

## CLASSES OF OPERATORS RELATED TO $m$ -ISOMETRIC OPERATORS

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*Abstract.* Isometries played a pivotal role in the development of operator theory, in particular with the theory of contractions and polar decompositions and has been widely studied due to its fundamental importance in the theory of stochastic processes, the intrinsic problem of modeling the general contractive operator via its isometric dilation and many other areas in applied mathematics. In this paper we present some properties of  $n$ -quasi- $(m, C)$ -isometric operators. We show that a power of a  $n$ -quasi- $(m, C)$ -isometric operator is again a  $n$ -quasi- $(m, C)$ -isometric operator and some products and tensor products of  $n$ -quasi- $(m, C)$ -isometries are again  $n$ -quasi- $(m, C)$ -isometries.

### 1. Introduction

Let  $\mathcal{H}$  be a separable infinite dimensional complex Hilbert space with inner product  $\langle \cdot | \cdot \rangle$ ,  $\mathcal{B}(\mathcal{H})$  be the set of all bounded linear operators on  $\mathcal{H}$ , and  $I = I_{\mathcal{H}}$  be the identity operator. For every  $T \in \mathcal{B}(\mathcal{H})$  its range is denoted by  $\mathcal{R}(T)$ , its null space by  $\mathcal{N}(T)$ . The adjoint of  $T$  is denoted by  $T^*$ . A subspace  $\mathcal{M} \subset \mathcal{H}$  is invariant for  $T$  (or  $T$ -invariant) if  $T\mathcal{M} \subset \mathcal{M}$ . As usual, the orthogonal complement and the closure of  $\mathcal{M}$  are denoted  $\mathcal{M}^{\perp}$  and  $\overline{\mathcal{M}}$  respectively. We denote by  $P_{\mathcal{M}}$  the orthogonal projection on  $\mathcal{M}$ .

A conjugation is a conjugate-linear operator  $C : \mathcal{H} \rightarrow \mathcal{H}$ , which is both involutive (i.e.,  $C^2 = I$ ) and isometric (i.e.,  $\langle Cx | Cy \rangle = \langle y | x \rangle \ (\forall x, y \in \mathcal{H})$ ).

Recall that if  $C$  is a conjugation on  $\mathcal{H}$ , then  $\|C\| = 1$ ,  $(CTC)^k = CT^kC$  and  $(CTC)^* = CT^*C$  for every positive integer  $k$  (see [14] and [15] for more details).

Throughout this paper, let  $m$  and  $n$  be natural numbers. An operator  $T \in \mathcal{B}(\mathcal{H})$  is said to be :

$m$ -isometry if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^{*m-k} T^{m-k} = 0, \quad (1.1)$$

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or equivalently if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 \quad \forall x \in \mathcal{H}. \tag{1.2}$$

where  $\binom{m}{k}$  is the binomial coefficient. These class of operators have been introduced and studied by J. Agler and M. Stankus in [2], [3] and [4]. In recent years, the  $m$ -isometric operators have received substantial attention. Several authors have studied the  $m$ -isometric operator. We refer the reader to [5, 6, 7, 8, 9, 10, 21] for further details.

It has been proved in [7] and [10] that the powers of an  $m$ -isometry are  $m$ -isometries and some products of  $m$ -isometries are again  $m$ -isometries. On the other hand, the perturbation of  $m$ -isometries by nilpotent operators has been considered in [9], [8], [5] and the dynamics of  $m$ -isometries has been explored in [6] and other papers. Furthermore, Duggal studied the tensor product of  $m$ -isometries in [13]. In addition,  $m$ -isometry weighted shift operators have been discussed in [1] and the reference therein. S. Mecheri and T. Parasad in [19] extended the notion of  $m$ -isometric operator to the case of  $n$ -quasi- $m$ -isometric operators of bounded linear operators on a Hilbert space. An operator  $T \in \mathcal{B}(\mathcal{H})$  is said to be  $n$ -quasi- $m$ -isometric operator if

$$T^{*n} \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} \right) T^n = 0. \tag{1.3}$$

The 1-quasi-isometries are shortly called quasi-isometries, such operators being firstly studied in [22] and [23]. The study of quasi-2-isometries was found in [20].

In [11], M. Chō, E. Ko and J. Lee introduced  $(m, C)$ -isometric operators with conjugation  $C$  and studied properties of such operators. For an operator  $T \in \mathcal{B}(\mathcal{H})$  and an integer  $m \geq 1$ ,  $T$  is said to be an  $(m, C)$ -isometric operator if there exists some conjugation  $C$  such that

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} C T^{m-k} C = 0. \tag{1.4}$$

According to definitions of  $m$ -isometry,  $n$ -quasi- $m$ -isometry and  $(m, C)$ -isometry, The authors in [24] define an  $n$ -quasi- $(m, C)$ -isometry as follows. An operator  $T$  is said to be an  $n$ -quasi- $(m, C)$ -isometric operator if there exists some conjugation  $C$  such that

$$T^{*n} \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} C T^{m-k} C \right) T^n = 0. \tag{1.5}$$

It is easy to see that the class of  $n$ -quasi- $(m, C)$ -isometry contains every  $(m, C)$ -isometric operators with conjugation  $C$ . In general, this inclusion relation is proper (see [24]). Many results about the class of  $n$ -quasi- $(m, C)$ -isometric operators have been found in [24].

In this paper it is shown that the operators in this class have many interesting properties in common with  $m$ -isometries,  $n$ -quasi- $m$ -isometries and  $(m, C)$ -isometric operators. In particular, we show that the powers of an  $n$ -quasi- $(m, C)$ -isometry are  $n$ -quasi- $(m, C)$ -isometries and some products and tensor products of  $n$ -quasi- $(m, C)$ -isometries are again  $n$ -quasi- $(m, C)$ -isometries. It has also been proved that the sum of an  $n$ -quasi- $(m, C)$ -isometry and a commuting nilpotent operator of degree  $p$  is a  $(n + p)$ -quasi- $(m + 2p - 2)$ -isometry.

### 2. Main results

We begin by the following theorem, which is a structure theorem for  $n$ -quasi- $(m, C)$ -isometric operators.

In [24], the authors studied the matrix representation of  $n$ -quasi- $(m, C)$ -isometric operator with respect to the direct sum of  $\overline{\mathcal{R}(T^n)}$  and its orthogonal complement. In the following we give an equivalent condition for  $T$  to be  $n$ -quasi- $(m, C)$ -isometric operator.

**THEOREM 2.1.** *Let  $C = C_1 \oplus C_2$  be a conjugation on  $\mathcal{H}$  where  $C_1$  and  $C_2$  are conjugation on  $\overline{\mathcal{R}(T^n)}$  and  $\mathcal{N}(T^{*n})$ , respectively. Assume that  $\mathcal{R}(T^n)$  is not dense, then the following statements are equivalent:*

- (1)  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator.
- (2)  $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$  on  $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ , where  $T_1$  is an  $(m, C_1)$ -isometric operator on  $\overline{\mathcal{R}(T^n)}$ ,  $T_3^n = 0$ , and  $\sigma(T) = \sigma(T_1) \cup \{0\}$  where  $\sigma(T)$  is the spectrum of  $T$ .

*Proof.* (1)  $\Rightarrow$  (2). Consider the matrix representation of  $T$  with respect to the decomposition  $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ :

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n}).$$

Let  $P$  be the projection of  $\mathcal{H}$  onto  $\overline{\mathcal{R}(T^n)}$ . Since  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator, it follows that

$$P \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} C T^{m-k} C \right) P = 0.$$

This means that

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T_1^{*m-k} C_1 T_1^{m-k} C_1 = 0.$$

Hence  $T_1$  is an  $(m, C_1)$ -isometric operator on  $\overline{\mathcal{R}(T^n)}$ .

Let  $x = x_1 \oplus x_2 \in \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n}) = \mathcal{H}$ . If  $x \in \mathcal{N}(T^{*n})$ , then

$$\begin{aligned} \langle T_3^n x_2, x_2 \rangle &= \langle T^n(I-P)x, (I-P)x \rangle \\ &= \langle (I-P)x, T^{*n}(I-P)x \rangle = 0. \end{aligned}$$

Hence  $T_3^n = 0$ . From [18, Corollary 7], it follows that  $\sigma(T) \cup \mathcal{S} = \sigma(T_1) \cup \sigma(T_3)$ , where  $\mathcal{S}$  is the union of certain of the holes in  $\sigma(T)$  which is a subset of  $\sigma(T_1) \cap \sigma(T_3)$ . Further  $\sigma(T_3) = \{0\}$  and  $\sigma(T_1) \cap \sigma(T_3)$  has no interior points. So we have  $\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}$ , by [18, Corollary 8].

(2)  $\Rightarrow$  (1) Suppose that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$$

where  $\overline{\mathcal{R}(T^n)}$  is the closure of  $\mathcal{R}(T^n)$ ,  $T_1$  is an  $(m, C_1)$ -isometry and  $T_3^n = 0$ . Since

$$T^n = \begin{pmatrix} T_1^n & \sum_{0 \leq j \leq n-1} T_1^j T_2 T_3^{n-1-j} \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} & T^{*n} \left( \sum_{0 \leq l \leq m} (-1)^k \binom{m}{k} T^{*m-k} C T^{m-k} C \right) T^n \\ &= \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*n} \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*m-k} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{m-k} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \right) \\ &\quad \times \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^n \\ &= \left( \begin{array}{cc} T_1^{*n} D T_1^n & T_1^{*n} D \sum_{0 \leq j \leq n-1} T_1^j T_2 T_3^{n-1-j} \\ \left( \sum_{0 \leq j \leq n-1} T_1^j T_2 T_3^{n-1-j} \right)^* D T_1^n & \left( \sum_{0 \leq j \leq n-1} T_1^j T_2 T_3^{n-1-j} \right)^* D \sum_{0 \leq j \leq n-1} T_1^j T_2 T_3^{n-1-j} \end{array} \right), \end{aligned}$$

where

$$D = \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T_1^{*m-k} C_1 T^{m-k} C_1.$$

Hence

$$T^{*n} \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} C T^{m-k} C \right) T^n = 0 \quad \text{on} \quad \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n}).$$

Thus  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator.

**COROLLARY 2.1.** *If  $T$  is a  $n$ -quasi- $(m, C)$ -isometric operator and  $\mathcal{R}(T^n)$  is dense, then  $T$  is an  $(m, C)$ -isometric operator.*

In [11], the authors showed that a power of an  $(m, C)$ -isometric operator is again a  $(m, C)$ -isometric operator. In the following theorem we show that this remains true for  $n$ -quasi- $(m, C)$ -isometric operators.

**THEOREM 2.2.** *Let  $C = C_1 \oplus C_2$  be a conjugation on  $\mathcal{H}$  where  $C_1$  and  $C_2$  are conjugation on  $\overline{\mathcal{R}(T^n)}$  and  $\mathcal{N}(T^{*n})$ , respectively. If  $T$  is a  $n$ -quasi- $(m, C)$ -isometric operator, then so is  $T^k$  for every natural number  $k$ .*

*Proof.* If  $\mathcal{R}(T^n)$  is dense, then  $T$  is an  $(m, C)$ -isometric operator and so is  $T^k$  for every positive integer  $k$ .

If  $\mathcal{R}(T^n)$  is not dense. By Theorem 2.1 we write the matrix representation of  $T$  on  $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$  as follows

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n}),$$

where  $T_1$  is an  $(m, C_1)$ -isometric operator. By [11, Theorem 2.1],  $T_1^k$  is an  $(m, C_1)$ -isometric operator. Since

$$T^k = \begin{pmatrix} T_1^k & \sum_{0 \leq j \leq k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n}).$$

Thus  $T^k$  for every natural number  $k$  is a  $n$ -quasi- $(m, C)$ -isometric operator by Theorem 2.1.

**REMARK 2.1.** The converse of Theorem 2.2 is not true in general as shown in the following example.

**EXAMPLE 2.1.** Let  $C$  be a conjugation on  $\mathbb{C}^2$  defined by  $C(x_1, x_2) = (\overline{x_2}, \overline{x_1})$  and consider the operator matrix  $T = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}$  on  $\mathbb{C}^2$ . A simple calculation shows that  $T^{*3} \left( T^{*3} C T^3 C - I \right) T^3 = 0$  and  $T^* \left( T^* C T C - I \right) T \neq 0$ . So, we obtain that  $T^3$  is a quasi- $(1, C)$ -isometric operator, but  $T$  it is not a quasi- $(1, C)$ -isometric operator.

It was observed that every  $(m, C)$ -isometric operator is an  $(k, C)$ -isometric operator for every integer  $k \geq m$ . In the following proposition we show that this remains true for  $n$ -quasi- $(m, C)$ -isometric operator.

**PROPOSITION 2.1.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and let  $C = C_1 \oplus C_2$  be a conjugation on  $\mathcal{H}$  where  $C_1$  and  $C_2$  are conjugation on  $\overline{\mathcal{R}(T^n)}$  and  $\mathcal{N}(T^{*n})$ , respectively. If  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator, then  $T$  is an  $l$ -quasi- $(k, C)$ -isometric operator for every positive integers  $k \geq m$  and  $l \geq n$ .*

*Proof.* If  $\mathcal{R}(T^n)$  is dense, then  $T$  is an  $(m, C)$ -isometric operator and hence  $T$  is an  $(k, C)$ -isometric operator for every positive integer  $k \geq m$ .

If  $\mathcal{R}(T^n)$  is not dense, by Theorem 2.1 we write the matrix representation of  $T$  on

$\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$  as follows  $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$  where  $T_1 = T_{|\overline{\mathcal{R}(T^n)}}$  is an  $(m, C_1)$ -isometric operator and  $T_3^n = 0$ . Obviously that  $T_1$  is an  $(k, C_1)$ -isometric operator for every integer  $k \geq m$ . The conclusion follows from the statement (2) of Theorem 2.1.

For an operator  $T \in \mathcal{B}(\mathcal{H})$  and a conjugation  $C$ , the operator  $\Lambda_m(T)$  is defined by

$$\Lambda_m(T) := \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} C T^{m-k} C.$$

Then  $T$  is an  $(m, C)$ -isometric operator if and only if  $\Lambda_m(T) = 0$ .

The following lemma gives another condition for which an  $n$ -quasi- $(m, C)$ -isometric operator became an  $n$ -quasi- $(k, C)$ -isometric operator for  $k \geq m$ .

**LEMMA 2.1.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be an  $n$ -quasi- $(m, C)$ -isometric operator where  $C$  is a conjugation on  $\mathcal{H}$ . If  $T(CTC) = (CTC)T$ , then  $T$  is an  $n$ -quasi- $(k, C)$ -isometric operator for every positive integer  $k \geq m$ .*

*Proof.* It is well known that  $\Lambda_{m+1}(T) = T^* \Lambda_m(T)(CTC) - \Lambda_m(T)$  ([11]). Under the assumptions that  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator and satisfies  $T(CTC) = (CTC)T$ , it follows

$$T^{*n} \Lambda_{m+1}(T) T^n = T^{*n+1} \Lambda_m(T) T^n (CTC) - T^{*n} \Lambda_m(T) T^n = 0.$$

Therefore  $T$  is an  $n$ -quasi- $(m + 1, C)$ -isometric operator.

Let  $T \in \mathcal{B}(\mathcal{H})$ . Denote by  $r(T)$  the spectral radius of  $T$ , that is,  $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ . We say that  $T$  is normaloid if  $r(T) = \|T\|$ .

**THEOREM 2.3.** *Let  $C = C_1 \oplus C_2$  be a conjugation on  $\mathcal{H}$  where  $C_1$  and  $C_2$  are conjugation on  $\overline{\mathcal{R}(T^n)}$  and  $\mathcal{N}(T^{*n})$  respectively. Let  $T \in \mathcal{B}(\mathcal{H})$  be an  $n$ -quasi- $(m, C)$ -isometric operator. Assume that  $T$  is power bounded and  $T_1 = T_{|\overline{\mathcal{R}(T^n)}}$  satisfies  $T_1 C_1 T_1 C_1 - I$  is normaloid, then  $T$  is an  $n$ -quasi- $(1, C)$ -isometric operator.*

*Proof.* We know that  $T$  admits the following matrix representation  $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$  on  $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ . Since  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator, it follows in view of Theorem 2.1 that  $T_1$  is an  $(m, C_1)$ -isometric operator and  $T_3^n = 0$ . Furthermore  $T$  is power bounded then it is easy that  $T_1$  is power bounded and satisfies  $T_1 C_1 T_1 C_1 - I$  is normaloid. By applying [11, Theorem 3.1] we obtain that  $T_1$  is an  $(1, C_1)$ -isometric operator. According to Theorem 2.1 we can deduce that  $T$  is an  $n$ -quasi- $(1, C)$ -isometric operator. Thus we complete the proof.

LEMMA 2.2. ([17, Lemma 3.15]) *If  $(a_j)_j$  is a sequence of complex numbers and  $r, s, m, l$  are positive integers satisfying*

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} a_{rk+j} = 0 \tag{2.1}$$

and

$$\sum_{0 \leq k \leq l} (-1)^k \binom{l}{k} a_{sk+j} = 0 \tag{2.2}$$

for all  $j \geq 0$ , then

$$\sum_{0 \leq k \leq q} (-1)^k \binom{q}{k} a_{pk} = 0, \tag{2.3}$$

where  $q$  is the greatest common divisor of  $r$  and  $s$ , and  $p$  is the minimum of  $m$  and  $l$ .

In [7] it was proved that if  $T^r$  is an  $m$ -isometry and  $T^s$  is an  $l$ -isometry, then  $T^q$  is a  $p$ -isometry, where  $q$  is the greatest common divisor of  $r$  and  $s$ , and  $p$  is the minimum of  $m$  and  $l$ . In the following theorem we extend this result as follows

THEOREM 2.4. *Let  $T \in \mathcal{B}(\mathcal{H})$  such that  $T^r$  is an  $(m, C)$ -isometry and  $T^s$  is an  $(l, C)$ -isometry, then  $T^q$  is a  $(p, C)$ -isometry, where  $q$  is the greatest common divisor of  $r$  and  $s$ , and  $p$  is the minimum of  $m$  and  $l$ .*

*Proof.*

$$\begin{aligned} T \text{ is an } (m, C) \text{ - isometry} &\Leftrightarrow \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^{*k} C T^k C = 0 \\ &\Leftrightarrow \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^{*k} C T^k = 0 \\ &\Leftrightarrow \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \langle C T^k x | T^k x \rangle = 0 \quad \forall x \in \mathcal{H}. \end{aligned}$$

Fix  $x \in \mathcal{H}$  and denote  $a_j = \langle C T^j x | T^j x \rangle$  for  $j = 1, 2, \dots$ . As  $T^r$  is an  $(m, C)$ -isometric operator the sequence  $(a_j)_{j \geq 0}$  verifies the recursive equation

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} a_{kr+j} = 0, \text{ for all } j \geq 0.$$

Analogously, as  $T^s$  is an  $(l, C)$ -isometric operator the sequence  $(a_j)_{j \geq 0}$  verifies the recursive equation

$$\sum_{0 \leq k \leq l} (-1)^{l-k} \binom{l}{k} a_{ks+j} = 0, \text{ for all } j \geq 0.$$

Applying Lemma 2.2 we obtain that

$$\sum_{0 \leq k \leq p} (-1)^{p-k} \binom{p}{k} a_{kq} = 0,$$

where  $q$  is the greatest common divisor of  $r$  and  $s$ , and  $p$  is the minimum of  $m$  and  $l$ . Finally  $T^q$  is an  $(p, C)$ -isometric operator.

The following corollary is direct consequence of preceding theorem.

**COROLLARY 2.2.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and let  $r, s, m, l$  be positive integers. The following properties hold:*

- (1) *If  $T$  is an  $(m, C)$ -isometric operator such that  $T^s$  is an  $(1, C)$ -isometric operator, then  $T$  is an  $(1, C)$ -isometric operator.*
- (2) *If  $T^r$  and  $T^{r+1}$  are  $(m, C)$ -isometries, then so is  $T$ .*
- (3) *If  $T^r$  is an  $(m, C)$ -isometric operator and  $T^{r+1}$  is an  $(l, C)$ -isometric operator with  $m < l$ , then  $T$  is an  $(m, C)$ -isometric operator.*

**THEOREM 2.5.** *Let  $S$  and  $T$  be in  $\mathcal{B}(\mathcal{H})$  and let  $C = C_1 \oplus C_2$  be a conjugation on  $\mathcal{H}$  where  $C_1$  and  $C_2$  are conjugation on  $\mathcal{B}(S^n)$  and  $\mathcal{N}(S^{*n})$ , respectively. Assume that  $T$  and  $S$  are doubly commuting and  $T(CTC) = (CTC)T$ ,  $T(CSC) = S(CTC)$  and  $S^*CTC = CTCS^*$ . If  $T$  is an  $n_1$ -quasi- $(k, C)$ -isometric operator and  $S$  is an  $n_2$ -quasi- $(m, C)$ -isometric operator, then  $TS$  is a  $n' = \max\{n_1, n_2\}$ -quasi- $(k+m-1, C)$ -isometric operator.-*

*Proof.* Since  $TS = ST$ ,  $T(CSC) = S(CTC)$  and  $S^*CTC = CTCS^*$ , it follows that

$$[T^*, S^*] = [CTC, CSC] = [CTC, S^*] = 0.$$

By taking into account [16, Lemma 12] we obtain that

$$\Lambda_{k+m-1}(TS) = \sum_{0 \leq j \leq k+m-1} \binom{k+m-1}{j} T^{*j} T^{*n'} \Lambda_{k+m-1-j}(T) T^{n'} CT^j CS^{*n'} \Lambda_j(S) S^{n'}.$$

Furthermore as  $[T, S^*] = [T, CSC] = [CTC, S^*] = 0$  we get

$$\begin{aligned} & (TS)^{*n'} \Lambda_{k+m-1}(TS) (TS)^{n'} \\ &= \sum_{0 \leq j \leq k+m-1} \binom{k+m-1}{j} T^{*j} T^{*n'} \Lambda_{k+m-1-j}(T) T^{n'} CT^j CS^{*n'} \Lambda_j(S) S^{n'}. \end{aligned}$$

Under the assumption that  $S$  is an  $n_2$ -quasi- $(k, C)$ -isometric operator, we get in view of Proposition 2.1  $S^{*n'} \Lambda_j(S) S^{n'} = 0$  for  $j \geq m$  and  $n' \geq n_2$ . On the other hand, if



$j \leq m - 1$ , then  $k + m - 1 - j \geq k + m - 1 - m + 1 = k$  and so  $T^{*n'} \Lambda_{k+m-1-j}(T) T^{n'} = 0$  by Lemma 2.1. Hence,  $TS$  is a  $n'$ -quasi- $(k + m - 1, C)$ -isometric operator.

**COROLLARY 2.3.** *Let  $S$  and  $T$  be in  $\mathcal{B}(\mathcal{H})$  are doubly commuting. Let  $C = C_1 \oplus C_2$  be a conjugation on  $\mathcal{H}$  where  $C_1$  and  $C_2$  are conjugation on  $\mathcal{R}(S^n)$  and  $\mathcal{N}(S^{*n})$ , respectively. Assume that  $T(CSC) = (CSC)T$ ,  $T(CTC) = (CTC)T$  and  $S^*CTC = CTCS^*$ . If  $T$  is an  $n_1$ -quasi- $(k, C)$ -isometric operator and  $S$  is an  $n_2$ -quasi- $(m, C)$ -isometric operator, then  $TS^q$  is a  $n' = \max\{n_1, n_2\}$ -quasi- $(k + m - 1, C)$ -isometric operator for some positive integer  $q$ .*

*Proof.* In view of Theorem 2.2 we have that  $S^q$  is an  $n_2$ -quasi- $(m, C)$ -isometric operator. Moreover  $T$  and  $S^q$  satisfy the conditions of Theorem 2.4. Hence  $TS^q$  is a  $n'$ -quasi- $(k + m - 1, C)$ -isometric operator.

**PROPOSITION 2.2.** *Let  $S$  and  $T$  be in  $\mathcal{B}(\mathcal{H})$  are doubly commuting. Assume that  $T(CSC) = (CSC)T$ ,  $T(CTC) = (CTC)T$ ,  $S^*CTC = CTCS^*$  and  $S(CSC) = (CSC)S$ . If  $T$  is an  $n_1$ -quasi- $(k, C)$ -isometric operator and  $S$  is an  $n_2$ -quasi- $(m, C)$ -isometric operator, then  $TS$  is a  $n' = \max\{n_1, n_2\}$ -quasi- $(k + m - 1, C)$ -isometric operator.*

*Proof.* Under the assumptions that  $T(CTC) = (CTC)T$  and  $S(CSC) = (CSC)S$ , it follows from Lemma 2.1 that  $T$  is an  $n_1$ -quasi- $(k + 1, C)$ -isometric operator and  $S$  is an  $n_2$ -quasi- $(m + 1, C)$ -isometric operator. By repeating the reasoning given in the proof of Theorem 2.5 we check that

$$(TS)^{*n'} \Lambda_{m+k-1}(TS) (TS)^{n'} = 0.$$

Therefore  $TS$  is a  $n'$ -quasi- $(m + k - 1)$ -isometric operator.

Let  $\mathcal{H} \overline{\otimes} \mathcal{H}$  denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product  $\mathcal{H} \otimes \mathcal{H}$  of  $\mathcal{H}$  and  $\mathcal{H}$ . It is well known that if  $x \in \mathcal{H} \overline{\otimes} \mathcal{H}$ , there exists linearly independent sets  $(u_i)_{i \in I}$  and  $(v_i)_{i \in I}$  such that  $x = \sum_{i \in I} u_i \otimes v_i$ . An inner product on  $\mathcal{H} \overline{\otimes} \mathcal{H}$  is defines as

$$\langle x \otimes y \mid u \otimes v \rangle := \langle x \mid u \rangle \langle y \mid v \rangle \text{ where } x, y, u, v \in \mathcal{H}.$$

We construct an operator  $\tilde{T}$  on the tensor product of Hilbert spaces. Let  $T$  be an operator on  $\mathcal{H}$  and  $S$  be an operator on  $\mathcal{H}$ . We define

$$\tilde{T} := T \otimes S : \mathcal{H} \overline{\otimes} \mathcal{H} \longrightarrow \mathcal{H} \overline{\otimes} \mathcal{H} \text{ by}$$

$$\tilde{T}(x) = (T \otimes S) \left( \sum_{i \in I} u_i \otimes v_i \right) = \sum_{i \in I} T(u_i) \otimes S(v_i).$$

In [12, Lemma 4.5], it was proved that if  $C$  and  $D$  be conjugations on  $\mathcal{H}$ . Then  $C \otimes D$  is a conjugation on  $\overline{\mathcal{H} \otimes \mathcal{H}}$ .

LEMMA 2.3. *If  $T \in \mathcal{B}(\mathcal{H})$  and  $C$  and  $D$  are conjugations on  $\mathcal{H}$  respectively. Then  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator if and only if the tensor product  $T \otimes I$  (resp.  $I \otimes T$ ) is an  $n$ -quasi- $(m, C \otimes D)$ -isometric operator.*

*Proof.* A straightforward computation gives

$$\begin{aligned} & (T \otimes I)^{*n} \left( \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} (T \otimes I)^{*k} (C \otimes D) (T \otimes I)^k (C \otimes D) \right) (T \otimes I)^n \\ &= T^{*n} \left( \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^{*k} C T^k C \right) \otimes I. \end{aligned}$$

From this we can get that  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator if and only if  $T \otimes I$  is an  $n$ -quasi- $(m, C \otimes D)$ -isometric operator.

As application of Lemma 2.3 and Proposition 2.2, we get the following theorem.

THEOREM 2.6. *Let  $T$  and  $S \in \mathcal{B}(\mathcal{H})$  such that  $T$  is an  $n_1$ -quasi- $(m, C)$ -isometric operator and  $S$  is an  $n_2$ - $(k, D)$ -isometric operator where  $C$  and  $D$  are conjugations on  $\mathcal{H}$ , respectively. If  $T(CTC) = (CTC)T$  and  $S(DSD) = (DSD)$ , then  $T \otimes S$  is an  $n' = \max\{n_1, n_2\}$ -quasi- $(m + k - 1, C \otimes D)$ -isometric operator.*

*Proof.* It is well known that  $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$ . In view of Lemma 2.3 we have that  $T \otimes I$  is an  $n_1$ -quasi- $(m, C \otimes D)$ -isometric operator and  $I \otimes S$  is an  $n_2$ -quasi- $(k, C \otimes D)$ -isometric operator. On the other hand, note that  $T \otimes I$  and  $I \otimes S$  satisfy all conditions in Proposition 2.2. We conclude that  $(T \otimes I)(I \otimes S)$  is a  $n'$ -quasi- $(m + k - 1, C \otimes D)$ -isometric operator.

LEMMA 2.4. *Let  $T, Q \in \mathcal{B}(\mathcal{H})$  such that  $TQ = QT$ , then for  $m \geq 2$*

$$\Lambda_m(T + Q) = \sum_{i+j+k=m} \binom{m}{i, j, k} (T + Q)^{*i} Q^{*j} \Lambda_k(T) C T^j C C Q^j C$$

where  $\binom{m}{i, j, k} = \frac{m!}{i! j! k!}$ .

*Proof.* The proof follows by similar arguments as in the proof of [25, Lemma 2].

It was proved in [8, Thoerem 3.1] that if  $T \in \mathcal{B}(\mathcal{H})$  is an  $m$ -isometry and  $Q \in \mathcal{B}(\mathcal{H})$  is an nilpotent operator of order  $p$  such that  $TQ = QT$ , then  $T + Q$ -is an  $(m + 2p - 2)$ -isometry. In the following theorem we show that this remains true for  $(m, C)$ -isometric operators.

**THEOREM 2.7.** *Let  $T, Q \in \mathcal{B}(\mathcal{H})$ . Assume  $T$  commutes with  $Q$ . If  $T$  is an  $(m, C)$ -isometric operator and  $Q$  is a nilpotent operator of order  $p$ . Then  $T + Q$  is an  $(m + 2p - 2, C)$ -isometric operator where  $C$  is a conjugation on  $\mathcal{H}$ .*

*Proof.* We need to show

$$\Lambda_{m+2p-2}(T + Q) = 0.$$

In view of Lemma 2.4 we have

$$\Lambda_{m+2p-2}(T + Q) = \sum_{i+j+k=m+2p-2} \binom{m+2p-2}{i, j, k} (T^* + Q^*)^i Q^{*j} \Lambda_k(T) C T^j C C Q^i C.$$

(i) If  $\max\{i, j\} \geq p$ , then  $CQ^iC = 0$  or  $Q^{*j} = 0$ .

(ii) If  $\max\{i, j\} \leq p - 1$ , then  $k \geq m$  and hence  $\Lambda_k(T) = 0$ .

From (i) and (ii) we get  $\Lambda_{m+2p-2}(T + Q) = 0$ .

In the following theorem we investigate the nilpotent perturbations of an  $n$ -quasi- $(m, C)$ -isometric operator.

**THEOREM 2.8.** *Let  $T$  and  $Q \in \mathcal{B}(\mathcal{H})$ . Assume that  $TQ = QT$  commutes,  $TCQC = CQCT$  and  $TCTC = CTCT$  where  $C$  is a conjugation on  $\mathcal{H}$ . If  $T$  is an  $n$ -quasi- $(m, C)$ -isometric operator and  $Q$  is a nilpotent operator of order  $p$ . Then  $T + Q$  is a  $(n + p)$ -quasi- $(m + 2p - 2, C)$ -isometric operator.*

*Proof.* We need to show

$$(T + Q)^\gamma \Lambda_{m+2p-2}(T + Q) (T + Q)^\gamma = 0 \text{ where } \gamma = n + p.$$

In view of Lemma 2.4 we have

$$\Lambda_{m+2p-2}(T + Q) = \sum_{i+j+k=m+2p-2} \binom{m+2p-2}{i, j, k} (T^* + Q^*)^i Q^{*j} \Lambda_k(T) C T^j C C Q^i C$$

and

$$\begin{aligned} & (T + Q)^{* \gamma} \Lambda_{m+2p-2}(T + Q) (T + Q)^\gamma \\ &= \left( \sum_{0 \leq r \leq 2\gamma} \binom{\gamma}{r} T^{*(\gamma-r)} Q^{*r} \right) \\ & \times \left( \sum_{i+j+k=m+2p-2} \binom{m+2p-2}{i, j, k} (T^* + Q^*)^i Q^{*j} \Lambda_k(T) C T^j C C Q^i C \right) \\ & \times \left( \sum_{0 \leq s \leq 2\gamma} \binom{\gamma}{s} T^{\gamma-s} Q^s \right). \end{aligned}$$

Now observe that if  $\max\{i, j\} \geq p$ , then  $CQ^jC = 0$  or  $Q^i = 0$ . and hence

$$(T^* + Q^*)^i Q^{*j} \Lambda_k(T) CT^j CCQ^i C = 0.$$

However, if  $\max\{i, j\} \leq p - 1$ , then  $k \geq m$ . Using the fact that  $T$  is an  $n$ -quasi- $m$ -isometry and  $TCTC = CTCT$ , we get

$$T^{*(n+p-r)} \Lambda_k(T) T^{n+p-s} = 0 \text{ for } r \in \{0, \dots, p\} \text{ and } s \in \{0, \dots, p\}$$

and

$$T^{*(n+p-r)} Q^{*r} \Lambda_k(T) T^{n+p-s} Q^s = 0 \text{ for } r \in \{p+1, \dots, n+p\} \text{ and } s \in \{p+1, \dots, n+p\}.$$

Combining the above arguments we deduce that

$$(T + Q)^{n+p} \Lambda_{m+2p-2}(T + Q) (T + Q)^{n+p} = 0.$$

Thus  $T + Q$  is a  $(n + p)$ -quasi- $(m + 2p - 2)$ -isometric operator. Therefore the theorem is proved.

EXAMPLE 2.2. Let  $C$  be a conjugation on  $\mathbb{C}^3$  defined by  $C(x_1, x_2, x_3) = (\overline{x_3}, \overline{x_2}, \overline{x_1})$  and consider the operator matrix  $T = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  on  $\mathbb{C}^3$ . Then  $T = I + Q$ . Since  $Q^2 = 0$ , it follows from Theorem 2.8 that  $T$  is a  $(n + 2)$ -quasi- $(m + 2, C)$ -isometric operator.

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REFERENCES

- [1] B. ABDULLAH T. LE, *The structure of m-isometric weighted shift operator*, Operators and Matrices, vol 10, Number 2 (2016), 319–334.
- [2] J. AGLER AND M. STANKUS, *m-Isometric transformations of Hilbert space I*, Integral Equations and Operator Theory, **21**(1995), 383–429.
- [3] J. AGLER AND M. STANKUS, *m-Isometric transformations of Hilbert space. II*, Integral Equ. Oper. Theory **23**(1), 1–48 (1995).
- [4] J. AGLER AND M. STANKUS, *m-Isometric transformations of Hilbert space. III*, Integral Equ. Oper. Theory **24**(4), 379–421 (1996).
- [5] M. F. AHMADI, S. YARMAHMOODI AND K. HEDAYATIAN, *Perturbation of (m, p)-isometries by nilpotent operators and their supercyclicity*, Oper. Matrices **11**(2017), 381–387.
- [6] F. BAYART, *m-Isometries on Banach spaces*, Math. Nachr. **284**(2011), 2141–2147.
- [7] T. BERMÚDEZ, C. D. MENDOZA AND A. MARTINÓN, *Powers of m-isometries*, Studia Mathematica **208** (3) (2012).
- [8] T. BERMÚDEZ, A. MARTINÓN, V.MVLLER, J.A. NODA, *Perturbation of m-Isometries by Nilpotent Operators*, Abstract and Applied Analysis, Volume 2014, Article ID 745479, 6 pages.
- [9] T. BERMÚDEZ, A. MARTINÓN, J.A. NODA, *An isometry plus a nilpotent operator is an m-isometry. Applications*, J. Math. Anal. Appl. **407**(2),(2013). 505–512.

- [10] T. BERMÚDEZ, A. MARTINÓN, J.A. NODA, *Products of  $m$ -isometries*, Linear Algebra and its Applications 438 (2013) 80–86.
- [11] M. CHO, E. KO AND J. E. LEE, *On  $(m, C)$ -Isometric Operators*, Complex Anal. Oper. Theory **10** (2016), 1679–1694.
- [12] M. CHO, J.E. LEE AND H. MOTOYOSHIA, *On  $[m, C]$ -Isometric Operators*, Filomat 31:7 (2017), 2073–2080.
- [13] B. P. DUGGAL, *Tensor product of  $n$ -isometries*, Linear Algebra and its Applications 437 (2012), 307–318.
- [14] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358**(2006), 1285–1315.
- [15] S. R. GARCIA, E. PRODAN AND M. PUTINAR, *Mathematical and physical aspects of complex symmetric operators*, J. Phys. A: Math. Theory **47**(2014) 353001 (54pp).
- [16] C. GU, *Structures of left  $n$ -invertible operators and their applications*, Studia Mathematica 226 (3) (2015).
- [17] K. HEDAYATIAN AND A. M. MOGHADDAM, *Some properties of the spherical  $m$ -isometries*, J. Operator 79:1(2018), 55–77.
- [18] J. KYU HAN, H. YOUL LEE AND W. YOUNG LEE, *Invertible completions of  $2 \times 2$  upper triangular operator matrices*, Proc. Amer. Math. Soc. 128 (1999), 119–123.
- [19] S. MECHERI AND T. PRASAD, *On  $n$ -quasi- $m$ -isometric operators*, Asian-European Journal of Mathematics **9** (2016), (8 pages).
- [20] S. MECHERI AND S. M. PATEL, *On quasi-2-isometric operators*, Linear and Multilinear Algebra (2017), <https://doi.org/10.1080/03081087.2017.1335283>.
- [21] S. M. PATEL, *2-isometric operators*, Glas. Mat. 37 (57) (2002), 141–145.
- [22] S. M. PATEL, *A note on quasi-isometries*, Glas. Mat. 35 (55) (2000) 307–312.
- [23] S. M. PATEL, *A note on quasi-isometries II*, Glas. Mat. 38 (58) (2003) 111–120.
- [24] O. A. M. SID AHMED, M. CHŌ AND J. E. LEE, *On  $n$ -quasi- $(m, C)$ -isometric operators*, Linear and Multilinear Algebra, <https://doi.org/10.1080/03081087.2018.1524437>.
- [25] O. A. M. SID AHMED, M. CHŌ AND J. E. LEE, *On  $(m, C)$ -Isometric Commuting Tuples of Operators on a Hilbert Space*, Results Math (2018) 73:51.

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