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SOME NEW OPERATOR INEQUALITIES

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Abstract. In this article, we present some new inequalities for positive linear mappings that can be viewed as super multiplicative inequalities. As applications, we deduce some numerical radius inequalities. Then several inequalities for the numerical radius are presented with the aid of convex functions.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a Hilbert space $\mathscr{H}$. If $A \in \mathscr{B}(\mathscr{H})$, the usual operator norm and the numerical radius of A are defined, respectively, by

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \text{ and } \omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

It can be also seen that $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$.

For two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, a linear mapping $\Phi : \mathscr{H}_1 \to \mathscr{H}_2$ is said to be a positive linear mapping if it maps positive operators to positive operators. In this context, recall that an operator $A \in \mathscr{B}(\mathscr{H})$ is said to be positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. Further, Φ is said to be unital if $\Phi(I_1) = I_2$; I_j being the identity operator in $\mathscr{B}(\mathscr{H}_j)$.

The simplest example of a unital positive linear mapping is the state $\Phi_x : \mathscr{B}(\mathscr{H}) \to \mathbb{C}$ defined by $\Phi_x(A) = \langle Ax, x\rangle$, where x is a fixed unit vector.

Notice that for such Φ_x, we do have $\omega(A) = \sup_{\|x\|=1} |\Phi_x(A)|$. Therefore, it is expected that inequalities for positive linear mappings have their roles in obtaining numerical radius inequalities.

This will be our first concern; to obtain new inequalities for positive linear mappings then to apply them and obtain numerical radius inequalities.

More precisely, it is known that a unital positive linear map Φ does not necessarily satisfy the inequality $\Phi(A^*)\Phi(A) \leqslant \Phi(A^*A)$ for an arbitrary A. However, in [5], it is shown that this inequality is true for all 2-positive unital mappings Φ. In this article,

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we prove a generalized form of this inequality that is valid for all unital positive linear mappings. Indeed, we show that (see [13, Exercise 4.1])

$$\Phi(A^*)\Phi(A)+\Phi(A)\Phi(A^*) \leqslant \Phi(A^*A+AA^*)$$

for all unital positive linear mappings $\Phi:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{K})$ and all $A\in\mathscr{B}(\mathscr{H})$. As a direct consequence of this inequality, we obtain

$$\omega(A)^2 \leqslant \frac{1}{2}\|A^*A+AA^*\|;$$

an inequality that has been shown in [10].

After that, reversed versions are shown with their applications.

Once this idea is finished, we move to the second target of this article, summarized as follows. It is well known that $\omega(A)\leqslant\|A\|\leqslant 2\omega(A)$ for any $A\in\mathscr{B}(\mathscr{H})$. Such inequality is important in the computation of $\omega(A)$ because $\|A\|$ is usually easier to compute. However, the two bounds of the inequality are a little distant. This observation led numerous researchers to look for better comparisons between $\omega(A)$ and $\|A\|$. We refer the reader to the recent references [15, 12] for more information.

Recall that if f is a convex function on a real interval J containing the spectrum of the self adjoint operator A, then

$$f(\langle Ax,x\rangle)\leqslant\langle f(A)x,x\rangle. \tag{1.1}$$

Looking at this inequality and the definition of $\omega(A)$, it is expected that theory of convex functions has an impact on inequalities governing the numerical radius. In this article, we study several inequalities between $f(\omega(A))$ and $f(\|A\|)$, where f is a certain function. Our interest will be the classes of geometrically convex and convex functions.

Recall that a function $f:J\subset(0,\infty)\to(0,\infty)$ is said to be geometrically convex if

$$f\left(\sqrt{ab}\right)\leqslant\sqrt{f(a)f(b)}$$

for all $a,b\in J$. We refer the reader to [16] as a reference treating numerical radius inequalities via geometrically convex functions.

The order $A\leqslant B$ for two self adjoint operators A and B means that $B-A$ is positive. That is, $\langle Ax,x\rangle\leqslant\langle Bx,x\rangle$ for all $x\in\mathscr{H}$.

Given a self adjoint operator $A\in\mathscr{B}(\mathscr{H})$ with spectrum in an interval J and a continuous function $f:J\to\mathbb{R}$, the operator $f(A)$ is defined via functional calculus.

In this context, it is important to recall the following observations.

(i) If $T\in\mathscr{B}(\mathscr{H})$ and f is a non-negative increasing function on $[0,\infty)$, then

$$f(\|T\|)=\|f(|T|)\|. \tag{1.2}$$

(ii) (See [2, Corollary 2.6]) If $S,T\in\mathscr{B}(\mathscr{H})$ are two positive operators, then

$$\left\|f\left(\frac{S+T}{2}\right)\right\|\leqslant\frac{1}{2}\|f(S)+f(T)\| \tag{1.3}$$

for every non-negative convex function f on $[0,\infty)$.

The main goal of this article is to present new sharp comparisons between the numerical radius and the operator norm, under the effect of a a certain function. For example, we show that for a certain class of functions f,

$$f(\omega(A)) \leqslant \frac{1}{\sqrt{2}} f(\|A\|)^{\frac{1}{2}} \|f(|A|)+f(|A^*|)\|^{\frac{1}{2}},$$

which will imply

$$\omega(A)^r \leqslant \frac{1}{\sqrt{2}} \|A\|^{\frac{r}{2}} \||A|^r+|A^*|^r\|^{\frac{1}{2}}, \ r \geqslant 1.$$

Indirect consequences of the these inequalities will be sharpened versions of the inequality $\omega(A) \leqslant \|A\|$.

With this theme, several relations will be presented for convex and geometrically convex functions.

Before proceeding to the main results, we present the following comparison between two well known results from [1] and [6].
In [6], the following inequality has been already shown

$$\omega(A)^2 \leqslant \frac{1}{2}\left(\|A\|^2+\omega\left(A^2\right)\right), \tag{1.4}$$

while in [1], the following inequality has been shown

$$\omega(A)^2 \leqslant \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}\omega\left(A^2\right). \tag{1.5}$$

On the other hand, it is shown in [9] that if $A \in \mathscr{B}(\mathscr{H})$, then

$$\left\||A|^2+|A^*|^2\right\| \leqslant \|A^2\|+\|A\|^2.$$

In fact, the inequality (1.5) is stronger than (1.4). Indeed we have the following chain of inequalities:

$$\begin{aligned}\omega(A)^2 &\leqslant \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}\omega\left(A^2\right)\\ &\leqslant \frac{1}{4}\left(\|A^2\|+\|A\|^2\right)+\frac{1}{2}\omega\left(A^2\right)\\ &\leqslant \frac{1}{2}\left(\|A\|^2+\omega\left(A^2\right)\right).\end{aligned}$$

Though, this is not pointed out in [1].

2. Inequalities involving positive linear maps

In this section, we present new super multiplicative inequalities for positive linear mappings with their applications. We know that [5, Corollary 2.8] if Φ is a unital 2-positive linear map, then for any $A \in \mathscr{M}_2$ (the class of all 2×2 matrices)

$$\Phi(A^*)\Phi(A) \leqslant \Phi(A^*A). \tag{2.1}$$

The extended version of this reads as follows.

THEOREM 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then for any unital positive linear map Φ,*

$$\Phi(A^*)\Phi(A)+\Phi(A)\Phi(A^*) \leqslant \Phi(A^*A+AA^*). \tag{2.2}$$

Proof. Let $A = B+iC$ be the Cartesian decomposition of A. Then B and C are self-adjoint and

$$A^*A+AA^* = 2\left(B^2+C^2\right). \tag{2.3}$$

Notice that

$$\begin{aligned}
&\Phi(A^*)\Phi(A)+\Phi(A)\Phi(A^*)\\
=&\Phi\left((B+iC)^*\right)\Phi(B+iC)+\Phi(B+iC)\Phi\left((B+iC)^*\right)\\
=&(\Phi(B)-i\Phi(C))(\Phi(B)+i\Phi(C))+(\Phi(B)+i\Phi(C))(\Phi(B)-i\Phi(C))\\
=&2\left(\Phi^2(B)+\Phi^2(C)\right)\\
\leqslant&2\left(\Phi\left(B^2\right)+\Phi\left(C^2\right)\right) \quad \text{(by (2.1))}\\
=&\Phi\left(2\left(B^2+C^2\right)\right)\\
=&\Phi(A^*A+AA^*) \quad \text{(by (2.3))}.
\end{aligned}$$

This completes the proof.

As an application of Theorem 2.1, we have the following numerical radius inequality due to Kittaneh [10].

COROLLARY 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega(A)^2 \leqslant \frac{1}{2}\|A^*A+AA^*\|. \tag{2.4}$$

Proof. Fix a unit vector $x \in \mathscr{H}$ and define for $T \in \mathscr{B}(\mathscr{H})$, $\Phi(T) = \langle Tx, x\rangle$. Then since Φ is positive unital, Theorem 2.1 assures that

$$2\langle A^*x,x\rangle\langle Ax,x\rangle \leqslant \langle (A^*A+AA^*)x,x\rangle.$$

Taking the supremum over $\|x\| = 1$ implies the desired inequality.

As a complementary result to Theorem 2.1 we have the following reversed version.

THEOREM 2.2. *Let $A \in \mathscr{B}(\mathscr{H})$. Then for any unital positive linear map Φ,*

$$\Phi(A^*A+AA^*)-\left(\Phi(A)^*\Phi(A)+\Phi(A)\Phi(A)^*\right) \leqslant 2\Delta(A)^2, \tag{2.5}$$

where $\Delta(A) = \inf_{z\in\mathbb{C}}\|A-z\|$.

Proof. Since A^*A+AA^* is self-adjoint, then $A^*A+AA^* \leqslant 2\|A\|^2$. Therefore,

$$\Phi(A^*A+AA^*)-\left(\Phi(A)^*\Phi(A)+\Phi(A)\Phi(A)^*\right) \leqslant 2\|A\|^2. \tag{2.6}$$

On the other hand, if we replace A by $A-z$ in (2.6) we get

$$\Phi\left(A^*A+AA^*\right)-\left(\Phi(A)^*\Phi(A)+\Phi(A)\Phi(A)^*\right)\leqslant 2\|A-z\|^2.$$

By taking infimum over $z\in\mathbb{C}$, we get (2.5).

COROLLARY 2.2. *Let $A\in\mathscr{B}(\mathscr{H})$. Then*

$$0\leqslant\frac{1}{2}\left\|A^*A+AA^*\right\|-\omega(A)^2\leqslant\Delta(A)^2, \tag{2.7}$$

where $\Delta(A)=\inf_{z\in\mathbb{C}}\|A-z\|$.

Proof. If we take $\Phi(T)=\langle Tx,x\rangle$ in (2.2) and (2.5), then we have

$$|\langle Ax,x\rangle|^2\leqslant\frac{1}{2}\left\langle A^*A+AA^*x,x\right\rangle,$$

and

$$\langle A^*A+AA^*x,x\rangle-2|\langle Ax,x\rangle|^2\leqslant 2\Delta(A)^2.$$

Now, by taking supremum over $x\in\mathscr{H}$ with $\|x\|=1$ we get the desired result.

REMARK 2.1. In [10], it has been shown that

$$\frac{1}{4}\|A^*A+AA^*\|\leqslant\omega(A)^2, \tag{2.8}$$

which is the reversed version of (2.4). Notice that this inequality is not optimal in the trivial case $A=I$, the identity.
At this stage, it is interesting to compare between our inequality (2.7) and (2.8). This reduces to the comparison between

$$\Delta(A)^2 \text{ and } \frac{1}{4}\|A^*A+AA^*\|. \tag{2.9}$$

The inequality (2.7) would be sharper than (2.8) provided that $\Delta(A)^2\leqslant\frac{1}{4}\|A^*A+AA^*\|$.

Unfortunately, we do not have an explicit comparison between the two quantities. However, we see that when $A=I$, the identity, we have $\Delta(A)^2=0$ and $\frac{1}{4}\|A^*A+AA^*\|=\frac{1}{2}$; making (2.7) sharper than (2.8) for the case $A=I$.

Moreover, it turns out that (2.7) is sharper than (2.8) when A is a positive definite matrix. Notice than, for such matrices, $\Delta(A)=r_A$ where r_A is the radius of the smallest interval containing the eigenvalues of A, see [4]. So, if A is positive with smallest eigenvalue m and largest eigenvalue M, we have

$$\Delta(A)=\frac{M-m}{2}\Rightarrow\Delta(A)^2\leqslant\frac{M^2}{4}.$$

On the other hand, for such A,

$$\frac{1}{4}\|A^*A+AA^*\|=\frac{\|A\|^2}{2}=\frac{M^2}{2}.$$

Therefore, for $0 < m \leqslant A \leqslant M$, we do have

$$\Delta(A)^2 \leqslant \frac{1}{4}\|A^*A + AA^*\|,$$

and hence (2.7) is sharper than (2.8).

Notice that (3.4) provides an additive reverse of (2.2) that is valid for all operators $A \in \mathscr{B}(\mathscr{H})$. In what follows, we present a multiplicative version that is valid only for accretive-dissipative operators. Recall that an operator $A \in \mathscr{B}(\mathscr{H})$ is said to be accretive-dissipative if $A = B + iC$ for positive operators B and C. First, a lemma from [11].

LEMMA 2.1. *Let Φ be a unital positive map on $\mathscr{B}(\mathscr{H})$. If $A \in \mathscr{B}(\mathscr{H})$ is a positive operator satisfying $0 < m \leqslant A \leqslant M$ for some scalars $m < M$, then*

$$\Phi\left(A^2\right) \leqslant \frac{(M+m)^2}{4Mm}\Phi(A)^2. \tag{2.10}$$

THEOREM 2.3. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ such that $0 < m \leqslant B, C \leqslant M$ for some scalars $m < M$. Then for any unital positive linear map Φ,*

$$\Phi\left(A^*A + AA^*\right) \leqslant \frac{(M+m)^2}{4Mm}\left(\Phi\left(A^*\right)\Phi\left(A\right) + \Phi\left(A\right)\Phi\left(A^*\right)\right). \tag{2.11}$$

Proof. By employing the inequality (2.10), one can write

$$\begin{aligned}\Phi\left(A^*A + AA^*\right) = 2\left(\Phi\left(B^2\right) + \Phi\left(C^2\right)\right) &\leqslant \frac{(M+m)^2}{2Mm}\left(\Phi(B)^2 + \Phi(C)^2\right)\\ &= \frac{(M+m)^2}{4Mm}\left(\Phi\left(A^*\right)\Phi\left(A\right) + \Phi\left(A\right)\Phi\left(A^*\right)\right),\end{aligned}$$

which proves (2.11).
As an application, we have the following comparison between $\|A\|$ and $\omega(A)$.

COROLLARY 2.3. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ such that $0 < m \leqslant B, C \leqslant M$ for some scalars $m < M$. Then*

$$\|A\| \leqslant \frac{M+m}{\sqrt{2mM}}\,\omega(A). \tag{2.12}$$

Proof. Letting $\Phi(A) = \langle Ax, x\rangle$ in Theorem 2.3 and taking the supremum over $\|x\| = 1$, we obtain

$$\frac{1}{2}\left\|A^*A + AA^*\right\| \leqslant \frac{(M+m)^2}{4Mm}\omega(A)^2. \tag{2.13}$$

On the other hand, we have [9, Inequality (33)] $\|A\|^2 \leqslant \|A^*A + AA^*\|$. Therefore, (2.13) implies

$$\|A\| \leqslant \frac{M+m}{\sqrt{2mM}}\,\omega(A),$$

as desired.

It is worth mentioning here that if $A \in \mathscr{B}(\mathscr{H})$ is a positive operator satisfying $0 < m \leqslant A \leqslant M$ for some scalars $m < M$, then (2.5) implies that (see [3])

$$\Phi\left(A^2\right) \leqslant \Phi(A)^2 + \frac{(M-m)^2}{4}. \tag{2.14}$$

By applying the same procedure as in the proof of Theorem 2.3, and using the inequality (2.14) we get

$$\Phi(A^*A + AA^*) \leqslant (M-m)^2 + \Phi(A^*)\Phi(A) + \Phi(A)\Phi(A^*). \tag{2.15}$$

REMARK 2.2. It follows from inequality (2.1) that

$$|\Phi(A)|^2 \leqslant \Phi\left(|A|^2\right), \tag{2.16}$$

whenever Φ is a unital 2-positive linear map. However, the inequality $|\Phi(A)| \leqslant \Phi(|A|)$ is not always true. If $0 < m \leqslant |A| \leqslant M$ and Φ is unital and 2-positive, then

$$|\Phi(A)| \leqslant \frac{M+m}{2\sqrt{Mm}}\Phi(|A|).$$

Actually, if Φ is unital and 2-positive, then we have

$$\begin{aligned} |\Phi(A)|^2 &\leqslant \Phi\left(|A|^2\right) \quad \text{(by (2.16))} \\ &\leqslant \frac{(M+m)^2}{4Mm}\Phi(|A|)^2 \quad \text{(by (2.10))}. \end{aligned}$$

Since the function $f(t) = \sqrt{t}$ is operator monotone, taking square root of both sides implies the desired inequality.

3. More estimates of numerical radii using convex functions

We begin this section by extending the inequality (1.5) to the following general form.

THEOREM 3.1. *Let $A \in \mathscr{B}(\mathscr{H})$ and let f be a non-negative increasing convex function. Then*

$$f\left(\omega(A)^2\right) \leqslant \frac{1}{4}\left\|f\left(|A|^2\right) + f\left(|A^*|^2\right)\right\| + \frac{1}{2}f\left(\omega\left(A^2\right)\right).$$

In particular, for any $r \geqslant 1$,

$$\omega(A)^{2r} \leqslant \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}\omega\left(A^2\right)^r.$$

Proof. Since f is increasing and convex, the inequality (1.5) implies

$$\begin{aligned}
f\left(\omega(A)^2\right) &\leqslant f\left(\frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}\omega\left(A^2\right)\right) \\
&\leqslant \frac{1}{2}\left(f\left(\left\|\frac{|A|^2+|A^*|^2}{2}\right\|\right)+f\left(\omega\left(A^2\right)\right)\right) \\
&= \frac{1}{2}\left\|f\left(\frac{|A|^2+|A^*|^2}{2}\right)\right\|+\frac{1}{2}f\left(\omega\left(A^2\right)\right) \quad \text{(by (1.2))} \\
&\leqslant \frac{1}{4}\left\|f\left(|A|^2\right)+f\left(|A^*|^2\right)\right\|+\frac{1}{2}f\left(\omega\left(A^2\right)\right) \quad \text{(by (1.3))},
\end{aligned}$$

as desired.
Notice that (1.5) follows from Theorem 3.1 by letting $f(t)=t$.

THEOREM 3.2. *Let $A\in\mathscr{B}(\mathscr{H})$ and let f be a non-negative increasing geometrically convex function. If in addition f is convex, then*

$$f(\omega(A)) \leqslant \frac{1}{\sqrt{2}} f(\|A\|)^{\frac{1}{2}} \|f(|A|)+f(|A^*|)\|^{\frac{1}{2}}. \tag{3.1}$$

In particular, for any $r\geqslant 1$,

$$\omega(A)^r \leqslant \frac{1}{\sqrt{2}} \|A\|^{\frac{r}{2}} \||A|^r+|A^*|^r\|^{\frac{1}{2}}. \tag{3.2}$$

The constant $\frac{1}{\sqrt{2}}$ is best possible in (3.1).

Proof. If $A\in\mathscr{B}(\mathscr{H})$ and $x,y\in\mathscr{H}$, then [8, pp. 75–76]

$$|\langle Ax,y\rangle| \leqslant \sqrt{\langle |A|x,x\rangle\langle |A^*|y,y\rangle}.$$

Let $x\in\mathscr{H}$ be a unit vector. We have

$$\begin{aligned}
&|\langle Ax,x\rangle| \\
&\leqslant \sqrt{\langle |A|x,x\rangle\langle |A^*|x,x\rangle} \\
&= \sqrt{\langle |A|x,x\rangle^{\frac{1}{2}}(\langle |A|x,x\rangle\langle |A^*|x,x\rangle)^{\frac{1}{4}}\langle |A^*|x,x\rangle^{\frac{1}{2}}(\langle |A|x,x\rangle\langle |A^*|x,x\rangle)^{\frac{1}{4}}} \\
&\leqslant \frac{1}{2}\left[\langle |A|x,x\rangle^{\frac{1}{2}}(\langle |A|x,x\rangle\langle |A^*|x,x\rangle)^{\frac{1}{4}}+\langle |A^*|x,x\rangle^{\frac{1}{2}}(\langle |A|x,x\rangle\langle |A^*|x,x\rangle)^{\frac{1}{4}}\right] \\
&\leqslant \frac{1}{2}\left[\langle |A|x,x\rangle^{\frac{1}{2}}\left(\frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right)^{\frac{1}{2}}+\langle |A^*|x,x\rangle^{\frac{1}{2}}\left(\frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right)^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\langle(|A|+|A^*|)x,x\rangle\right)^{\frac{1}{2}}\left(\langle |A|x,x\rangle^{\frac{1}{2}}+\langle |A^*|x,x\rangle^{\frac{1}{2}}\right)\right].
\end{aligned}$$

That is,

$$|\langle Ax,x\rangle| \leqslant \frac{1}{2\sqrt{2}}\left[\langle(|A|+|A^*|)x,x\rangle^{\frac{1}{2}}\left(\langle|A|x,x\rangle^{\frac{1}{2}}+\langle|A^*|x,x\rangle^{\frac{1}{2}}\right)\right].$$

Taking the supremum over $x\in\mathscr{H}$ with $\|x\|=1$, and using the fact that $\||A|\| = \||A^*|\| = \|A\|$, we get

$$\omega(A) \leqslant \frac{1}{\sqrt{2}}(\|A\|\,\||A|+|A^*|\|)^{\frac{1}{2}}. \tag{3.3}$$

Now, since f is increasing geometrically convex and convex, it follows from the inequality (3.3) that

$$\begin{aligned}
f(\omega(A)) &\leqslant f\left(\sqrt{\|A\|\left\|\frac{|A|+|A^*|}{2}\right\|}\right) \leqslant \sqrt{f(\|A\|)\,f\left(\left\|\frac{|A|+|A^*|}{2}\right\|\right)} \\
&= \sqrt{f(\|A\|)\left\|f\left(\frac{|A|+|A^*|}{2}\right)\right\|} \quad (\text{by } (1.2)) \\
&\leqslant \sqrt{\frac{1}{2}f(\|A\|)\,\|f(|A|)+f(|A^*|)\|} \quad (\text{by } (1.3))
\end{aligned}$$

as required.

The inequality (3.2) follows directly from (3.1) by letting $f(t)=t^r\,(r\geqslant 1)$.

It remains to show that the constant $\frac{1}{\sqrt{2}}$ in (3.1) is the best constant. Assume that a constant $C>0$ exists such that

$$f(\omega(A)) \leqslant C f(\|A\|)^{\frac{1}{2}}\|f(|A|)+f(|A^*|)\|^{\frac{1}{2}} \tag{3.4}$$

for all $A\in\mathscr{B}(\mathscr{H})$. Now if A is a normal operator, then $\|f(|A|)+f(|A^*|)\|=2\|f(|A|)\| = 2f(\|A\|)$ and $f(\omega(A))=f(\|A\|)$. Consequently, (3.4) implies $\frac{1}{\sqrt{2}}\leqslant C$; proving the sharpness of the constant $\frac{1}{\sqrt{2}}$.

The first consequence of Theorem 3.2 is the following two-term refinement of $\omega(A)\leqslant\|A\|$. For the proof, we need to recall the basic inequality of Davidson and Power [7, Lemma 3.3], which asserts that for positive operators $A,B\in\mathscr{B}(\mathscr{H})$, we have

$$\|A+B\| \leqslant \max\{\|A\|,\|B\|\}+\|AB\|^{\frac{1}{2}}. \tag{3.5}$$

COROLLARY 3.1. *Let $A\in\mathscr{B}(\mathscr{H})$. Then*

$$\omega(A) \leqslant \sqrt{\frac{1}{2}\left(\|A\|\,\||A|+|A|^*\|\right)} \leqslant \sqrt{\frac{1}{2}\|A\|\left(\|A\|+\|A^2\|^{\frac{1}{2}}\right)} \leqslant \|A\|.$$

Proof. The first inequality is already shown in (3.3). The second inequality follows from (3.5) and taking into account that $\|\,|A|\;|A^*|\,\| = \left\|A^2\right\|$. For the third inequality we used the fact that $\left\|A^2\right\|\leqslant\|A\|^2$.

In the following result, we present a relation for the numerical radius of the product of two operators.

PROPOSITION 3.1. *Let $A,B\in\mathscr{B}(\mathscr{H})$. Then*

$$\omega(B^*A)\leqslant\frac{1}{2\sqrt{2}}\left((\|A\|+\|B\|)\left\||A|^2+|B|^2\right\|^{\frac{1}{2}}\right). \tag{3.6}$$

Proof. It follows from the Cauchy–Schwarz inequality that

$$|\langle B^*Ax,x\rangle|=|\langle Ax,Bx\rangle|\leqslant\|Ax\|\,\|Bx\|=\sqrt{\langle Ax,A^*x\rangle\,\langle Bx,B^*x\rangle}=\sqrt{\left\langle|A|^2x,x\right\rangle\left\langle|B|^2x,x\right\rangle}.$$

By the same arguments as before one can get

$$|\langle B^*Ax,x\rangle|\leqslant\frac{1}{2\sqrt{2}}\left(\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle^{\frac{1}{2}}\left(\left\langle|A|^2x,x\right\rangle^{\frac{1}{2}}+\left\langle|B|^2x,x\right\rangle^{\frac{1}{2}}\right)\right).$$

Taking supremum over $x\in\mathscr{H}$ with $\|x\|=1$, we deduce the desired inequality (3.6).

COROLLARY 3.2. *Let $A\in\mathscr{B}(\mathscr{H})$. Then*

$$\omega\left(A^2\right)\leqslant\frac{1}{\sqrt{2}}\|A\|\left\||A|^2+|A^*|^2\right\|^{\frac{1}{2}}\leqslant\frac{1}{\sqrt{2}}\|A\|\left(\left\|A^2\right\|+\|A\|^2\right)^{\frac{1}{2}}.$$

The constant $\frac{1}{\sqrt{2}}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. The first inequality follows directly from (3.6) by letting $B^*=A$, while the second inequality is obtained from the fact that $\left\||A|^2+|A^*|^2\right\|\leqslant\left\|A^2\right\|+\|A\|^2$, see for example [1, Lemma 2.3].

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