A)$. If $\ker(\lambda I-A)$ is not a reduced subspace of A, then, there exists $r>\varepsilon$ such that $\overline{\mathbb{D}}_r(\lambda)\subset\sigma_\varepsilon(A)$.*

Recall that if H is infinite-dimensional, then A may have no eigenvalues. So, we cannot directly use the previous lemma. Note that $A\in\mathscr{B}(H)$ is not invertible if it is not bounded below: that is, if there is no $c>0$ such that $\|Ax\|\geqslant c\|x\|$ for all $x\in H$. The spectrum thus includes the set of *approximate eigenvalues*, which are those $\lambda\in\mathbb{C}$ such that $\lambda I-A$ is not bounded below: equivalently, it is the set of λ for which there is sequence of unit vectors $x_n\in H$, known as approximate eigenvectors, such that $\lim\limits_{n\to\infty}\|(\lambda I-A)x_n\|=0$. The set of approximate eigenvalues is called the *approximate point spectrum* and denoted by $\sigma_{ap}(A)$. Clearly, $\sigma_p(A)\subseteq\sigma_{ap}(A)$ and it is well-known that $\partial\sigma(A)\subseteq\sigma_{ap}(A)\subseteq\sigma(A)$. In particular, $\sigma_{ap}(A)$ is always nonempty, see [1, Theorem 6.18]. Let K be a Hilbert space containing H as a subspace. A map $\pi:\mathscr{B}(H)\to\mathscr{B}(K)$ is called a *faithful $*-$representation* if $\pi(A+\lambda B)=\pi(A)+\lambda\pi(B)$, $\pi(AB)=\pi(A)\pi(B)$, $\pi(A^*)=\pi(A)^*$, $\pi(I)=I$, and $\|\pi(A)\|=\|A\|$ for all $\lambda\in\mathbb{C}$ and

$A, B \in \mathscr{B}(H)$. Using the approximate eigenvectors, Berberian [5] constructed a Hilbert space K extension of H and a faithful $*-$representation $\pi : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sigma_{ap}(A) = \sigma_{ap}(\pi(A)) = \sigma_p(\pi(A))$. Using Berberian's construction, it becomes natural to speak of eigenvectors and eigenvalues of the representation of every operator $A \in \mathscr{B}(H)$ in $\mathscr{B}(K)$.

We may now use the previous lemmas and present our first main result which gives a characterization of the normality of a class of operators or matrices in terms of only one ε_0-pseudospectrum.

THEOREM 2.1. *Let $A \in \mathscr{B}(H)$ be an operator satisfying the condition (β_1). Then A is normal if and only if $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. In particular, if $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$, then $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$.*

Proof. Assume first that A is normal. By the spectral theorem, the operator A satisfies the condition (G_1). Thus Lemma 2.1 implies that $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$.

Assume now that $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. Berberian's construction implies that there exist a Hilbert space K containing H as a subspace and a faithful $*-$representation $\pi : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sigma_{ap}(A) = \sigma_{ap}(\pi(A)) = \sigma_p(\pi(A))$. The condition (β_1) implies that $\operatorname{int}(\sigma(A)) = \emptyset$, and then $\sigma(A) = \partial\sigma(A)$. Thus, the relation $\partial\sigma(A) \subseteq \sigma_{ap}(A) \subseteq \sigma(A)$ implies that $\sigma(A) = \sigma_{ap}(A)$. Since π is in particular a homomorphism, we obtain $\sigma(A) = \sigma(\pi(A))$ and then

$$\sigma_p(\pi(A)) = \sigma_{ap}(A) = \sigma(A) = \sigma(\pi(A)).$$

This shows that the operator $\pi(A)$ also satisfies the condition (β_1). We will now show that $\pi(A)$ is normal. Since π is a unital homomorphism and $\|\pi(A)\| = \|A\|$ for all $A \in \mathscr{B}(H)$, we get

$$\left\|(zI - \pi(A))^{-1}\right\| = \left\|(\pi(zI - A))^{-1}\right\| = \left\|\pi\left((zI - A)^{-1}\right)\right\| = \left\|(zI - A)^{-1}\right\|$$

for all $z \in \mathbb{C}$. Then Definition (1) implies that $\sigma_\varepsilon(A) = \sigma_\varepsilon(\pi(A))$ for all $\varepsilon > 0$. Hence, the hypothesis $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ is equivalent to

$$\sigma_{\varepsilon_0}(\pi(A)) = \sigma(\pi(A)) + \mathbb{D}_{\varepsilon_0}.$$

Therefore, the condition (β_1) implies that there is no $r > \varepsilon_0$ such that $\overline{\mathbb{D}}_r(\lambda) \subset \sigma_{\varepsilon_0}(\pi(A))$ for all $\lambda \in \sigma(\pi(A))$. By Lemma 2.2, we deduce that the eigenspace $\ker(\lambda I - \pi(A))$ reduces $\pi(A)$ for every $\lambda \in \sigma(\pi(A))$. It follows that $\ker(\mu I - \pi(A))$ and $\ker(\lambda I - \pi(A))$ are orthogonal for distinct complex numbers $\mu, \lambda \in \sigma(\pi(A))$, and then K is spanned by $\left\{\ker(\lambda I - \pi(A)) : \lambda \in \sigma_p(\pi(A)) = \sigma(\pi(A))\right\}$. Therefore, $\pi(A)$ is a diagonal normal operator. That is the underlying Hilbert space of $\pi(A)$ is the closed linear span of the eigenspaces.

Recall that $\pi(A)$ is normal if and only if $\pi(A)^*\pi(A) = \pi(A)\pi(A)^*$. Since π is a faithful $*-$representation, we obtain $\pi(A^*A) = \pi(AA^*)$, and then

$$\|A^*A - AA^*\| = \|\pi(A^*A - AA^*)\| = \|\pi(A^*A) - \pi(AA^*)\| = 0.$$

We deduce that $A^*A = AA^*$, and then A is normal. □

Note that the class of operators satisfying the condition (β) is a subclass of the operators satisfying (β_1). Thus, the same proof of Theorem 2.1 leads to the following result.

COROLLARY 2.1. *Let $A \in \mathscr{B}(H)$ be an operator satisfying the condition (β). Then A is normal if and only if $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$.*

A bounded operator on a Hilbert space is said to be *quasi-nilpotent* if its spectrum is the singleton $\{0\}$. Using the definition (2) of the pseudospectra and Theorem 2.1, we obtain a simple proof of the non-stability of the spectra of quasi-nilpotent operators under perturbations.

COROLLARY 2.2. *Let $A \in \mathscr{B}(H)$ be a quasi-nilpotent operator, but not the zero operator. Then for all $\varepsilon > 0$, there exists $r > \varepsilon$ such that $\mathbb{D}_r \subset \sigma_\varepsilon(A)$.*

Proof. Assume for a contradiction that there exists $\varepsilon_0 > 0$ such that $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$. Thus, $\sigma_{\varepsilon_0}(A) = \mathbb{D}_{\varepsilon_0}$ and so Theorem 2.1 implies that A is normal. By the spectral theorem, we get

$$\|A\| = \max\{|\lambda| : \lambda \in \sigma(A)\} = 0.$$

Thus, $A = 0$, a contradiction. □

As mentioned in the Introduction, a finite-dimensional matrix A is normal if and only if $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$. For an operator satisfying (β_1), a characterization of the normality in terms of one ε_0-pseudospectrum is given in Theorem 2.1. We may ask if it is possible to determine the normality of an arbitrary complex matrix by studying only one pseudospectrum. The following theorems and example address this question. Recall that, for a subset $X \subset \mathbb{C}$, $\mathrm{co}(X)$ denotes the convex hull of X in $\mathbb{C}$ and $\mathrm{int}(X)$ denotes the interior of X in $\mathbb{C}$.

THEOREM 2.2. *Let $A \in \mathbb{C}^{N\times N}$ be a complex matrix such that $N \leqslant 4$. Then A is normal if and only if $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$.*

Proof. If A is normal, then $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for all $\varepsilon_0 > 0$. Assume that there exists some $\varepsilon_0 > 0$ such that $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$. We want to show that A is normal. Note that if A satisfies the condition (β_1), Theorem 2.1 implies that A is normal. Thus, we only need to assume that A is a 4×4 complex matrix which does not satisfy the condition (β_1). Hence, A must have four distinct eigenvalues and only one of them is in $\mathrm{int}(\mathrm{co}(\sigma(A)))$. Let $\lambda_i, 1 \leqslant i \leqslant 3$, be the eigenvalues of A that lie on the boundary of $\mathrm{co}(\sigma(A))$. Since $\sigma_{\varepsilon_0}(A) = \sigma(A) + \mathbb{D}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$, we get $\overline{\mathbb{D}}_r(\lambda_i) \not\subset \sigma_{\varepsilon_0}(A)$

for all $r > \varepsilon_0$. Thus, Lemma 2.2 implies that the eigenspaces $\ker(\lambda_i I - A)$ are reducing subspaces of A. This shows that A is unitarily similar to a diagonal matrix, and then A is normal. □

Below, we construct an example to show that the previous result is not valid in general if $N \geqslant 5$.

EXAMPLE 1. Let $S := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be the 2×2 shift matrix. We have $\sigma_\varepsilon(S) = \mathbb{D}_{\varepsilon'}$, where $\varepsilon' = \sqrt{\varepsilon^2+\varepsilon}$ (see [10, Proposition 2.4]). Let D be a diagonal matrix with eigenvalues $-4, -4i$ and $4+4i$. Note that D is normal, and then its pseudospectra are the union of the disks centered at the eigenvalues with radii ε.
Let now $A := S \oplus D \in \mathbb{C}^{5\times 5}$. The $\varepsilon-$pseudospectrum of A is then given by

$$\sigma_\varepsilon(A) = \sigma_\varepsilon(S) \cup \sigma_\varepsilon(D) = \mathbb{D}_{\varepsilon'} \cup (\sigma(D) + \mathbb{D}_\varepsilon).$$

For $\varepsilon = 10$, we have

$$\sigma_{10}(A) = \sigma_{10}(S) \cup \sigma_{10}(D) = \mathbb{D}_{\sqrt{110}} \cup (\mathbb{D}_{10}(-4) \cup \mathbb{D}_{10}(-4i) \cup \mathbb{D}_{10}(4+4i)).$$

Let $\mathbb{T}_r(\lambda)$ denote the circle of radius r centred at $\lambda \in \mathbb{C}$, i.e. $\mathbb{T}_r(\lambda) = \partial\mathbb{D}_r(\lambda)$. The points of intersection between the circles of radius 10 about the three eigenvalues of D are $\mathbb{T}_{10}(-4) \cap \mathbb{T}_{10}(-4i) = \left\{(-2 \pm \sqrt{46})(1+i)\right\}$, $\mathbb{T}_{10}(-4) \cap \mathbb{T}_{10}(4+4i) = \{-4+10i, 4-6i\}$ and $\mathbb{T}_{10}(-4i) \cap \mathbb{T}_{10}(4+4i) = \{10-4i, -6+4i\}$. In particular, note that $|(-2-\sqrt{46})(1+i)| = (2+\sqrt{46})\sqrt{2} > \sqrt{110}$ and $|-4+10i| = |10-4i| = \sqrt{116} > \sqrt{110}$. Thus

$$\mathbb{D}_{\sqrt{110}} \subset \mathbb{D}_{10}(-4) \cup \mathbb{D}_{10}(-4i) \cup \mathbb{D}_{10}(4+4i),$$

that is $\sigma_{10}(S) = \mathbb{D}_{\sqrt{110}} \subset \sigma_{10}(D)$, see Figure 1. Therefore

$$\sigma_{10}(A) = \sigma_{10}(D) = \sigma(D) + \mathbb{D}_{10} = \sigma(A) + \mathbb{D}_{10},$$

but A is not normal. In fact, for $\varepsilon < \frac{16}{9}$, $\sigma_\varepsilon(A)$ is the union of four disjoint disks such that the radius of one of them is $\sqrt{\varepsilon^2+\varepsilon} > \varepsilon$.

If A is a finite-dimensional normal matrix or a linear bounded normal operator such that $\sigma(A)$ consists of isolated points, then the $\varepsilon-$pseudospectrum of A is the union of disjoint disks of radii ε for some $\varepsilon > 0$. The following theorem shows that this necessary condition is also sufficient.

THEOREM 2.3. *Let $A \in \mathscr{B}(H)$. If there exists $\varepsilon_0 > 0$ such that the ε_0-pseudospectrum $\sigma_{\varepsilon_0}(A)$ of A consists of disjoint disks of radii ε_0, then A is normal.*

Proof. The proof is similar to the proof of Theorem 2.1. The only difference is that the operator A may not satisfy the condition (β_1). Assume that there exists $\varepsilon_0 > 0$ such that $\sigma_{\varepsilon_0}(A)$ consists of disjoint disks of radii ε_0. Then the spectrum $\sigma(A)$ of A consists of isolated points. So we obtain

$$\sigma(A) = \partial\sigma(A) \subseteq \sigma_{ap}(A) \subseteq \sigma(A).$$

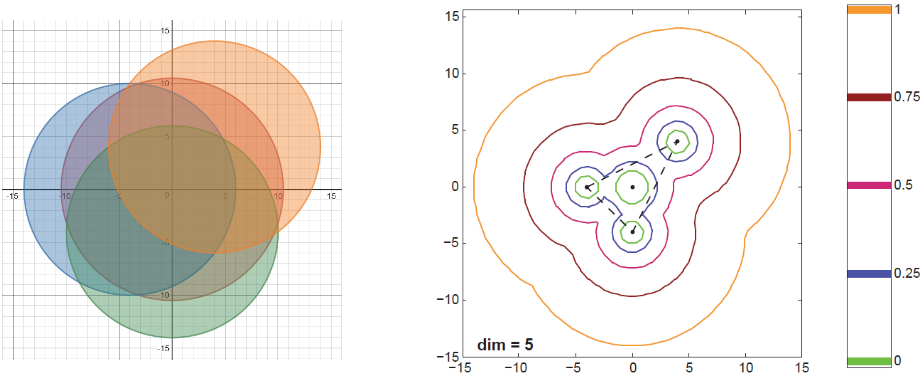

Figure 1: The image on the left shows the 10-pseudospectrum of $A = S \oplus D$, which is the union of the disks centred at $-4, -4i, 4+4i$ with radii 10 and the disk centred at the origin with radius $\sqrt{110}$. The image on the right shows the boundaries of $\sigma_\varepsilon(A)$ for $\varepsilon = 10^0, 10^{0.5}, 10^{0.75}, 10^1$. The solid dots are the eigenvalues of A. The dashed triangle is the boundary of the numerical range of A.

Thus $\sigma(A) = \sigma_{ap}(A)$. By Berberian's construction, there exist a Hilbert space K and a faithful $*-$representation $\pi : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sigma(\pi(A)) = \sigma_p(\pi(A)) = \sigma(A)$. Since $\sigma_{\varepsilon_0}(\pi(A)) = \sigma_{\varepsilon_0}(A)$, Lemma 2.2 implies that all the eigenspaces of $\pi(A)$ are reducing subspaces. This implies that $\pi(A)$ is diagonal normal operator, and then A is normal. □

3. Numerical ranges of operators with minimal pseudospectra

Let $\langle \cdot, \cdot \rangle$ denote the inner product in H. We recall that the numerical range of A is defined by

$$W(A) := \{\langle Ax, x\rangle : x \in H, \ \|x\| = 1\},$$

and let $\overline{W}(A)$ denote its closure in $\mathbb{C}$.

LEMMA 3.1. [21, Theorem 2] *Let $A \in \mathscr{B}(H)$ and $X \subset \mathbb{C}$ be a closed convex subset of the complex plane. Then $\overline{W}(A) \subseteq X$ if and only if*

$$\left\| (zI - A)^{-1} \right\| \leqslant \frac{1}{\mathrm{d}(z, X)} \quad \textit{for all} \quad z \notin X.$$

Using Lemmas 2.1 and 3.1, we obtain the following result.

THEOREM 3.1. *Let $A \in \mathscr{B}(H)$. If $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$, then $\overline{W}(A) = \mathrm{co}(\sigma(A))$.*

Proof. By the spectral inclusion and the convexity of the numerical range, we have $\mathrm{co}(\sigma(A)) \subseteq \overline{W}(A)$. Assume that $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$. By Lemma

2.1, we obtain the following inequality

$$\left\|(zI-A)^{-1}\right\| = \frac{1}{\mathrm{d}(z,\sigma(A))} \leqslant \frac{1}{\mathrm{d}(z,\mathrm{co}(\sigma(A)))}, \quad \text{for all} \quad z \notin \mathrm{co}(\sigma(A)).$$

The inclusion $\overline{W}(A) \subseteq \mathrm{co}(\sigma(A))$ follows then from Lemma 3.1. □

The converse of Theorem 3.1 is true if $\dim(H) \leqslant 4$. This is a consequence of Johnson's result [15, Theorem 3].

PROPOSITION 3.1. [15, Theorem 3] *Let $A \in \mathbb{C}^{N\times N}$. We have $W(A) = \mathrm{co}(\sigma(A))$ if and only if A is normal or A is unitarily similar to $A_1 \oplus A_2$, where A_1 is normal and $W(A_2) \subseteq W(A_1)$.*

Suppose that $A \in \mathbb{C}^{4\times 4}$ is not normal but $W(A) = \mathrm{co}(\sigma(A))$. Then Johnson's proposition implies that A is unitarily similar to $A_1 \oplus A_2$ with A_1 normal and $W(A_2) \subseteq W(A_1)$. This is impossible if A_2 is nonnormal: if A_2 is nonnormal, then it must be 2×2 or 3×3. In these cases, $W(A_1)$ is either a line segment or a point, while $W(A_2)$ has a nontrivial interior in $\mathbb{C}$, making the inclusion $W(A_2) \subseteq W(A_1)$ impossible.

PROPOSITION 3.2. *Let $A \in \mathbb{C}^{N\times N}$ such that $N \leqslant 4$ and $W(A) = \mathrm{co}(\sigma(A))$. Then $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$.*

The converse of Theorem 3.1 is not valid in general if $\dim(H) > 4$. Counterexamples may be constructed via the direct sum similar to the construction of Example 1. In fact, if S is the 2×2 shift matrix and $D := \mathrm{diag}(-4,-4i,4+4i)$, then the numerical range of $A = S \oplus D$ is given by

$$W(A) = \mathrm{co}(W(S) \cup W(D)) = \mathrm{co}\left(\overline{\mathbb{D}}_{\frac{1}{2}} \cup \mathrm{co}(\sigma(D))\right) = \mathrm{co}(\sigma(D)) = \mathrm{co}(\sigma(A))$$

since $\overline{\mathbb{D}}_{\frac{1}{2}} \subset \mathrm{co}(\sigma(D))$ and $0 \in \mathrm{co}(\sigma(D))$, see Figure 1. However, there exist some $\varepsilon > 0$ such that $\sigma_\varepsilon(A) \neq \sigma(A) + \mathbb{D}_\varepsilon$. Indeed, we have $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ if and only if $\sigma_\varepsilon(S) \subseteq \sigma_\varepsilon(D)$.

Although, the converse of Theorem 3.1 is not true in general, we observe the following particular case.

THEOREM 3.2. *Let $A \in \mathscr{B}(H)$ such that $\sigma(A) = \overline{W}(A)$. Then $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$.*

Proof. We always have

$$\sigma(A) + \mathbb{D}_\varepsilon \subseteq \sigma_\varepsilon(A) \text{ for all } \varepsilon > 0. \tag{3}$$

For the converse inclusion, note that (3) is equivalent to

$$\left\|(zI-A)^{-1}\right\| \geqslant \frac{1}{\mathrm{d}(z,\sigma(A))} \quad \text{for all} \quad z \notin \sigma(A).$$

By Lemma 3.1, we get

$$\left\|(zI-A)^{-1}\right\| \leqslant \frac{1}{\mathrm{d}(z,W(A))} = \frac{1}{\mathrm{d}(z,\sigma(A))} \quad \text{for all} \quad z \notin \sigma(A).$$

Therefore,

$$\left\|(zI-A)^{-1}\right\| = \frac{1}{\mathrm{d}(z,\sigma(A))} \quad \text{for all} \quad z \notin \sigma(A).$$

which is equivalent to $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$ by Lemma 2.1. □

REMARK 1. It was shown in [15] that if $\sigma(A) \subseteq \partial W(A)$, then A is normal. In particular, we have $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$. The operators satisfying the hypothesis of Theorem 3.2 may not be normal. Consider for instance the shift operator S in the infinite-dimensional complex Hilbert space $H = \ell^2(\mathbb{N})$. We have $\sigma(S) = \overline{W}(S)$, but S is not normal.

We conclude this section by an application of Theorem 3.1 to Triangular Toeplitz operators. Recall that an *upper triangular Toeplitz operator* T on the Hilbert space $H = \ell^2(\mathbb{N})$ is defined by the infinite matrix

$$T = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ & a_0 & a_1 & a_2 & \cdots \\ & & a_0 & a_1 & \ddots \\ & & & a_0 & \ddots \\ & & & & \ddots \end{pmatrix},$$

where $a_k \in \mathbb{C}$ for $k \geqslant 0$. A Toeplitz operator T is *lower triangular* if its transpose T^{t} is upper triangular. We say T is *triangular* if it is lower or upper triangular.

Let T be a triangular Toeplitz operator. The *symbol* $f(z)$ of T is defined by the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We assume that T has absolutely summable coefficients, i.e. $\sum_{k=0}^{\infty} |a_k| < \infty$. Thus f is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. It is well-known [19] that $\sigma(T) = f\left(\overline{\mathbb{D}}\right)$ and $\sigma_\varepsilon(T) = f\left(\overline{\mathbb{D}}\right) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$. For related results on pseudospectra of Toeplitz operators, we refer the reader to [6, 7].

COROLLARY 3.1. *Let T be a triangular Toeplitz operator with absolutely summable coefficients. Then $\overline{W}(T) = \mathrm{co}(\sigma(T))$.*

Proof. Assume first that T is upper triangular. Then $\sigma_\varepsilon(T) = \sigma(T) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$, and so Theorem 3.1 implies that $\overline{W}(T) = \mathrm{co}(\sigma(T))$. If T is lower triangular, then its transpose T^{t} is upper triangular. Therefore,

$$\overline{W}(T) = \overline{W}(T^{\mathrm{t}}) = \mathrm{co}\left(\sigma(T^{\mathrm{t}})\right) = \mathrm{co}(\sigma(T)). \quad \square$$

We note that Corollary 3.1 is already known in the Toeplitz operator literature, see [6, Theorem 7.11]. It is presented here as an immediate application of Theorem 3.1.

4. Normal $\partial W(A)-$dilation

The observation of this section is motivated by the results obtained in [23] and the fact that similarity transformations do not preserve the numerical range in general. Using pseudospectral tools, we show the existence of an operator S satisfying interesting properties such as the inclusion $\overline{W}(SAS^{-1}) \subseteq \overline{W}(A)$.

Let X be a compact set in the complex plane. We say $A \in \mathscr{B}(H)$ has *a normal* $\partial X-$*dilation* if there is a larger Hilbert space K containing H as a closed subspace and a normal operator N on K with spectrum $\sigma(N) \subseteq \partial X$ such that

$$r(A) = P_H r(N)_{|_H} \quad \text{for all} \quad r \in \mathscr{R}(X),$$

where $P_H : K \to H$ denotes the orthogonal projection of K onto H and $\mathscr{R}(X)$ denotes the algebra of rational functions whose poles lie off X.

THEOREM 4.1. *Let $A \in \mathscr{B}(H)$. Then there exists an invertible operator $S \in \mathscr{B}(H)$ satisfying the following properties.*

(i) $\|S\|\,\|S^{-1}\| \leqslant 1+\sqrt{2}$,

(ii) The operator SAS^{-1} has a normal $\partial W(A)-$dilation,

(iii) $\overline{W}(SAS^{-1}) \subseteq \overline{W}(A)$.

We need some definitions and known results for the proof of the previous theorem. Let X be a compact set in the complex plane. Let $A \in \mathscr{B}(H)$ with $\sigma(A) \subseteq X$, and define $\rho : \mathscr{R}(X) \longrightarrow \mathscr{B}(H)$ by $\rho(r) := r(A)$. The homomorphism ρ is *bounded* if there exists a constant $k > 0$ such that

$$\|\rho\| := \sup_{\|r\|_X = 1} \|\rho(r)\| \leqslant k,$$

where $\|r\|_X := \sup_{z\in X} |r(z)|$. We denote the algebra of matrix-valued rational functions whose poles lie off X by $\mathscr{R}_m(X)$, and set

$$\|\rho\|_{\text{cb}} := \inf\left\{ k > 0 : \|R(A)\| \leqslant k \sup_{z\in X} \|R(z)\| \text{ for all } R \in \mathscr{R}_m(X) \right\}.$$

Then we say that ρ is *completely bounded* if $\|\rho\|_{\text{cb}}$ is finite. If ρ is contractive ($\|\rho\| \leqslant 1$), then X is called a *spectral set* for A. When ρ is only bounded with $\|\rho\| \leqslant k$, then X is called a *k-spectral set* for A. If ρ is completely contractive ($\|\rho\|_{\text{cb}} \leqslant 1$), then we shall call X a *complete spectral set* for A. When ρ is only completely bounded with $\|\rho\|_{\text{cb}} \leqslant k$, then we shall call X a *complete k-spectral set* for A.

Crouzeix-Palencia's Theorem [8]: Let $A \in \mathscr{B}(H)$. Then the homomorphism $f \mapsto f(A)$, from the algebra of all holomorphic functions on $W(A)$ and continuous on

$\overline{W}(A)$ into $\mathscr{B}(H)$, is completely bounded by $1+\sqrt{2}$. In other words, the closure of the numerical range $\overline{W}(A)$ is a complete $1+\sqrt{2}-$spectral set for the operator A.

Paulsen's Theorem [16, Theorem 9.1]: Let $\mathscr{A}$ be an operator algebra, and let $\rho : \mathscr{A} \longrightarrow \mathscr{B}(H)$ be a unital completely bounded homomorphism. Then there exists an invertible operator S with $\|S\|\,\|S^{-1}\| = \|\rho\|_{\mathrm{cb}}$ such that $S\rho(\cdot)S^{-1}$ is a completely contractive homomorphism. In particular, if X is a complete $k-$spectral set for A, then there exists an invertible operator $S \in \mathscr{B}(H)$ such that $\|S\|\,\|S^{-1}\| \leqslant k$ and X is a complete spectral set for SAS^{-1}.

Arveson's Theorem [16, Corollary 7.8]: Let $A \in \mathscr{B}(H)$, and let X be a spectral set for A. Then X is a complete spectral set for A if and only if A has a normal $\partial X-$dilation.

Proof of Theorem 4.1. Crouzeix-Palencia's Theorem shows that $\overline{W}(A)$ is a complete $1+\sqrt{2}-$spectral set for A. Then Paulsen's Theorem implies that there exists an invertible operator $S \in \mathscr{B}(H)$ with $\|S\|\,\|S^{-1}\| \leqslant 1+\sqrt{2}$ such that $\overline{W}(A)$ is a complete spectral set for the operator SAS^{-1}. Therefore, by Arveson's Theorem, the operator SAS^{-1} has a normal $\partial W(A)-$dilation. Hence, there exist a larger Hilbert space K containing H as a subspace and a normal operator $N \in \mathscr{B}(K)$ with spectrum $\sigma(N) \subseteq \partial W(A)$ such that

$$r\left(SAS^{-1}\right) = P_H r(N)_{|H} \quad \text{for all} \quad r \in \mathscr{R}\left(\overline{W}(A)\right),$$

where P_H is the orthogonal projection of K onto H. This completes the proof of (i) and (ii). Let now $x \in H$. Then, we have

$$\langle SAS^{-1}x, x\rangle = \langle P_H N_{|H}x, x\rangle = \langle Nx, x\rangle.$$

Thus, we obtain $\overline{W}(SAS^{-1}) \subseteq \overline{W}(N) = \mathrm{co}(\sigma(N))$. Therefore,

$$\overline{W}(SAS^{-1}) \subseteq \mathrm{co}(\sigma(N)) \subseteq \mathrm{co}(\partial W(A)) = \overline{W}(A).$$

This completes the proof of (iii). $\square$

REMARK 2. The proof of (i) and (ii) in the previous theorem was provided in [9] for the upper bound 12 instead of the sharper constant $1+\sqrt{2}$. We present it here for the sake of completeness.

REMARK 3. Using the pseudospectra, we obtain a different proof of (iii) in Theorem 4.1. We have

$$\left(zI - SAS^{-1}\right)^{-1} = P_H\,(zI-N)^{-1}_{|H} \quad \text{for all} \quad z \notin \overline{W}(A).$$

Then, we get, for all $z \notin \overline{W}(A)$,

$$\left\|\left(zI - SAS^{-1}\right)^{-1}\right\| = \left\|P_H\,(zI-N)^{-1}_{|H}\right\| \leqslant \left\|(zI-N)^{-1}\right\| = \frac{1}{\mathrm{d}(z,\sigma(N))} \leqslant \frac{1}{\mathrm{d}(z,\overline{W}(A))}.$$

Therefore Lemma 3.1 implies that $\overline{W}(SAS^{-1}) \subseteq \overline{W}(A)$.

In the case that $A \in \mathscr{B}(H)$ satisfies $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$, the containment $\overline{W}(SAS^{-1}) \subseteq \overline{W}(A)$ holds with equality.

COROLLARY 4.1. *Let $A \in \mathscr{B}(H)$ be a bounded operator with pseudospectra $\sigma_\varepsilon(A) = \sigma(A) + \mathbb{D}_\varepsilon$ for all $\varepsilon > 0$. Then there exists an invertible operator $S \in \mathscr{B}(H)$ satisfying the following properties.*

(i) $\|S\|\,\|S^{-1}\| \leqslant 1 + \sqrt{2}$,

(ii) The operator SAS^{-1} has a normal $\partial W(A)-$dilation,

(iii) $\overline{W}(SAS^{-1}) = \overline{W}(A)$.

Proof. The properties (i) and (ii) are established in the proof of Theorem 4.1, so we only need to prove (iii). Theorems 3.1 and 4.1 imply that $\overline{W}(SAS^{-1}) \subseteq \overline{W}(A) = \operatorname{co}(\sigma(A))$. On the other hand, we have $\sigma(A) = \sigma(SAS^{-1}) \subseteq \overline{W}(SAS^{-1})$. Thus $\operatorname{co}(\sigma(A)) \subseteq \overline{W}(SAS^{-1})$, and so $\overline{W}(SAS^{-1}) = \overline{W}(A) = \operatorname{co}(\sigma(A))$. $\square$

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