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MATRIX VALUED p–CONVOLUTION OPERATORS

ALI EBADIAN AND ALI JABBARI

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Abstract. Let G be a locally compact group equipped with the left (or right) Haar measure m_G, M_n be an $n \times n$ matrix with entries in $\mathbb{C}$ and let $M(G,M_n)$ be the Banach algebra consists all M_n-valued measures on G. We define left and right p-convolution operators on $L^p(G,M_n)$, where $1 < p < \infty$ and investigate some properties of these operators. For a locally compact abelian group G, we consider the Fourier transforms of M_n-valued functions and measures and show that there is an isometric $*$-homomorphism ($*$-anti-homomorphism) from $L^\infty(\widehat{G},M_n)$ onto the space of all p-convolution operators.

1. Introduction

Let G be a locally compact group and $1 < p < \infty$. A bounded operator $T : L^p(G) \longrightarrow L^p(G)$ is called a p-convolution operator of G if $T({}_af) = {}_aT(f)$, for all $a \in G$ and $f \in L^p(G)$. These operators are also called translation invariant operators, see [6], where Hörmander's studied these operators on $\mathbb{R}^n$. Following [4], we denote the set of all p-convolution operators of G by $CV_p(G)$. Suppose that $\mathscr{B}(L^p(G))$ denotes the space of all maps from $L^p(G)$ into itself and $B(L^p(G))$ denotes the Banach algebra consists all linear bounded operators from $L^p(G)$ into itself. Then clearly, $CV_p(G)$ is a subalgebra of $B(L^p(G))$.

Let G be a locally compact group, m_G be the unique Haar measure on G and $1 < p,q < \infty$ such that $1/p + 1/q = 1$. Let M_n be an $n \times n$, $n \in \mathbb{N}$, matrix with entries in $\mathbb{C}$. We equip M_n with the C^*-norm. The trace $\mathrm{Tr} : M_n \longrightarrow \mathbb{C}$ is a positive linear functional of norm n. Suppose that $\mathscr{B}$ is a σ-algebra of Borel sets in G, $\mu : G \longrightarrow M_n$ is countably additive function that we call it an M_n-valued measure on G and denote by an $n \times n$ matrix $\mu = (\mu_{ij})$ of complex valued measures μ_{ij} on G. The variation of μ is $|\mu|$ that is a positive real finite measure on G defined by

$$|\mu|(E) = \sup_{\mathscr{P}} \left\{ \sum_{E_i \in \mathscr{P}} \|\mu(E_i)\| : E \in \mathscr{B} \right\},$$

where $\mathscr{P}$ is a partition of E into a finite number of pairwise disjoint Borel sets. Define the norm of μ as $\|\mu\| = |\mu|(G)$. Following [1, 2], μ has a polar representation $\mathrm{d}\mu =$

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$\omega \cdot \mathrm{d}|\mu|$ where $\omega : G \longrightarrow M_n$ is a Bochner integrable function with $\|\omega(\cdot)\| = 1$. A function $f = (f_{ij}) : G \longrightarrow M_n$ is called μ-integrable if each f_{ij} is a Borel function and the integral $\int_G f_{ij}\,\mathrm{d}\mu_{k\ell}$ exist in which case. For any $E \in \mathscr{B}$, the integral $\int_E f\,\mathrm{d}\mu$ is an $n \times n$ matrix with ij-th entry

$$\sum_k \int_E f_{ik}\,\mathrm{d}\mu_{kj}.$$

Then, we have the norm of f,

$$\left\| \int_G f\,\mathrm{d}\mu \right\| = \left\| \int_G f(x)\omega(x)\,\mathrm{d}|\mu|(x) \right\| \leqslant \int_G \|f(x)\|\,\mathrm{d}|\mu|(x). \tag{1.1}$$

The trace-norm $\|\cdot\|_{tr}$ is equivalent to the C^*-norm on M_n and $M_n^* = (M_n, \|\cdot\|_{tr})$, by this, we can regard an M_n^*-valued measure on G as an M_n-valued measure on G, and vice versa. We denote the space of all M_n^*-valued measures on G by $M(G, M_n^*)$ with the total variation norm $\|\cdot\|_{tr}$. This space is linearly isomorphic to the space $(M(G,M_n), \|\cdot\|)$. By $C_0(G,M_n)$, we man the Banach space of continuous M_n-valued functions on G vanishing at infinity with the supremum norm and $C_C(G,M_n)$ denotes the subspace of $C_0(G,M_n)$ consists all M_n-valued continuous functions with compact supports. By [1, Lemma 5], $M(G,M_n^*)$ is linearly isomorphic order-isomorphic to the dual of $C_0(G,M_n)$, with the following duality formula:

$$\langle \cdot, \cdot \rangle : C_0(G,M_n) \times M(G,M_n^*) \longrightarrow \mathbb{C}$$

$$\langle f, \mu \rangle = \mathrm{Tr}\left(\int_G f\mathrm{d}\mu \right) = \sum_{i,k} \int_G f_{ik}\mathrm{d}\mu_{k,i}, \tag{1.2}$$

for any $f = (f_{ij}) \in C_0(G,M_n)$ and $\mu = (\mu_{ij}) \in M(G,M_n^*)$. By [3, Proposition 2.4], $(M(G,M_n^*), \|\cdot\|_{tr})$ is a Banach algebra with the following convolution product:

$$\langle f, \mu * \nu \rangle = \mathrm{Tr}\left(\int_G \int_G f(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \right), \tag{1.3}$$

for all $f \in C_0(G,M_n)$ and $\mu, \nu \in M(G,M_n^*)$. Also, $(M(G,M_n), \|\cdot\|)$ becomes a Banach algebra with the convolution product and is algebraically isomorphic to $(M(G,M_n^*), \|\cdot\|_{tr})$. Let $f = (f_{ij})$ be a Borel M_n-valued function on G and $\mu = (\mu_{ij})$ be a M_n-valued measure on G. An M_n-valued convolution $f * \mu$, if exists at $x \in G$, is defined by

$$(f * \mu)(x) = \int_G f(xy^{-1})\,\mathrm{d}\mu(y). \tag{1.4}$$

The left convolution $\mu *_\ell f$ is the following integral if it exists:

$$(\mu *_\ell f)(x) = \int_G \mathrm{d}\mu(y) f(y^{-1}x) \qquad (x \in G). \tag{1.5}$$

The transposed integral $\int_G \mathrm{d}\mu(x) f(x)$ which is defined to have ij-entry

$$\left(\int_G \mathrm{d}\mu(x) f(x) \right)_{ij} = \sum_k \int_G f_{kj}(x)\,\mathrm{d}\mu_{ik}(x).$$

Also,

$$\left\| \int_G \mathrm{d}\mu(x) f(x) \right\| \leqslant \int_G \|f(x)\| \,\mathrm{d}|\mu|(x). \tag{1.6}$$

For given $\mu \in M(G, M_n)$, following [2, Page 24], we consider $\widetilde{\mu} \in (G, M_n)$ by $\mathrm{d}\widetilde{\mu}(x) = \mathrm{d}\mu(x^{-1})$, for all $x \in G$. Consider the complex vector space $L^p(G, M_n)$. Then by [5], the dual of $L^p(G, M_n)$ is identified by $L^q(G, M_n^*)$ with the following duality formula:

$$\langle \cdot, \cdot \rangle : L^p(G, M_n) \times L^q(G, M_n^*) \longrightarrow \mathbb{C}$$
$$\langle f, g \rangle = \mathrm{Tr}\left(\int_G f(x) g(x) \mathrm{d}m_G(x) \right). \tag{1.7}$$

For any $f \in L^p(G, M_n)$ and $g \in L^q(G, M_n^*)$, we have

$$\langle f * \mu, g \rangle = \mathrm{Tr}\left(\int_G \int_G g(xy) f(x) \,\mathrm{d}\mu(y) \,\mathrm{d}m_G(x) \right) = \langle f, \widetilde{\mu} * g \rangle. \tag{1.8}$$

Following [2], by (1.1) and (1.6) we have

$$\|f * \mu\|_p \leqslant \|f\|_p \|\mu\|, \quad \text{and} \quad \|\mu *_\ell f\|_p \leqslant \|f\|_p \|\mu\|, \tag{1.9}$$

for all $f \in L^p(G, M_n)$ and $\mu \in M(G, M_n)$, where

$$\|f\|_p = \left(\int_G \|f(x)\|_{tr}^p \,\mathrm{d}m_G(x) \right)^{\frac{1}{p}}.$$

In the next section, we introduce the notions of left and right p-convolution operators on $L^p(G, M_n)$ Banach spaces, where G is a locally compact group equipped with the left or right Haar measure. We give some results and properties of aforementioned operators. In the Section 3, we consider Fourier transforms of matrix valued functions and show that for abelian locally compact group there are isometric homomorphism and isometric anti-homomorphism from $L^\infty(\widehat{G}, M_n)$ onto the space of all p-convolution operators.

2. Matrix valued p-convolution operators

In this section we define matrix valued left and right p-convolution operators and give some results related to these operators defined on $L^p(G, M_n)$. We start with the following definition.

DEFINITION 2.1. Let G be a locally compact group, $1 < p < \infty$. A bounded operator $T : L^p(G, M_n) \longrightarrow L^p(G, M_n)$ is called a matrix valued left p-convolution operator of G if $T({}_a f) = {}_a T(f)$, for all $a \in G$ and $f \in L^p(G, M_n)$. We denote the set of all matrix valued left p-convolution operators of G by $LCV_p(G, M_n)$. Similarly, we define the right p-convolution operator with entries in M_n, if $T(f_a) = T(f)_a$, for all $a \in G$, $f \in L^p(G, M_n)$ and we denote the set of all such operators by $RCV_p(G, M_n)$. We denote the space of matrix valued p-convolution operators by $CV_p(G, M_n)$ that is $LCV_p(G, M_n) \cap RCV_p(G, M_n)$.

Clearly if G is a locally compact abelian group then $CV_p(G,M_n) = LCV_p(G,M_n) = RCV_p(G,M_n)$.

DEFINITION 2.2. Let G be a locally compact group and $1 < p < \infty$.

(i) Define $\leftthreetimes_G^p : M(G,M_n) \longrightarrow \mathscr{B}(L^p(G,M_n))$ such that for every $\mu \in M(G,M_n)$, $\leftthreetimes_G^p(\mu) : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is defined by $\leftthreetimes_G^p(\mu)(f) = f * \widetilde{\mu}$, for every $f \in L^p(G,M_n)$.

(ii) Define $\rightthreetimes_G^p : M(G,M_n) \longrightarrow \mathscr{B}(L^p(G,M_n))$ such that for every $\mu \in M(G,M_n)$, $\rightthreetimes_G^p(\mu) : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is defined by $\rightthreetimes_G^p(\mu)(f) = \widetilde{\mu} *_\ell f$, for all $f \in L^p(G,M_n)$.

LEMMA 2.3. *Let G be a locally compact group, m_G be the left or right Haar measure and $1 < p < \infty$. Then $\leftthreetimes_G^p(\mu) : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is a linear bounded left p-convolution operator such that*

$$\leftthreetimes_G^p(\mu)(f)(x) = \int_G f(xy)\, \mathrm{d}\mu(y), \tag{2.1}$$

for all $x \in G$, $f \in L^p(G,M_n)$ and $\mu \in M(G,M_n)$. Furthermore, $\leftthreetimes_G^p$ has the following properties:

(i) for all $a \in G$ and $f \in L^p(G,M_n)$, $\leftthreetimes_G^p(\delta_a)(f) = f_a$.

(ii) $\leftthreetimes_G^p(\delta_{ab}) = \leftthreetimes_G^p(\delta_a) \circ \leftthreetimes_G^p(\delta_b)$ for all $a,b \in G$.

(iii) for every $\mu \in M(G,M_n)$, $\leftthreetimes_G^p(\mu) = \int_G \leftthreetimes_G(\delta_y)\, \mathrm{d}\mu(y)$.

Proof. For all $x \in G$, $f \in C_C(G,M_n)$ and $\mu \in M(G,M_n)$,

$$(f * \widetilde{\mu})(x) = \int_G f\left(xy^{-1}\right) \mathrm{d}\widetilde{\mu}(y) = \int_G f\left(xy^{-1}\right) \mathrm{d}\mu(y^{-1}) = \int_G f(xy)\, \mathrm{d}\mu(y).$$

It is easy to see that $f * \widetilde{\mu}$ is continuous and by (1.9), we have $\|f * \widetilde{\mu}\|_p \leqslant \|f\|_p \|\mu\|$. Since $C_C(G,M_n)$ is dense in $L^P(G,M_n)$, the formula (2.1) holds. We now show that $\leftthreetimes_G^p$ is a p-convolution operator. By (2.1), we have

$$\begin{aligned} {}_a\left(\leftthreetimes_G^p(\mu)(f)\right)(x) &= {}_a(f * \widetilde{\mu})(x) = (f * \widetilde{\mu})(ax) \\ &= \int_G f(axy)\mathrm{d}\mu(y) = \int_G {}_af(xy)\mathrm{d}\mu(y) \\ &= ({}_af * \widetilde{\mu})(x) = \left(\leftthreetimes_G^p(\mu)({}_af)\right)(x), \end{aligned}$$

for all $x \in G$, $f \in L^p(G,M_n)$ and $\mu \in M(G,M_n)$. Hence, $\leftthreetimes_G^p(\mu)$ is a p-convolution operator. The implications (i) and (ii) hold by applying (2.1).

(iii) For any $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$, we have

$$\begin{aligned}
\langle \leftthreetimes_G^p(\mu)(f), g\rangle &= \operatorname{Tr}\left(\int_G \leftthreetimes_G^p(\mu)(f)(x)g(x)\mathrm{d}m_G(x)\right) \\
\text{by (2.1)} &= \operatorname{Tr}\left(\int_G\int_G f(xy)g(x)\mathrm{d}\mu(y)\mathrm{d}m_G(x)\right) \\
\text{by (i)} &= \operatorname{Tr}\left(\int_G\int_G \left(\leftthreetimes_G(\delta_y)(f)\right)(x)g(x)\mathrm{d}\mu(y)\mathrm{d}m_G(x)\right) \\
&= \operatorname{Tr}\left(\int_G\left(\int_G \left(\leftthreetimes_G(\delta_y)(f)\right)(x)\mathrm{d}\mu(y)\right)g(x)\mathrm{d}m_G(x)\right) \\
&= \langle \int_G \leftthreetimes_G(\delta_y)(f)\mathrm{d}\mu(y), g\rangle.
\end{aligned}$$

Thus, $\leftthreetimes_G^p(\mu) = \int_G \leftthreetimes_G(\delta_y)\,\mathrm{d}\mu(y)$, for every $\mu \in M(G,M_n)$.
Similar to the above Lemma, we have the following result for $\rightthreetimes_G^p$ where we omit its proof because it is similar to the aforementioned result.

LEMMA 2.4. *Let G be a locally compact group, m_G be the left or right Haar measure and $1<p<\infty$. Then $\rightthreetimes_G^p(\mu): L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is a linear bounded right p-convolution operator such that*

$$\leftthreetimes_G^p(\mu)(f)(x) = \int_G \mathrm{d}\mu(y)f(yx), \tag{2.2}$$

for all $x \in G$, $f \in L^p(G,M_n)$ and $\mu \in M(G,M_n)$. Furthermore, $\rightthreetimes_G^p$ has the following properties:

(i) for all $a \in G$ and $f \in L^p(G,M_n)$, $\leftthreetimes_G^p(\delta_a)(f) =_a f$.

(ii) $\rightthreetimes_G^p(\delta_{ab}) = \rightthreetimes_G^p(\delta_a)\circ \rightthreetimes_G^p(\delta_b)$, for all $a,b \in G$.

(iii) for every $\mu \in M(G,M_n)$, $\rightthreetimes_G^p(\mu) = \int_G \mathrm{d}\mu(y)\rightthreetimes_G(\delta_y)$.

Let G be a locally compact group, $L^p(G,M_n)$ and $L^q(G,M_n)$ be as before. For all $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$, $\int_G f(x)g(x)\,\mathrm{d}m_G(x)$ is in M_n. We denote this M_n-valued integral by

$$\int_G f(x)g(x)\,\mathrm{d}m_G(x) = \langle f,g\rangle_{M_n},$$

indeed it is M_n-valued duality formula and $\operatorname{Tr}(\langle f,g\rangle_{M_n}) = \operatorname{Tr}\langle f,g\rangle$. By this notation we have the following result:

PROPOSITION 2.5. *Let G be a locally compact group, m_G be the left or right Haar measure and $1<p<\infty$. Then $\leftthreetimes_G^p(\mu*\nu) = \leftthreetimes_G^p(\mu)\circ\leftthreetimes_G^p(\nu)$ and $\rightthreetimes_G^p(\mu*\nu) = \rightthreetimes_G^p(\mu)\circ\rightthreetimes_G^p(\nu)$, for all $\mu,\nu \in M(G,M_n)$.*

Proof. We investigate only the case $\leftthreetimes_G^p$ and the case $\rightthreetimes_G^p$ is similar. For all $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$, Lemma 2.3 implies

$$
\begin{aligned}
&\left\langle \left(\leftthreetimes_G^p(\mu)\circ\leftthreetimes_G^p(\nu)\right)(f),g\right\rangle = \left\langle \leftthreetimes_G^p(\nu)(f),\leftthreetimes_G^p(\mu)^*(g)\right\rangle \\
&= \mathrm{Tr}\left(\int_G \leftthreetimes_G^p(\nu)(f)(x)\leftthreetimes_G^p(\mu)^*(g)(x)\mathrm{d}m_G(x)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G \leftthreetimes_G^p(\delta_y)(f)(x)\leftthreetimes_G^p(\mu)^*(g)(x)\mathrm{d}\nu(y)\mathrm{d}m_G(x)\right)\\
&= \mathrm{Tr}\left(\int_G \langle\leftthreetimes_G^p(\delta_y)(f),\leftthreetimes_G^p(\mu)^*(g)\rangle_{M_n}\mathrm{d}\nu(y)\right)\\
&= \mathrm{Tr}\left(\int_G \langle\left(\leftthreetimes_G^p(\mu)\circ\leftthreetimes_G^p(\delta_y)\right)(f),g\rangle_{M_n}\mathrm{d}\nu(y)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G\int_G \leftthreetimes_G^p(\delta_x)\left(\leftthreetimes_G^p(\delta_y)\right)(f)(t)g(t)\mathrm{d}\mu(x)\mathrm{d}m_G(t)\mathrm{d}\nu(y)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G\int_G \leftthreetimes_G^p(\delta_{xy})(f)(t)g(t)\mathrm{d}m_G(t)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G \left\langle\leftthreetimes_G^p(\delta_{xy})(f),g\right\rangle_{M_n}\mathrm{d}\mu(x)\mathrm{d}\nu(y)\right).
\end{aligned}
$$

Let $f \in L^p(G,M_n)$, $g \in L^q(G,M_n^*)$, and set $h(x) = \leftthreetimes_G^p(\delta_x)(f)$, for every $x \in G$. Then by (1.2) and (1.3) we have

$$
\begin{aligned}
\left\langle \left(\leftthreetimes_G^p(\mu*\nu)\right)(f),g\right\rangle &= \mathrm{Tr}\left(\int_G \leftthreetimes_G^p(\mu*\nu)(f)(t)g(t)\mathrm{d}m_G(t)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G \leftthreetimes_G^p(\delta_y)(f)(t)g(t)\mathrm{d}m_G(t)\mathrm{d}(\mu*\nu)(y)\right)\\
&= \mathrm{Tr}\left(\int_G \langle\leftthreetimes_G^p(\delta_y)(f),g\rangle_{M_n}\mathrm{d}(\mu*\nu)(y)\right)\\
&= \mathrm{Tr}\left(\int_G \langle h(y),g\rangle_{M_n}\mathrm{d}(\mu*\nu)(y)\right) = \langle\langle h,g\rangle_{M_n},\mu*\nu\rangle\\
&= \mathrm{Tr}\left(\int_G \langle h(xy),g\rangle_{M_n}\mathrm{d}\mu(x)\mathrm{d}\nu(y)\right)\\
&= \mathrm{Tr}\left(\int_G \langle\leftthreetimes_G^p(\delta_{xy})(f),g\rangle_{M_n}\mathrm{d}\mu(x)\mathrm{d}\nu(y)\right).
\end{aligned}
$$

This implies that $\left\langle \left(\leftthreetimes_G^p(\mu)\circ\leftthreetimes_G^p(\nu)\right)(f),g\right\rangle = \left\langle \left(\leftthreetimes_G^p(\mu*\nu)\right)(f),g\right\rangle$, for all $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$. Hence, $\leftthreetimes_G^p(\mu*\nu) = \leftthreetimes_G^p(\mu)\circ\leftthreetimes_G^p(\nu)$, for all $\mu,\nu \in M(G,M_n)$.

Let $f \in L^1(G,M_n)$, then $f\cdot m_G \in M(G,M_n)$ with the following total variation

$$
\|f\cdot m_g\| = |f\cdot m_G|(G) = \int_G \|f(x)\|\,\mathrm{d}m_G(x) = \|f\|_1.
$$

We identify $L^1(G,M_n)$ as a closed subspace of $M(G,M_n)$ such that contains all absolutely continuous M_n-valued measures on G and it also is a right ideal of $M(G,M_n)$, because $(f\cdot m_G)*\mu=(f*\mu)\cdot m_G$, for all $f\in L^1(G,M_n)$ and $\mu\in M(G,M_n)$. For any $f\in L^1(G,M_n)$, we put $\lambda_G^p(f)=\lambda_G^p(f\cdot m_G)$. From (1.5), we have the following left convolution product

$$(f*_\ell g)(x)=\int_G \mathrm{d}m_G(y)f(y)g(y^{-1}x), \tag{2.3}$$

for all $f\in L^1(G,M_n)$ and $g\in L^p(G,M_n)$. This together (1.9) implies that $\|f*_\ell g\|_p\leqslant \|g\|_p\|f\|_1$, for all $f\in L^1(G,M_n)$ and $g\in L^p(G,M_n)$. Thus, $L^p(G,M_n)$ is a left Banach $L^1(G,M_n)$-module.

THEOREM 2.6. *Let G be a locally compact group, m_G be the left or right Haar measure on G, $1<p<\infty$ and $T\in B(L^p(G,M_n))$. Then $T\in LCV_p(G,M_n)$ if and only if $T(f*_\ell g)=f*_\ell T(g)$, for all $f\in L^1(G,M_n)$ and $g\in L^p(G,M_n)$.*

Proof. As we discussed the above, $L^p(G,M_n)$ is a left Banach $L^1(G,M_n)$-module. Now, suppose that $T\in LCV_p(G,M_n)$ with the left convolution product. Then

$$\begin{aligned}
\langle f*_\ell T(g),h\rangle &= \mathrm{Tr}\left(\int_G (f*_\ell T(g))(x)h(x)\mathrm{d}m_G(x)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y)f(y)T(g)(y^{-1}x)h(x)\mathrm{d}m_G(x)\right)\\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y)f(y)T({}_{y^{-1}}g)(x)h(x)\mathrm{d}m_G(x)\right)\\
&= \mathrm{Tr}\left(\int_G \mathrm{d}m_G(y)f(y)\int_G T({}_{y^{-1}}g)(x)h(x)\mathrm{d}m_G(x)\right)\\
&= \mathrm{Tr}\left(\int_G \mathrm{d}m_G(y)f(y)\langle T({}_{y^{-1}}g),h\rangle_{M_n}\right)\\
&= \mathrm{Tr}\left(\int_G \mathrm{d}m_G(y)f(y)\langle {}_{y^{-1}}g,T^*(h)\rangle_{M_n}\right)\\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y)f(y)g(y^{-1}x)T^*(h)(x)\mathrm{d}m_G(x)\right)\\
&= \langle f*_\ell g,T^*(h)\rangle=\langle T(f*_\ell g),h\rangle,
\end{aligned}$$

for all $f\in L^1(G,M_n)$, $g\in L^p(G,M_n)$ and $h\in L^q(G,M_n^*)$. Thus, $T(f*_\ell g)=f*_\ell T(g)$, for all $f\in L^1(G,M_n)$ and $g\in L^p(G,M_n)$, where $1<q<\infty$ and $1/p+1/q=1$.

Conversely, assume that $T(f*_\ell g)=f*_\ell T(g)$, for all $f\in L^1(G,M_n)$ and $g\in L^p(G,M_n)$. For every $\varepsilon>0$, we have

$$\|{}_xf-f\|_p<\varepsilon, \tag{2.4}$$

for every $f \in L^p(G,M_n)$. Let $(e_\alpha) \subseteq L^1(G)$ be a bounded approximate identity for $L^1(G)$, where e_α's have compact supports and $\|e_\alpha\|_1 \leqslant 1$, then $L^1(G,M_n)$ has a bounded approximate identity (E_α) as follows:

$$E_\alpha = \begin{pmatrix} e_\alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_\alpha \end{pmatrix}$$

where E_α's are M_n-valued functions such that $\|E_\alpha\|_{L^1(G,M_n)} \leqslant 1$. This together (2.4) implies that one can choose an element $f \in L^1(G,M_n)$ with compact support and $\|f\|_{L^1(G,M_n)} \leqslant 1$ such that $\|f *_\ell g - g\|_p < \varepsilon_1$, where ε_1 depends on ε. Thus, for every $a \in G$, we have $\|_a(f *_\ell g) - {}_ag\|_p < \varepsilon_1$. This implies that

$$\|_aT(f *_\ell g) - {}_aT(g)\|_p = \|_a(f *_\ell T(g)) - {}_aT(g)\|_p < \varepsilon_1, \tag{2.5}$$

for every $a \in G$. Since T is a bounded linear operator, by (2.5), we have

$$\|T({}_a(f *_\ell g)) - T({}_ag)\|_p < \|T\|\varepsilon_1, \quad \text{and} \quad \|_aT((f *_\ell g)) - {}_aT(g)\|_p < \|T\|\varepsilon_1, \tag{2.6}$$

for every $a \in G$. Also, for all $g \in L^p(G,M_n)$ and $a,x \in G$,

$$\begin{aligned} {}_a(f *_\ell g)(x) &= (f *_\ell g)(ax) = \int_G \mathrm{d}m_G(y) f(y) g(axy^{-1}) = \int_G \mathrm{d}m_G(y) f(y)\, {}_ag(xy^{-1}) \\ &= (f *_\ell {}_ag)(x). \end{aligned} \tag{2.7}$$

The above argument implies that

$$T({}_a(f *_\ell g)) = T(f *_\ell {}_ag) = (f *_\ell T({}_ag)) = (f *_\ell {}_aT(g)) = {}_a(f *_\ell T(g)), \tag{2.8}$$

for all $g \in L^p(G,M_n)$ and $a \in G$. Now, set $\varepsilon_1 = \varepsilon/2\|T\|$. Then by (2.6) and (2.8), we have

$$\begin{aligned} \|T({}_ag) - {}_aT(g)\|_p &\leqslant \|T({}_ag) - T({}_a(f *_\ell g))\|_p + \|T({}_a(f *_\ell g)) - {}_aT((f *_\ell g))\|_p \\ &\quad + \|_aT((f *_\ell g)) - {}_aT(g)\|_p \\ &< \varepsilon, \end{aligned}$$

for all $g \in L^p(G,M_n)$ and $a \in G$.
In light of (1.4), we define the following convolution product

$$(g * f)(x) = \int_G g(xy^{-1}) f(y)\, \mathrm{d}m_G(y), \tag{2.9}$$

for all $g \in L^p(G,M_n)$, $f \in L^1(G,M_n)$ and $x \in G$.

THEOREM 2.7. *Let G be a locally compact group, m_G be the left or right Haar measure on G, $1 < p < \infty$ and $T \in B(L^p(G,M_n))$. Then $T \in RCV_p(G,M_n)$ if and only if $T(g * f) = T(g) * f$, for all $f \in L^1(G,M_n)$ and $g \in L^p(G,M_n)$.*

Proof. By (1.9) we get $\|g*f\|_p \leqslant \|g\|_p\|f\|_1$, for all $g\in L^p(G,M_n)$ and $f\in L^1(G,M_n)$. This shows that $L^p(G,M_n)$ is a right Banach $L^1(G,M_n)$-module. Assume that $T\in RCV_p(G,M_n)$. By a similar argument in the proof of Theorem 2.6, we have

$$\langle T(g)*f,h\rangle = \langle T(g*f),h\rangle,$$

for all $g\in L^p(G,M_n)$, $h\in L^q(G,M_n^*)$ and $f\in L^1(G,M_n)$, where $1<q<\infty$ and $1/p+1/q=1$.

Conversely, suppose that $T(g*f)=T(g)*f$, for all $g\in L^p(G,M_n)$ and $f\in L^P(G,M_n)$. We do similar to the converse case of the proof of Theorem 2.6. For every $\varepsilon>0$, we have

$$\|f_x-f\|<\varepsilon \quad (f\in L^P(G,M_n),\ x\in G). \tag{2.10}$$

We now choose an element $f\in L^1(G,M_n)$ with $\|f\|_{L^1(G,M_n)}\leqslant 1$ and the compact support such that for every $\varepsilon_1>0$, we have $\|g*f-g\|_p<\varepsilon_1$, where ε_1 depends on ε. Thus, for every $a\in G$, we have $\|\,(g*f)_a-\ g_a\|_p<\varepsilon_1$. This implies that

$$\|T\,(g*f)_a-T(g)_a\|_p=\|\,(T(g)*f)_a-T(g)_a\|_p<\varepsilon_1, \tag{2.11}$$

for every $a\in G$. Then by (2.11) and since T is a linear bounded operator, we have

$$\|T\,((g*f)_a)-T\,(g_a)\,\|_p<\|T\|\varepsilon_1,\quad \text{and}\quad \|T\,((g*f))_a-T\,(g)_a\,\|_p<\|T\|\varepsilon_1, \tag{2.12}$$

for every $a\in G$. On the other hand, for all $g\in L^p(G,M_n)$ and $a,x\in G$, we get

$$\begin{aligned}(g*f)_a(x) &= (g*f)(xa)=\int_G g(y^{-1}xa)f(y)\mathrm{d}m_G(y)=\int_G g_a(xy^{-1})f(y)\mathrm{d}m_G(y)\\ &=(g_a*f)(x).\end{aligned} \tag{2.13}$$

Then

$$T\,((g*f)_a)=T\,(g_a*f)=(T\,(g_a)*f)=(T\,(g)_a*f)=(T\,(g)*f)_a\,, \tag{2.14}$$

for all $g\in L^p(G,M_n)$ and $a\in G$. We now set $\varepsilon_1=\varepsilon/2\|T\|$. Then by (2.12) and (2.14), we have

$$\begin{aligned}\|T(g_a)-T(g)_a\|_p &\leqslant \|T(g_a)-T\,((g*f)_a)\,\|_p+\|T\,((g*f)_a)-T\,((g*f))_a\,\|_p\\ &\quad +\|T\,((g*f))_a-T(g)_a\|_p\\ &<\varepsilon,\end{aligned}$$

for all $g\in L^p(G,M_n)$ and $a\in G$.

By the following results we investigate the duals of left and right p-convolution operators.

PROPOSITION 2.8. *Let G be a locally compact group with the left Haar measure m_G and let $1<p,q<\infty$ such that $1/p+1/q=1$. If $T:L^p(G,M_n)\longrightarrow L^p(G,M_n)$ is in $LCV_p(G,M_n)$, then $T^*:L^q(G,M_n^*)\longrightarrow L^q(G,M_n^*)$ is in $LCV_q(G,M_n^*)$.*

Proof. For all $a \in G$, $f \in L^p(G,M_n)$ and $h \in L^q(G,M_n)$,

$$\begin{aligned}
\langle f, {}_aT^*(h)\rangle &= \mathrm{Tr}\left(\int_G {}_aT^*(h)(x)f(x)\mathrm{d}m_G(x)\right) = \mathrm{Tr}\left(\int_G T^*(h)(x)\ {}_{a^{-1}}f(x)\mathrm{d}m_G(x)\right)\\
&= \langle {}_{a^{-1}}f, T^*(h)\rangle = \langle T\left({}_{a^{-1}}f\right), h\rangle = \langle {}_{a^{-1}}T\left(f\right), h\rangle\\
&= \mathrm{Tr}\left(\int_G T(f)(a^{-1}x)h(x)\mathrm{d}m_G(x)\right) = \mathrm{Tr}\left(\int_G T(f)(x)\ {}_ah(x)\mathrm{d}m_G(x)\right)\\
&= \langle T\left(f\right), {}_ah\rangle = \langle f, T^*\left({}_ah\right)\rangle.
\end{aligned}$$

This implies that $T^* \in LCV_q(G,M_n^*)$.
The proof of the following result is similar to the above proof exactly and we so omit it.

PROPOSITION 2.9. *Let G be a locally compact group with the right Haar measure m_G and let $1 < p,q < \infty$ such that $1/p+1/q=1$. If $T : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is in $RCV_p(G,M_n)$, then $T^* : L^q(G,M_n^*) \longrightarrow L^q(G,M_n^*)$ is in $RCV_q(G,M_n^*)$.*

By Propositions 2.8 and 2.9 we have:

COROLLARY 2.10. *Let G be a unimodular locally compact group and let $1 < p,q < \infty$ such that $1/p+1/q=1$. If $T : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is in $CV_p(G,M_n)$, then $T^* : L^q(G,M_n^*) \longrightarrow L^q(G,M_n^*)$ is in $CV_q(G,M_n^*)$.*

3. Fourier transform and p-convolution operators

In this section, we consider Fourier transform of matrix valued functions, we suppose that G is a locally compact abelian group with the Haar measure m_G and by $\widehat{G}$, we denote the dual of G. Following [2, Section 3. §3], for any $\pi \in \widehat{G}$, $\mu \in M(G,M_n)$ and $f \in L^1(G,M_n)$, we define their Fourier transforms by

$$\widehat{\mu}(\pi) = \int_G \pi\left(x^{-1}\right)\mathrm{d}\mu(x), \tag{3.1}$$

and

$$\widehat{f}(\pi) = \int_G f(x)\pi\left(x^{-1}\right)\mathrm{d}m_G(x). \tag{3.2}$$

Also,

$$\widehat{\mu *_\ell f} = \widehat{\mu}\widehat{f} \quad \text{and} \quad \widehat{f*\mu} = \widehat{f}\widehat{\mu}, \tag{3.3}$$

for all $\mu \in M(G,M_n)$ and $f \in L^1(G,M_n)$. We denote by $M_{n,2}$ the vector space M_n equipped with the Hilbert-Schmidt norm and we also consider $L^2(G,M_{n,2})$ as a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. For every $f \in L^1(G,M_{n,2}) \cap L^2(G,M_{n,2})$, by [2, Lemma 3.3.9], $\|\widehat{f}\|_2 = \|f\|_2$ and there is a unique continuous map $\mathscr{F} : L^2(G,M_{n,2}) \longrightarrow L^2(\widehat{G},M_{n,2})$ such that $\mathscr{F}(f) = \widehat{f}$. Similar to [4, Definition 1.3.2], for any $\phi \in L^\infty(\widehat{G},M_n)$, we define

$$\Lambda_{\widehat{G}}(\phi)(f) = \mathscr{F}^{-1}\left(\phi\mathscr{F}(f)\right), \quad (f \in L^2(G,M_{n,2})). \tag{3.4}$$

We now give a result similar to [4, Theorem 1.3.2] as follows:

THEOREM 3.1. *Let G be a locally compact abelian group. Then $\Lambda_{\widehat{G}}$ is an isometric $*$-isomorphism from $L^\infty(\widehat{G},M_n)$ onto $CV_2(G,M_n)$.*

Proof. By a similar argument in the proof of [4, Theorem 1.3.2], $\Lambda_{\widehat{G}}$ is an algebraic homomorphism i.e., for all $\phi,\psi \in L^\infty(\widehat{G},M_n)$ and $f \in L^2(G,M_{n,2})$,

$$\Lambda_{\widehat{G}}(\phi\psi)(f) = \Lambda_{\widehat{G}}(\phi)\left(\Lambda_{\widehat{G}}(\psi)(f)\right). \tag{3.5}$$

Also, it is injective and for any $\phi \in L^\infty(\widehat{G},M_n)$ and $f,g \in L^2(G,M_{n,2})$,

$$\begin{aligned}\langle g,\Lambda_{\widehat{G}}(\phi^*)(f)\rangle &= \langle g,\mathscr{F}^{-1}(\phi^*\mathscr{F}(f))\rangle = \langle \mathscr{F}(g),\phi^*\mathscr{F}(f)\rangle \\ &= \mathrm{Tr}\left(\int_G \phi\mathscr{F}(g)\mathscr{F}(f)^*\mathrm{d}m_G\right) \\ &= \langle \phi\mathscr{F}(g),\mathscr{F}(f)\rangle = \langle \mathscr{F}^{-1}(\phi\mathscr{F}(g)),f\rangle = \langle \Lambda_{\widehat{G}}(\phi)(g),f\rangle \\ &= \langle g,\Lambda_{\widehat{G}}(\phi)^*(f)\rangle.\end{aligned}$$

This shows that $\Lambda_{\widehat{G}}$ is a $*$-homomorphism.

By proof of [2, Proposition 3.3.12], $\Lambda_{\widehat{G}}$ is a $*$-isomorphism. On the other hand, every $*$-isomorphism between C^*-algebras is an isometry [7, Corollary I.5.4], thus, $\Lambda_{\widehat{G}}$ is an isometry. Denote by ι_G the canonical map of G onto $\widehat{\widehat{G}}$. For every $a \in G$, $\pi \in \widehat{G}$ and $f \in L^2(G,M_{n,2})$,

$$\begin{aligned}\mathscr{F}({}_af)(\pi) &= \int_G f(ax)\pi(x^{-1})\mathrm{d}m_G(x) = \int_G f(x)\pi(x^{-1})\pi(a)\mathrm{d}m_G(x) \\ &= \pi(a)\int_G f(x)\pi(x^{-1})\mathrm{d}m_G(x) = \iota_G(a)(\pi)\mathscr{F}(f)(\pi) \\ &= (\iota_G(a)\mathscr{F}(f))(\pi).\end{aligned} \tag{3.6}$$

One can identify $L^\infty(\widehat{G},M_n)$ as a C^*-subalgebra of $B\left(L^2(G,M_{n,2})\right)$. Then $\widehat{\mu} \in L^\infty(\widehat{G},M_n)$. From this fact, Lemma 2.4, (3.3) and (3.6), we have

$$\begin{aligned}{}_a\left(\Lambda_{\widehat{G}}(\widehat{\mu})(f)\right) &= {}_a\left(\mathscr{F}^{-1}(\widehat{\mu}\mathscr{F}(f))\right) = {}_a\left(\mathscr{F}^{-1}(\mathscr{F}(\mu *_\ell f))\right) = {}_a(\mu *_\ell f) \\ &= \mu *_\ell {}_af = \mathscr{F}^{-1}(\mathscr{F}(\mu *_\ell {}_af)) = \mathscr{F}^{-1}(\widehat{\mu}\mathscr{F}({}_af)) \\ &= \Lambda_{\widehat{G}}(\widehat{\mu})({}_af),\end{aligned} \tag{3.7}$$

for all $a \in G$, $f \in L^2(G,M_{n,2})$ and $\widehat{\mu} \in L^\infty(\widehat{G},M_n)$. Similarly, we have

$$\left(\Lambda_{\widehat{G}}(\widehat{\mu})(f)\right)_a = \Lambda_{\widehat{G}}(\widehat{\mu})(f_a), \tag{3.8}$$

for all $a \in G$, $f \in L^2(G,M_{n,2})$ and $\widehat{\mu} \in L^\infty(\widehat{G},M_n)$. Thus $\Lambda_{\widehat{G}}(\widehat{\mu})$ is in $CV_2(G,M_n)$, for all $\widehat{\mu} \in L^\infty(\widehat{G},M_n)$.

Similar to (3.4), for every $\phi \in L^\infty(\widehat{G},M_n)$, we define

$$\Theta_{\widehat{G}}(\phi)(f) = \mathscr{F}^{-1}(\mathscr{F}(f)\phi), \quad (f \in L^2(G,M_{n,2})). \tag{3.9}$$

Similar to Theorem 3.1, we have the following result for $\Theta_{\widehat{G}}$.

THEOREM 3.2. *Let G be a locally compact abelian group. Then $\Theta_{\widehat{G}}$ is an isometric $*$-anti-isomorphism from $L^\infty(\widehat{G},M_n)$ onto $CV_2(G,M_n)$.*

Proof. For all $\phi,\psi\in L^\infty(\widehat{G},M_n)$ and $f\in L^2(G,M_{n,2})$,

$$\begin{aligned}\Theta_{\widehat{G}}(\phi\psi)(f) &= \mathscr{F}^{-1}(\mathscr{F}(f)\phi\psi) = \mathscr{F}^{-1}((\mathscr{F}(f)\phi)\psi) = \mathscr{F}^{-1}\left(\mathscr{F}\left(\Theta_{\widehat{G}}(\phi)(f)\right)\psi\right)\\ &= \Theta_{\widehat{G}}(\psi)\left(\Theta_{\widehat{G}}(\phi)(f)\right).\end{aligned}$$

Thus, $\Theta_{\widehat{G}}$ is an anti-homomorphism. Also, similar to $\Lambda_{\widehat{G}}$ one can show that $\Theta_{\widehat{G}}$ is a $*$-map and consequently, it is a $*$-anti-homomorphism between C^*-algebras. This implies that it is an isometry. By Lemma 2.3 and (3.3), similar to the relations (3.7) and (3.8), we obtain for $\Theta_{\widehat{G}}$. Hence it is in $CV_2(G,M_n)$.

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Ali Ebadian
Faculty of Science
Department of Mathematics, Urmia University
Urmia, Iran
e-mail: ebadian.ali@gmail.com

Ali Jabbari
Department of Mathematics
Payame Noor University
Tehran, Iran
e-mail: jabbari_al@yahoo.com