Operators
and
Matrices

Volume 14, Number 1 (2020), 129–138

doi:10.7153/oam-2020-14-10

ASYMPTOTIC STABILITY OF A PERTURBED ABSTRACT DIFFERENTIAL EQUATIONS IN BANACH SPACES

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(*Communicated by F. Kittaneh*)

Abstract. This paper is mainly concerned with the asymptotic stability of the solutions of a perturbed abstract differential equation in Banach spaces. Let A be a generator of an exponentially stable operator semigroup and let $C(t)$, $t \geqslant 0$ be a linear bounded variable operator. Assuming that the perturbation $F(t,x)$ is sufficiently small norm for the equation $\frac{dx}{dt} = Ax + C(t)x + F(t,x)$, we derive the Lyapunov asymptotic stability conditions. These results are applied to partial differential equations.

1. Introduction

The problem of Lyapunov stability of infinite-dimensional dynamical systems has received a considerable amount of interest in the past, see [3, 4, 5, 7, 10, 13, 14, 16]. One of the basic methods for the stability analysis is the direct Lyapunov method. By that method, many strong result were established, see [5, 8], but finding the Lyapunov functionals is usually a difficult mathematical problem. A fundamental approach to the stability of diffusion parabolic equations is the method of upper and lower solutions. A systematical treatment of that approach is given in [11].
Many problems in partial differential equations which arise from physical models can be considered as ordinary differential equations in appropriate infinite dimensional spaces, for which elegant theories and powerful techniques have recently been developed, see [1, 2, 9, 12]. The asymptotic stability of zero solution of the non-perturbed parabolic equation was studied in [13, 14]. In the hypotheses from [13] and by assuming that the perturbation is of sufficient small norm, the author proved the Lyapunov stability and asymptotic stability of the zero solution of the perturbed equation. However, we extended in [15] previous results concerning the Lyapunov stability of some non-autonomous nonlinear evolution equations in Banach spaces. Moreover, Gil [4] investigated the stability of linear non-autonomous equations in a Banach space, which can be considered as integrally small perturbations of autonomous equations. Recently, Marx et al. [10] introduced the asymptotic behavior for nonlinear perturbed systems with nominal linear part and the perturbed term is a nonlinear damping function via

Mathematics subject classification (2010): 12H20, 47D03, 93D20, 35R15.
Keywords and phrases: Abstract differential equations, semigroups of linear operators, asymptotic stability, partial differential equations on infinite-dimensional.

Paper OaM-14-10

Lyapunov techniques in infinite-dimensional spaces. This result has been applied to the Linearized Korteweg-de Vries equation and wave equation.
The aim of this paper is to study the Lyapunov stability and asymptotic stability of the perturbed abstract differential equation based on the direct Lyapunov method when the origin is an equilibrium point. We prove the existence and uniqueness of a mild solution of the perturbed problem. In particular, we consider evolution equations with periodic boundary conditions.
The remainder of this work is organized as follows. In Section 2, some preliminary results are summarized and assumptions are provided. The statement of the mains result are provided in Section 3-4. Finally, an application is given in Section 5 to show the effectiveness of our result. Our conclusion is given in Section 6.

2. Mathematical preliminaries

We will use the following notation throughout this paper: $\mathbb{R}^+$ denotes the set of all non-negative real numbers, X denotes a real or complex Banach space with the norm $\|.\|_X$. $D(A)$ denotes the domain of the operator A, clM denotes the closure of a set M and I the identity operator. For a bounded operator K, $\|K\|$ is the operator norm. Everywhere below A is a linear operator in X with domain $D(A)$, generating a strongly continuous semigroup $T(t)$, that is,

$$A = \lim_{h \longrightarrow 0} \frac{T(h) - I}{h}$$

in the strong topology, and $C(t)$, $t \geqslant 0$ is a linear bounded variable operator mapping $D(A)$ into itself. Put $B(t) = A + C(t)$.
Consider the non-autonomous abstract differential equation

$$\begin{cases} \dfrac{dx}{dt} = B(t)x(t) + F(t,x), \qquad t \geqslant s \geqslant 0, \\ \\ x(s) = a, \end{cases} \tag{1}$$

where $x(t) \in X$ is the system state and a fixed $s \in \mathbb{R}^+$. Suppose now that the system (1) satisfies the following assumptions:
$(\mathscr{H}_1)$ The operator $A : D(A) \subset X \to X$ is a closed linear operator with $cl(D(A)) = X$, generates a strongly continuous semigroup on X, $T(t)$, exponentially stable, that is, there exist $M \geqslant 1$ and $\alpha > 0$, such that $\|T(t)\| \leqslant Me^{-\alpha t}$, for any $t \in \mathbb{R}^+$.
$(\mathscr{H}_2)$ The nonlinear mapping F, defined on the Cartesian product of $\mathbb{R}^+$ with a neighborhood of 0 in X, is continuous, $F(t,0) = 0$ for all $t \geqslant 0$, there exist $\beta > 0$, $D > 0$ and x_1, x_2 in a neighborhood of 0 in X, satisfying the following inequality

$$\|F(t,x_1) - F(t,x_2)\| \leqslant D\, max^{\beta}(\|x_1\|, \|x_2\|)\|x_1 - x_2\|. \tag{2}$$

Put

$$J(t) = \int_0^t C(u+s)du, \quad t \geqslant 0,$$

and

$$m(t) = \|AJ(t) - J(t)B(t)\|, \quad t \geqslant 0.$$

$(\mathscr{H}_3)$ $L := \sup_{t\geqslant 0}(\|J(t)\| + \int_0^t \|T(t-\theta)\| m(\theta) d\theta) < 1.$

REMARK 1. It is shown in [12] that the fitness of the L^1-norm of T, that is,

$$\int_0^{+\infty} \|T(t)\| dt < \infty,$$

implies that $T(t)$ is exponentially stable.

Obviously, $x(t) = 0$ is a solution of equation (1). We study its stability (in the sense made precise below).
With the non-autonomous equation (1), we associate the integral equation

$$x(t) = T(t-s)a + \int_s^t T(t-\theta)C(\theta)x(\theta)d\theta + \int_s^t T(t-\theta)F(\theta, x(\theta))d\theta. \tag{3}$$

Let s' be a real number with $s' > s$.

DEFINITION 1. We say that a function $x(.;s,a) : [s,s'[\to X$ is an A-mild solution of problem (1) on $[s,s'[$ if it is a solution of (3), for any $t \in [s,s'[$.

Obviously, a classical solution of (1) on $[s,s'[$ is also a solution of (3), hence an A-mild solution of the above problem.

REMARK 2. Note that in [7], Ion studied this problem when $C(t), \quad t \geqslant 0$ is a linear bounded constant operator and proved the stability and asymptotic stability of the zero solution. Moreover, Gil [4] assumed the equation (1) when $F(t,x) = 0$ and under some conditions he proved that the system is exponentially stable.

Our concern regards the stability behavior of the equilibrium point $\bar{x} = 0$. First, we recall some definitions about stability in Lyapunov sense.

DEFINITION 2. ([6]) A classical (resp, A-mild) solution $x(.;s_0,a_0)$ of problem (1) is called stable if for every $\varepsilon > 0$ and every $s > s_0$ there is a $\delta = \delta(\varepsilon, s)$, such that for every $y \in X$ with $\|y - x(s;s_0,a_0)\| \leqslant \delta$, the classical (resp. A-mild) solution $x(.;s,y)$ exits, is defined on $[s,+\infty[$, and

$$\|x(t;s_0,a_0) - x(t;s,y)\| < \varepsilon, \quad \forall t \geqslant s \geqslant 0.$$

For the asymptotic stability we take the following definition.

DEFINITION 3. A classical (resp, A-mild) solution $x(.;s_0,a_0)$ of problem (1) is called asymptotically stable if it is stable and for every $s > s_0$ there is a $\delta = \delta(s) > 0$, such that for $y \in X$ with $\|y - x(s;s_0,a_0)\| \leqslant \delta$, the classical (resp. A-mild) solution $x(.;s,y)$ exits, is defined on $[s,+\infty[$, and

$$\|x(t;s_0,a_0) - x(t;s,y)\| \longrightarrow 0, \quad as\, t \longrightarrow \infty.$$

The following Lemma will also be required in our investigations.

LEMMA 1. *Let $w(t)$, $f(t)$ and $v(t)$ $(0 \leqslant s \leqslant t \leqslant b \leqslant \infty)$ be functions whose values are bounded operators. Assume that $w(t)$ is integrable and $f(t)$ and $v(t)$ have integrable derivatives on $[0,b]$. Then, with the notation $j_w(t) = \int_0^t w(u)du$, one has*

$$\int_0^t f(u)w(u)v(u)du = f(t)j_w(t)v(t) - \int_0^t [f'(u)j_w(u)v(u) + f(u)j_w(u)v'(u)]du, \quad t \leqslant b.$$

Proof. Clearly,

$$\frac{d}{dt} f(t)j_w(t)v(t) = f'(t)j_w(t)v(t) + f(t)w(t)v(t) + f(t)j_w(t)v'(t).$$

Integrating this equality and taking into account that $j_w(0) = 0$, we arrive at the required result. □

3. Existence and stability of the origin of the perturbed problem

We consider $\mathscr{C}_b = \mathscr{C}_b([0,+\infty[)$ the space of continuous bounded functions from $[0,+\infty[$ to X, endowed with the supremum norm $\|x\|_0 = \sup_{t \geqslant 0} \|x(t)\|$.
We define the operators H, E_s, G_s as follows:

$$H : X \to \mathscr{C}_b$$

associating with $a \in X$ the function $H(a)$ is given by

$$H(a)(\tau) = T(\tau)a, \ \tau \geqslant 0;$$

for every $s \geqslant 0$,

$$E_s : \mathscr{C}_b \to \mathscr{C}_b,$$

given by

$$E_s(x)(\tau) = \int_0^\tau T(\tau - \theta)C(s + \theta)x(\theta)d\theta;$$

and

$$G_s : \mathscr{C}_b \to \mathscr{C}_b,$$

defined by

$$G_s(x)(\tau) = \int_0^\tau T(\tau - \theta)F(s + \theta, x(\theta))d\theta.$$

The fact that the operators H and G_s take values in the space $\mathscr{C}_b$ is a consequence of Lemma 1 and 2 from [13]. On the other hand, we have the following inequalities:

$$\|H\|_{\mathscr{L}(X,\mathscr{C}_b)} \leqslant M,$$

$$\|G_s(x)\|_0 \leqslant K\|x\|_0^{\beta+1}, \quad \forall x \in \mathscr{C}_b,$$

and

$$\|G_s(x) - G_s(y)\|_0 \leqslant Kmax(\|x\|_0^{\beta}, \|y\|_0^{\beta})\|x-y\|_0, \quad \forall x, y \in \mathscr{C}_b,$$

with $K = \dfrac{MD}{\alpha}\cdot$
Assume that

$$\|J(t)\| \leqslant q < 1, \ (q = const; t \geqslant 0).$$

Then, by Lemma 1, we get

$$\begin{aligned}&\int_0^\tau T(\tau-\theta)C(s+\theta)x(\theta)d\theta\\ =&T(0)J(\tau)x(\tau) - \int_0^\tau \left[\left(\frac{dT(\tau-\theta)}{d\theta}\right)J(\theta)x(\theta) + T((\tau-\theta)J(\theta)\dot{x}(\theta)\right]d\theta.\end{aligned}$$

But,

$$\frac{dT(\tau-\theta)}{d\theta} = -AT(\tau-\theta).$$

Then,

$$\begin{aligned}&\int_0^\tau T(\tau-\theta)C(s+\theta)x(\theta)d\theta\\ =&J(\tau)x(\tau) + \int_0^\tau T(\tau-\theta)[AJ(\theta) - J(\theta)B(\theta) - J(\theta)F(\theta, x(\theta))]x(\theta)d\theta.\end{aligned}$$

Thus, for $x \in \mathscr{C}_b$,

$$\begin{aligned}\|E_s(x)(\tau)\| \leqslant &\|J(\tau)x(\tau)\| + \int_0^\tau \|T(\tau-\theta)\|m(\theta)\|x(\theta)\|d\theta\\ &+ \int_0^\tau \|T(\tau-\theta)\|\|J(\tau)\|\|F(\theta, x(\theta))\|x(\theta)\|d\theta\\ \leqslant &L\|x\|_0 + Kq\|x\|_0^{\beta+2}.\end{aligned}$$

By taking the supremum for $\tau \in [0, +\infty[$, we obtain

$$\|E_s(x)\|_0 \leqslant L\|x\|_0 + Kq\|x\|_0^{\beta+2}.$$

The same method leads us to

$$\|E_s(x_1) - E_s(x_2)\|_0 \leqslant L\|x_1 - x_2\|_0 + Kq \ \max(\|x_1\|_0^{\beta}, \|x_2\|_0^{\beta})\|x_1 - x_2\|_0^2, \quad \forall x_1, x_2 \in \mathscr{C}_b.$$

In the integral equation (3), where $t > s$, let us set $t - s = \tau$ and make the change variable $\theta - s = \theta'$. We obtain,

$$\begin{aligned}x(s+\tau) =&T(\tau)a + \int_0^\tau T(\tau-\theta')C(s+\theta')x(s+\theta')d\theta'\\ &+ \int_0^\tau T(\tau-\theta')F(s+\theta', x(s+\theta'))d\theta'.\end{aligned} \tag{4}$$

For every continuous function $x:\mathbb{R}^+\to X$ and every $s\in\mathbb{R}^+$, we define the function $x|_s:\mathbb{R}^+\to X$, by $x|_s(\theta)=x(s+\theta)$, such that the previous integral equation can be written as

$$x|_s = Ha + E_s(x|_s) + G_s(x|_s) \tag{5}$$

Since $x|_s$ belongs to $\mathscr{C}_b$, equation (5) is equivalent to the fixed point problem

$$\phi = \varphi(a,\phi),$$

where $\varphi : X\times\mathscr{C}_b\to\mathscr{C}_b$ is given by

$$\phi = \varphi(a,\phi) = Ha + E_s(\phi) + G_s(\phi).$$

In the sequel, we will assumed some conditions that guarantee the existence and uniqueness of solutions for the system (1).

THEOREM 1. *Let* $r_1^\beta + 2qr_1^{\beta+1} < \dfrac{1-L}{K}$. *If* $\|a\| < \dfrac{r_1}{M}(1-L-Kr_1^\beta - 2Kqr_1^{\beta+1})$, *the mapping* $\varphi(a,.)$ *is an uniform contraction from* $B(0,r_1)\subset\mathscr{C}_b$ *to itself.*

Proof. For any $a\in X$ and any $\phi_1,\phi_2\in B(0,r)\subset\mathscr{C}_b$, we have

$$\|\varphi(a,\phi_1)-\varphi(a,\phi_2)\|_0 \leqslant [L+Kr^\beta+2Kqr^{\beta+1}]\|\phi_1-\phi_2\|_0.$$

We choose a positive real number r_1 satisfying the condition

$$L+Kr_1^\beta+2Kqr_1^{\beta+1}<1 \Longleftrightarrow r_1^\beta+2qr_1^{\beta+1}<\frac{1-L}{K},$$

and find that, for $\|\phi\|_0\leqslant r_1$, $\varphi(a,\phi)$ is an uniform contraction with respect to a. If $\|a\|\leqslant r_0$, then

$$\|\varphi(a,\phi)\|_0 \leqslant Mr_0 + (Kr_1^\beta + Kqr_1^{\beta+1})r_1.$$

By imposing this last quantity to be less than r_1, we find the restriction

$$r_0 < \frac{r_1}{M}(1-L-Kr_1^\beta-2Kqr_1^{\beta+1}).$$

This ends the proof of Theorem 1. □

PROPOSITION 1. *For every* $s\geqslant 0$ *and any* $\|a\|$ *small enough, there is an unique A-mild solution of problem (1).*

Proof. We consider $\|a\|<r_0$, with r_0 defined in the proof of Theorem 1. By The Uniform Contraction Principle and Theorem 1, we have the existence of an unique fixed point $\phi^*(a)\in B(0,r_1)\subset\mathscr{C}_b$ of the mapping $\varphi(a,.)$. We define the function $x: [s,+\infty[\to X$,

$$x(t;s,a) = \phi^*(a)(t-s),\ t\geqslant s.$$

This is the solution of problem (3), that is, the A-mild solution of problem (1). This ends the proof. □

THEOREM 2. *Assume that* $(\mathscr{H}_1)-(\mathscr{H}_3)$ *hold. Then, the A-mild solution* $x(t)=0$, $t\geqslant 0$, *of equation* (1) *is stable.*

Proof. The solution $x(t)=0$ can be regarded as $x(.;s_0,0)$ for any $s_0\geqslant 0$. For $s\geqslant 0$ and $\varepsilon>0$, we choose $\delta<\min\{\varepsilon,r_1\}$, with

$$r_1^{\beta}+2qr_1^{\beta+1}<\frac{1-L}{K},$$

and

$$\delta_1=\min\{\delta,\frac{\delta}{M}(1-L-K\delta^{\beta}-2Kq\delta^{\beta+1})\}.$$

By Theorem 1, we have for $\|a\|<\delta_1$ the A-mild solution $x(t;s,a)$ exists for $t\geqslant s$, is unique and $\|x(t;s,a)\|<\varepsilon$, for any $t\geqslant s$. This finished the proof. □

4. Asymptotic stability of the origin of the perturbed problem

The main result of this paper is given by the following Theorem:

THEOREM 3. *Under assumptions* $(\mathscr{H}_1)-(\mathscr{H}_3)$, *the A-mild solution* $x(.;s,0)=0$, $s\geqslant 0$, *of equation* (1) *is asymptotically stable.*

Proof. Let $\varepsilon>0$ be given such that $K\varepsilon^{\beta}+Kq\varepsilon^{\beta+1}<\frac{1-L}{3}\cdot$ We take δ_1 as in the proof of Theorem 2 and a with $\|a\|<\delta_1$. Using the equation (4), one has

$$\begin{aligned}\|x(s+\tau)\| \leqslant&\|T(\tau)a\|+\|\int_0^{\tau}T(\tau-\theta')C(s+\theta')x(s+\theta')d\theta'\|\\ &+\|\int_0^{\tau}T(\tau-\theta')F(s+\theta',x(s+\theta'))d\theta'\|\\ \leqslant& Me^{-\alpha\tau}\|a\|+L\|x|_s\|_0+K\|x|_s\|_0^{\beta+1}+Kq\|x|_s\|_0^{\beta+2}.\end{aligned}$$

From the proof of Theorem 2, it follows that $\|x|_s\|_0<\varepsilon$, and thus

$$\|x(s+\tau)\|\leqslant Me^{-\alpha\tau}\|a\|+L\varepsilon+K\varepsilon^{\beta+1}+Kq\varepsilon^{\beta+2}.$$

We choose τ_1, such that $Me^{-\alpha\tau_1}<\frac{1-L}{3}\cdot$ Then, for $\tau\geqslant\tau_1$,

$$\|x(s+\tau)\|\leqslant 2\frac{1-L}{3}\varepsilon+L\varepsilon=\left(\frac{2}{3}+\frac{L}{3}\right)\varepsilon.$$

Hence, for $\tau\geqslant\tau_1$, we have

$$\|x(s+\tau)\|\leqslant\lambda\varepsilon,\quad \lambda=\frac{2}{3}+\frac{L}{3}<1,$$

that is

$$\|x|_{s+\tau_1}\|_0\leqslant\lambda\varepsilon.$$

Due to equation (4), we obtain

$$\|x(s+\tau_1+\tau)\| \leqslant Me^{-\alpha\tau}\|x(s+\tau_1)\| + L\|x|_{s+\tau_1}\|_0 + K\|x|_{s+\tau_1}\|_0^{\beta+1} + Kq\|x|_{s+\tau_1}\|_0^{\beta+2}$$

and

$$\|x(s+\tau_1+\tau)\| \leqslant \lambda^2\varepsilon, \quad \tau > \tau_1.$$

That is

$$\|x|_{s+2\tau_1}\| \leqslant \lambda^2\varepsilon.$$

By induction, we obtain that for $t > s+n\tau_1$, the inequality $\|x(t)\| \leqslant \lambda^n\varepsilon$ holds, implying that

$$\|x(t;s,x_s)\| \longrightarrow 0, \quad as\, t \longrightarrow \infty.$$

Hence, A-mild null solution of problem (1) is asymptotically stable. This ends the proof. □

To illustrate Theorem 3, we consider the following equation

$$\begin{cases} \dot{x}(t) = Ax(t) + c(t)C_0x(t) + F(t,x), \\ x(0) = a \end{cases} \tag{6}$$

where A and F two operators satisfying the hypothesis $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$. C_0 is a constant operator and $c(t)$ is a scalar real continuous function bounded on $[0,+\infty[$. So, $C(t) = c(t)C_0$. Let, $\theta_0 = \sup_t i_c(t)$ with $i_c(t) = \left|\int_0^t c(u)du\right|$. We obtain,

$$\begin{aligned} m(t) = \|AJ(t) - J(t)B(t)\| &\leqslant i_c(t)(\|AC_0 - C_0A\| + |c(t)|\|C_0^2\|) \\ &\leqslant i_c(t)(\|AC_0 - C_0A\| + \|C_0^2\|). \end{aligned}$$

On the other hand,

$$\int_0^t \|T(t-\theta)\|d\theta \leqslant \frac{M}{\alpha}, \quad t \geqslant 0.$$

As a consequence of Theorem 3, we have the following result.

COROLLARY 1. *Under assumptions* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$, *if the inequality*

$$\theta_0\left(\|C_0\| + \frac{M}{\alpha}(\|AC_0 - C_0A\| + \|C_0^2\|)\right) < 1$$

holds, then A-mild null solution of problem (6) is asymptotically stable.

EXAMPLE 1. Consider the system (6) with $c(t) = \cos(wt)(w > 0)$. Then, $i_c(t) \leqslant \frac{1}{w}$ and $m(t) \leqslant \frac{1}{w}(\|AC_0 - C_0A\| + \|C_0^2\|)$. Therefore, if

$$\|C_0\| + \frac{M}{\alpha}(\|AC_0 - C_0A\| + \|C_0^2\|) < w,$$

then A-mild null solution of problem (6) is asymptotically stable.

5. A partial differential equation with periodic boundary conditions

As an application, we study a partial differential equation to illustrate the applicability of our result.
Consider the perturbed problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial u(x,t)}{\partial x} + (-b + c(t)a(x))u(x,t) + xu^2(x,t), \quad u(x,0) = u_0(x), \\ u(0,t) = u(1,t), \quad (t \geqslant 0), \quad (0 \leqslant x \leqslant 1) \end{cases} \tag{7}$$

with a positive constant b, $a(x)$ is a real differentiable function and $c(t)$ is the same as in the previous section. The partial differential equation can be formulated to an abstract differential equation on $X = L^2(0,1)$ of the form

$$\frac{du}{dt}(t) = Au(t) + C(t)u(t) + F(t,u(t)), \quad t \geqslant 0, \quad u(0) = u_0,$$

where the operator A is defined by

$$Au(x,t) = \frac{\partial u(x,t)}{\partial x} - bu(x,t), \quad u \in D(A),$$

with $D(A) = \{f \in L^2(0,1) : f' \in L^2(0,1); f(0) = f(1)\}$, $C(t) = c(t)a(x)I$, and

$$F(t,u(x,t)) = xu^2(x,t), \quad t \geqslant 0, \quad 0 \leqslant x \leqslant 1.$$

We have,

$$\begin{aligned} < Au, u > &= \int_0^1 \left(\frac{du(x)}{dx} - bu(x) \right) u(x)dx = \int_0^1 \left(\frac{1}{2}\frac{du^2(x)}{dx} - bu^2(x) \right) dx \\ &= \frac{1}{2}(u^2(1) - u^2(0)) - b\int_0^1 u^2(x)dx = -b\int_0^1 u^2(x)dx. \end{aligned}$$

Let $v(t,x)$ is the solution of $\dfrac{dv}{dt} = Av$. Then,

$$\frac{d}{dt} < v, v >= 2 < Av, v > \leqslant -2b < v, v > .$$

Therefore,

$$\|T(t)\| \leqslant e^{-bt}, \quad t \geqslant 0.$$

Moreover, the perturbation F satisfies the condition (2). On the other hand, we have

$$(Aa(x) - a(x)A)u(x) = \frac{d(a(x)u(x))}{dx} - a(x)\frac{du(x)}{dx} = a'(x)u(x), \quad u \in D(A).$$

Let, $\theta_0 = \sup_t \left| \int_0^t c(u)du \right|$. Due to the hypothesis $(\mathscr{H}_3)$, we obtain the following.

COROLLARY 2. *Under assumptions* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$, *if the inequality*

$$\theta_0(|a(x)| + \frac{1}{b}(|a'(x)| + |a(x)|^2)) < 1, \quad 0 \leqslant x \leqslant 1,$$

holds, then A-mild null solution of problem (7) is asymptotically stable.

6. Conclusion

In this paper, we have studied the problem of Lyapunov stability of a perturbed abstract differential equation in a Banach space. In this case, we introduced the existence and uniqueness of solutions of system. Moreover, sufficient conditions have been derived to guarantee the asymptotic stability for a class of perturbed systems. As an illustration, we gave an example of partial differential equation with periodic boundary conditions.

REFERENCES

[1] N. CHAFEE, *A stability analysis for a semilinear parabolic partial differential equation*, Journal of Differential Equations **15** (1974), 522–540.

[2] R.F. CURTAIN, *Stability of stochastic partial differential equation*, Journal of Mathematical Analysis and Applications **79** (1981), 352–369.

[3] H. DAMAK AND M.A. HAMMAMI, *Stabilization and Practical Asymptotic Stability of Abstract Differential Equations*, Numerical Functional Analysis and Optimization **37** (2016), 1235–1247.

[4] M. GIL, *Integrally small perturbation of semigroups and stability of partial differential equations*, International Journal of Partial Differential Equations **2013** (2013).

[5] M.I. GIL, *Stability of Finite and Infinite Dimensional Systems*, Kluwer Academic Publishers, Boston, Mass, USA (1998).

[6] G. GUHRING AND F. RABIGER, *Linearized stability for semilinear non-autonomous evolution equations with applications to retarded differential equations*, Ulmer Seminare **2** (1997).

[7] A.V. ION, *Lyapunov stability of the zero solution of a perturbed abstract parabolic non-autonomous equation*, ROMAI Journal **4** (2008), 105–114.

[8] V. LAKSHMIKANTHAM, V.M. MASTROV AND S. SIVASUNDARAM, *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Kluwer Academic, Dordrecht, The Netherlands (1991).

[9] Z. LUO, B. GUO AND O. MORGUL, *Stability and Stabilization of Infinite Dimensional Systems With Applications*, Springer, London (1999).

[10] S. MARX, Y. CHITOUR AND C. PRIEUR, *Stability analysis of dissipative systems subject to nonlinear damping via Lyapunov techniques*, arXiv:1808.05370v1, (2018).

[11] C.V. PAO, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, NY, USA (1992).

[12] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, NY, USA (1983).

[13] V.A. TRENOGIN, *Lyapunov theorem about the stability by linear approximation as the consequence of the implicit operators theorem*, Doklady Academii Nauk **407** (2006), 742–746.

[14] V.A. TRENOGIN, *Asymptotic stability of trivial solution of differential equations with unbounded linearity*, in Banach spaces, ROMAI Journal **3** (2007), 207–212.

[15] V.A. TRENOGIN AND ANCA-V.ION, *New results in the stability study of non-autonomous evolution equations in Banach spaces*, Journal of Mathematics and Applications (2010), No 33, 117–127.

[16] A.L. ZUEV, *Partial asymptotic stability of abstract differential equations*, Ukrainian Mathematical Journal **58** (2006), 709–717.

(Received March 6, 2019)

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