Operators
and
Matrices
Volume 14, Number 1 (2020), 139–144 doi:10.7153/oam-2020-14-11

A NOTE ON ORLICZ ALGEBRAS

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(*Communicated by N.-C. Wong*)

Abstract. The purpose of this paper is to give a necessary and sufficient condition for an Orlicz space $L^{\Phi}(G)$ to be a convolution Banach algebra, where G is a compactly generated locally compact abelian group and Φ is a Young function satisfying Δ_2-condition and an extra sequence condition.

1. Introduction and preliminaries

Let $1 < p < \infty$ and G be a locally compact group. This is an old question to find equivalence conditions for a Lebesgue space $L^p(G)$ to be a convolution Banach algebra. The first result related to this question is due to [8, 7], and it was formulated in 1963 by Rajagopalan in his Ph.D. thesis. So far, this question has been studied in several papers.

The main motivation for writing this work is the paper [1] in which it is proved that for a given $2 < p < \infty$, if $f * g$ exists for all $f, g \in L^p(G)$, then automatically $L^p(G)$ is closed under the convolution product, and so $L^p(G)$ will be a convolution Banach algebra; see [1, Theorem 1.1].

In fact, we intend to study this question for Orlicz spaces as a generalization of the usual Lebesgue spaces. Hudzik, Kaminska and Musielak in [4] gave some equivalent conditions for an Orlicz space $L^{\Phi}(G)$ to be a convolution algebra, where Φ is a Young function satisfying Δ_2-condition; see Theorem 1. Recently, in [6] the weighted Orlicz algebras have been studied. In this paper, by some ideas from [1, Theorem 1.1] we prove that if Φ is a Young function with Δ_2-condition and satisfying the sequence condition (1), then $L^{\Phi}(G)$ is a Banach algebra if and only if $f * g$ exists for all $f, g \in L^{\Phi}(G)$. In our main result, we assume that G is a compactly generated abelian group to use some key facts about its periodic (or compact) elements. Finally, after some examples, we give a result about logarithmic Zygmund spaces.

First, let us recall some basic definitions and notations about Orlicz spaces; see the book [5] as a monograph on this topic. In sequel, G is a second countable locally compact group with a (left) Haar measure m.

Mathematics subject classification (2010): 46E30, 43A15.

Keywords and phrases: Locally compact group, Orlicz algebra, convolution, Young function, N-function, Zygmund space.

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DEFINITION 1. A convex even mapping $\Phi:\mathbb{R}\to[0,\infty]$ is called a *Young function* if $\Phi(0)=\lim_{x\to 0}\Phi(x)=0$ and $\lim_{x\to\infty}\Phi(x)=\infty$.

A continuous Young function $\Phi:\mathbb{R}\to[0,\infty)$ is called an *N-function* if $\Phi(x)=0$ implies $x=0$,

$$\lim_{x\to 0}\frac{\Phi(x)}{x}=0, \quad \text{and} \lim_{x\to\infty}\frac{\Phi(x)}{x}=\infty.$$

DEFINITION 2. The *complementary* of a Young function Φ is defined by

$$\Psi(x):=\sup\{y|x|-\Phi(y): y\geqslant 0\}, \quad (x\in\mathbb{R}).$$

In this case, (Φ,Ψ) is called a *Young pair*.

In sequel, (Φ,Ψ) is a Young pair. The set of all Borel measurable functions $f:G\to[-\infty,\infty]$ such that for some $\alpha>0$,

$$\int_G \Phi(\alpha|f(x)|)\,dm(x)<\infty,$$

is denoted by $L^\Phi(G)$. Two elements $f,g\in L^\Phi(G)$ are considered the same if they are equal m-almost everywhere.

For each $f\in L^\Phi(G)$ we define

$$\|f\|_\Phi:=\sup\left\{\int_G|fg|\,dm:\int_G\Psi(|g(x)|)\,dm(x)\leqslant 1\right\}.$$

Since G is locally compact and m is a regular measure on G, the measure m has the finite subset property, and so by [5, Chapter III, Proposition 11], $(L^\Phi(G),\|\cdot\|_\Phi)$ (named *Orlicz space*) is a Banach space.

In special case, if $p\geqslant 1$ and the Young function Φ is given by $\Phi(x):=|x|^p$ for all $x\in\mathbb{R}$, then $L^\Phi(G)$ is the usual Lebesgue space $L^p(G)$.

The mapping $\|\cdot\|_\Phi^\circ$ defined by

$$\|f\|_\Phi^\circ:=\inf\left\{\lambda>0:\int_G\Phi(\frac{1}{\lambda}f(x))\,dm(x)\leqslant 1\right\}, \quad (f\in L^\Phi(G)),$$

is also a norm on $L^\Phi(G)$, called the *Luxemburg* norm [5, Chapter III, Theorem 3]. For each $f\in L^p(G)$ we have

$$\|f\|_\Phi^\circ\leqslant\|f\|_\Phi\leqslant 2\|f\|_\Phi^\circ,$$

and so the above norms are equivalent [5, Chapter III, Proposition 4].

The *convolution* of any two functions $f,g\in L^\Phi(G)$ is defined by

$$(f*g)(x):=\int_G f(y)g(y^{-1}x)\,dm(y), \quad (x\in G),$$

while this integral exists.

The Orlicz space $L^\Phi(G)$ is a *convolution Banach algebra* if there exist a constant $c>0$ such that for all $f,g\in L^\Phi(G)$, $f*g\in L^\Phi(G)$ and

$$\|f*g\|_\Phi\leqslant c\,\|f\|_\Phi\,\|g\|_\Phi.$$

DEFINITION 3. We say that a Young function Φ satisfies Δ_2-*condition* (and denote $\Phi \in \Delta_2$) if there exist constants $c > 0$ and $x_0 \geqslant 0$ such that

$$\Phi(2x) \leqslant c\Phi(x), \quad (x \geqslant x_0).$$

2. Main result

Here, first we recall a result from [4, Theorem 2] which is a key tool in the proof of Theorem 2.

THEOREM 1. *Let G be a locally compact group and Φ be a Young function with $\Phi \in \Delta_2$. The following conditions are equivalent:*

1. *$L^\Phi(G)$ is a Banach algebra under convolution;*
2. *$L^\Phi(G) \subseteq L^1(G)$;*
3. *$\lim_{x\to 0^+} \frac{\Phi(x)}{x} > 0$ or G is compact.*

DEFINITION 4. Let G be a locally compact group. An element $a \in G$ is called *periodic* (or *compact*) if the closed subgroup generated by a is compact. If $a \in G$ is not periodic, it is called *aperiodic*.

Now, the main result is stated.

THEOREM 2. *Let G be a compactly generated locally compact abelian group and Φ be a Young function with $\Phi \in \Delta_2$ such that*

$$\lim_{x\to 0^+} \frac{\Phi(x)}{x} = 0.$$

Let there exist two nonnegative sequences (α_n) and (β_n) such that

$$\sum_{n=1}^{\infty} \Phi(\alpha_n) < \infty, \quad \sum_{n=1}^{\infty} \Phi(\beta_n) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty. \tag{1}$$

Then, $L^\Phi(G)$ is a Banach algebra under the convolution if and only if for each $f,g \in L^\Phi(G)$, $(f*g)(x)$ exists for almost every $x \in G$.

Proof. Let for each $f,g \in L^\Phi(G)$, $(f*g)(x)$ exists for almost every $x \in G$. Since G is a compactly generated locally compact abelian group, by [3, 9.26(b)], the set of all periodic elements of G is a compact subgroup of G. So, in contrast, if G is non-compact, it contains an aperiodic element a. Let U be a compact symmetric neighborhood of e in G. Then, by [2, Lemma 2.1], there is a constant natural number N_0 such that for each $n \geqslant N_0$, $U \cap Ua^n = \varnothing$. Let V be a compact symmetric neighborhood of

e with $VV \subseteq U$. Then, we have also $V \cap Va^n = \varnothing$ for all $n \geqslant N_0$. By (1), there is a constant natural number N_1 such that

$$\sum_{n=N_1}^{\infty} \Phi(\alpha_n) < \frac{1}{m(V)} \quad \text{and} \quad \sum_{n=N_1}^{\infty} \Phi(\beta_n) < \frac{1}{m(VV)}.$$

Setting $N := \max\{N_0, N_1\}$, we define

$$f(x) := \sum_{n=N}^{\infty} \alpha_n \chi_{Va^{-nN}}(x),$$

and

$$g(x) := \sum_{n=N}^{\infty} \beta_n \chi_{VVa^{nN}}(x),$$

for all $x \in G$. For each distinct $m, n \geqslant N$ we have

$$Va^{-mN} \cap Va^{-nN} = VVa^{mN} \cap VVa^{nN} = \varnothing.$$

So, since $\Phi(0) = 0$,

$$\begin{aligned}\int_G \Phi(f(x))\,dm(x) &= \int_{\bigcup_{n=N}^{\infty} Va^{-nN}} \Phi(f(x))\,dm(x) = \sum_{n=N}^{\infty} \int_{Va^{-nN}} \Phi(f(x))\,dm(x)\\ &= \sum_{n=N}^{\infty} \int_{Va^{-nN}} \Phi(\alpha_n)\,dm(x) = m(V) \sum_{n=N}^{\infty} \Phi(\alpha_n) < 1.\end{aligned}$$

Similarly,

$$\int_G \Phi(g(x))\,dm(x) = m(VV) \sum_{n=N}^{\infty} \Phi(\beta_n) < 1.$$

So, $f, g \in L^{\Phi}(G)$. But, for each $x \in V$,

$$\begin{aligned}(f*g)(x) &= \int_G f(y) g(y^{-1}x)\,dm(y)\\ &= \sum_{n=N}^{\infty} \int_{Va^{-nN}} f(y) g(y^{-1}x)\,dm(y) = \sum_{n=N}^{\infty} \alpha_n \int_{Va^{-nN}} g(y^{-1}x)\,dm(y)\\ &= \sum_{n=N}^{\infty} \alpha_n \int_{Va^{-nN}} \sum_{m=N}^{\infty} \beta_m \chi_{VVa^{mN}}(y^{-1}x)\,dm(y) = \sum_{n=N}^{\infty} \alpha_n \int_{Va^{-nN}} \beta_n\,dm(y)\\ &= m(V) \sum_{n=N}^{\infty} \alpha_n \beta_n = \infty,\end{aligned}$$

since G is abelian and m is a Haar measure. This implies that $f*g$ does not exist on a set with positive measure, a contradiction. $\square$

Thanks to [5, Chapter II, Theorem 3], one can change some assumptions of Theorem 2:

COROLLARY 1. *Let Φ be an N-function satisfying the sequence condition (1), and for some constants $c>0$ and $x_0 \in \mathbb{R}$,*

$$\frac{x\varphi(x)}{\Phi(x)} \leqslant c, \quad (x \geqslant x_0),$$

where φ is the right derivative of Φ. Then, $L^{\Phi}(G)$ is a convolution Banach algebra if and only if $f*g$ exists for each $f,g \in L^{\Phi}(G)$.

EXAMPLE 1. 1. Let $a>0$ and $p>2$. Then, the function Φ defined by $\Phi(x) := a|x|^p$ for all $x \in \mathbb{R}$, satisfies the hypothesis of Theorem 2. This example will cover the Lebesgue space $L^p(G)$.

2. Let $p>2$, $\gamma>0$, and the function Φ defined by $\Phi(x) := |x|^p(\ln(1+|x|))^{\gamma}$ for all $x \in \mathbb{R}$, be an N-function. Then,

$$\frac{x\Phi'(x)}{\Phi(x)} = p + \frac{\gamma x}{(1+x)\ln(1+x)} \leqslant p+\gamma^2, \quad (x \geqslant e^{\frac{1}{\gamma}}-1),$$

and so by [5, Chapter II, Theorem 3], $\Phi \in \Delta_2$. Also, setting $\alpha_n = \beta_n := \frac{1}{\sqrt{n}}$, we see that for each $1<\beta<\frac{p}{2}$, the relation $\Phi(\alpha_n) = \Phi(\beta_n) \leqslant \frac{1}{n^{\beta}}$ holds for sufficiently large $n \in \mathbb{N}$, and so, Φ satisfies the sequence condition (1).

3. Let $\alpha>0$, and the function Φ defined by $\Phi(x) := |x|(\ln(1+|x|))^{\alpha}$ for all $x \in \mathbb{R}$, be an N-function. Then,

$$\frac{x\Phi^{'}(x)}{\Phi(x)} = 1 + \frac{\alpha x}{(1+x)\ln(1+x)} < 1+\alpha^2,$$

whenever $x \geqslant e^{\frac{1}{\alpha}}-1$. Now, let $\alpha_n = \beta_n := \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Then, for each $1<\beta\leqslant 2$ and $n \in \mathbb{N}$ we have $\Phi(\alpha_n) = \Phi(\beta_n) \leqslant \frac{\alpha}{2\beta-\alpha}\frac{1}{n^{\beta}}$. This implies that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the condition (1) whenever $1<\beta\leqslant 2$ and $\alpha<2\beta$.

COROLLARY 2. *Let $2<p<3$ and $-2<\alpha\leqslant -1$. Then, the Zygmund space $L^p(\ln L)^{\alpha}$ is a Banach algebra if and only if $f*g$ exists for all $f,g \in L^p(\ln L)^{\alpha}$.*

Proof. For each $x \in \mathbb{R}$, we define $\Phi_{p,\alpha}(x) := |x|^p(\ln(e+|x|))^{\alpha}$. Then, the Orlicz space $L^{\Phi_{p,\alpha}}(G)$ will recover the logarithmic Zygmund spaces $L^p(\ln L)^{\alpha}$. Setting $\alpha_n = \beta_n = \frac{1}{\sqrt{n}}$ for each $n=1,2,\ldots$, by some calculation one can see that the function $\Phi_{p,\alpha}$ satisfies the assumptions of Theorem 2, and the proof is complete. □

We end this paper with an open problem: Let (α_n) and (β_n) be two sequences of positive numbers with $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$. Is there an N-function $\Phi \in \Delta_2$ such that

$$\sum_{n=1}^{\infty}\Phi(\alpha_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}\Phi(\beta_n) < \infty?$$

Acknowledgement. The authors would like to thank the referee of this paper for helpful remarks and suggestions to improve this paper.

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(Received March 14, 2019)

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