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ON THE CLASS OF n–POWER D–m–QUASI–NORMAL OPERATORS ON HILBERT SPACES

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Abstract. As a continuation of our previous work [22], this paper is devoted to the study for further properties of the class of (n,m)-power D-normal operators($[(n,m)D\mathbf{N}]$) and introduce some classes of operators on Hilbert space called D-m-quasi-normal operators and it is denoted by $(\big([D(\mathbf{QN})^m]\big)$, n-power D-m-quasi-normal operators and it is denoted by $\big([nD(\mathbf{QN})^m]\big)$, associated with a Drazin invertible operator using its Drazin inverse. Some characterizations of D-m-quasi-normal and n-power D-m-quasi-normal operators are discussed. Inclusion relations among the various classes of normal operators are characterized.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators defined in $\mathscr{H}$. For every $T \in \mathscr{B}(\mathscr{H})$, denote by $\mathscr{R}(T)$, $\mathscr{N}(T)$ and T^* the range, the null space and the adjoint of T, respectively. If $\mathscr{M} \subset \mathscr{H}$ is a closed subspace of $\mathscr{H}$ satisfying $T\mathscr{M} \subset \mathscr{M}$, then $\mathscr{M}$ is called an invariant subspace of T. In addition, if $\mathscr{M}$ also is invariant subspace of T^*, then $\mathscr{M}$ is called a reducing subspace of T.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be

(1) normal if $T^*T = TT^*$ ([9, 20]),

(2) quasi-normal if $T\big(T^*T\big) = \big(T^*T\big)T$ ([3]),

(3) n-power normal if $T^nT^* = T^*T^n$ ([18]),

(4) n-power quasi-normal if $T^n\big(T^*T\big) = \big(T^*T\big)T^n$ ([23, 24]),

(5) (n,m)-power normal if $T^nT^{*m} = T^{*m}T^n$ ([1]),

(6) m-quasi-normal if $T\big(T^*T\big)^m = \big(T^*T\big)^mT$ ([21]),

(7) n-power m-quasi-normal if $T^n\big(T^*T\big)^m = \big(T^*T\big)^mT^n$ ([25]),

(8) (n,m)-power quasi-normal if $T^n\big(T^{*m}T\big) = \big(T^{*m}T\big)T^n$ ([2]).

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For any arbitrary operator $T \in \mathscr{B}(\mathscr{H})$ may always be expressed as $T = S + iR$ with $S, R \in \mathscr{B}(\mathscr{H})$ being both self-adjoint. Necessarily, $S = \frac{1}{2}\left(T + T^*\right)$ which will be denoted by ReT and it is called the real part of T. Also, $R = \frac{1}{2i}\left(T - T^*\right)$ is the imaginary part of T, written ImT. We shall write $C^2 = T^*T$ and $B = TT^*$, where B and C are non-negative definite. For any operator $T \in \mathscr{B}(\mathscr{H})$, $|T| = (T^*T)^{\frac{1}{2}}$ and

$$[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2.$$

An element $T \in \mathscr{B}(\mathscr{H})$ will be called Drazin invertible, if there exists a necessarily unique $S \in \mathscr{B}(\mathscr{H})$ and some $k \in \mathbb{N}$ such that

$$T^kST = T^k, \;\; STS = S, \;\; TS = ST.$$

If the Drazin inverse of T exists, then it will be denoted by T^D . In addition, the index of T, which will be denoted by $\mathrm{ind}(T)$, is the least non-negative integer k for which the above equations hold. For more details see [4, 5, 6, 8, 6].

Next consider the set

$$D\mathscr{R}(\mathscr{H}) = \{T \in \mathscr{B}(\mathscr{H}) : T \text{ is Drazin invertible}\}.$$

LEMMA 1.1. *([5], [27]) Let $T, S \in D\mathscr{R}(\mathscr{H})^D$. Then the following properties hold.*

(1) *TS is Drazin invertible if and only if ST is Drazin invertible. Moreover*

$$(TS)^D = T[(ST)^D]^2S \;\; and \;\; ind(TS) \leqslant ind(ST) + 1.$$

(2) *If $TS = ST$, then $(TS)^D = S^DT^D = T^DS^D, T^DS = ST^D$ and $TS^D = S^DT$.*

(3) *If $TS = ST = 0$, then $\left(T + S\right)^D = T^D + S^D$.*

The authors M. Dana and R. Yousfi [10, 11, 12] has introduced and studied the following classes of operators. Let $T \in D\mathscr{B}(\mathscr{H})$. Then T is said to be

(i) D-normal if $T^DT^* = T^*T^D$.

(ii) D-quasi-normal if $T^D(T^*T) = (T^*T)T^D$.

(iii) n-power D-normal if $\left(T^D\right)^nT^* = T^*\left(T^D\right)^n$.

(iv) n-power D-quasi-normal if $\left(T^D\right)^n(T^*T) = (T^*T)\left(T^D\right)^n$.

(v) Skew D-quasi-normal if $T^*TT^D = TT^DT^*$.

Let $[D\mathbf{N}],[nD\mathbf{N}],[D\mathbf{QN}]$, $[nD\mathbf{QN}]$ and $[\mathfrak{S}\mathfrak{D}]$ denote the classes constituting of D-normal, n-power D-normal, D-quasi-normal,n-power D-quasi-normal operators and skew D-normal operators, respectively. The following inclusions hold:

(i) $[D\mathbf{N}] \subset [D\mathbf{QN}] \subset [nD\mathbf{QN}]$.

(ii) $[D\mathbf{N}] \subset [nD\mathbf{N}] \subset [nD\mathbf{QN}]$.

(iii) $[D\mathbf{N}] \subset [nD\mathbf{N}] \subset [\mathfrak{S}\mathfrak{D}]$.

For more details on these classes, we refer the interested reader to [10, 11, 12].

Recently the present authors [22] has introduced a new classes of operators called (n,m)-power D-normal and (n,m)-power D-quasi normal as follows.

Let $T \in D\mathscr{R}(\mathscr{H})$. We said that

(1) T is (n,m)-power D-normal if

$$\left(T^D\right)^n T^{*m} = T^{*m}\left(T^D\right)^n, \tag{1.1}$$

for some positive integers n and m. This class of operators will denoted by $[(n,m)D\mathbf{N}]$.

(2) T is (n,m)-power D-quasi-normal if

$$\left(T^D\right)^n\left(T^{*m}T\right) = \left(T^{*m}T\right)\left(T^D\right)^n, \tag{1.2}$$

for some positive integers n and m. This class of operators will denoted by $[(n,m)D\mathbf{QN}]$.

From the above definitions, we get that the class of n-power D-normal (n-power D-quasi-normal) operators form a subclass of the class of (n,m)-power D-normal ((n,m)-power D-quasi-normal) operators for all positive integers m and n.

Many results about the classes of (n,m)-power D-normal and (n,m)-power D-quasi-normal operators have been found in [22].

2. Some results on the class $[(n,m)D\mathbf{N}]$

In this section we give some results about (n,m)-power D-normal operators where the authors in [22] also have depicted some properties of such operators.

In [12, Proposition 2.8] it has proved that $[nD\mathbf{N}] \subset [\mathfrak{S}\mathfrak{D}]$. In the following proposition, we extend this inclusion to the class $[(n,m)D\mathbf{N}]$.

PROPOSITION 2.1. *Let $T \in D\mathscr{R}(\mathscr{H})$ be a (n,m)-power D-normal operator for some positive integers n and m. Then T is skew D-quasi normal. i.e.; $[(n,m)D\mathbf{N}] \subset [\mathfrak{S}\mathfrak{D}]$.*

Proof. Since T is (n,m)-power D-normal operator, it follows that T^{nm} is D-normal and hence T is nm-power D-normal or equivalently $T \in [nmD\mathbf{N}]$. So, T is skew-D-quasi normal by the statement (2) of Proposition 2.8 in [12].

COROLLARY 2.1. *Let $T \in D\mathscr{R}(\mathscr{H})$ be a (n,m)-power D-normal operator for some positive integers n and m. Then the following statements hold.*

(1) $\mathscr{N}(T^k) \subset \mathscr{N}(T^{*D})$, *for every* $k \in \mathbb{N}$.

(2) $\mathscr{N}(T^D) \subset \mathscr{N}(T^{*D})$.

Proof. The statement (1) follows form Proposition 2.1 and [12, Lemma 2.29]. However the statement (2) follows form Proposition 2.1 and [12, Corollary 2.30].

PROPOSITION 2.2. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T^n is normal operator, then T is of class $[(n,m)D\mathbf{N}]$.*

Proof. Indeed, since T^n is normal and $T^mT^n = T^nT^m$, it follows from Fuglede theorem([15]) that $T^{*m}T^n = T^nT^{*m}$. Taking in consideration Lemma 1.1 we get

$$T^{*m}\left(T^D\right)^n = \left(T^D\right)^n T^{*m}.$$

Hence T is (n,m)-power D-normal.

THEOREM 2.1. *If $T,S \in D\mathscr{R}(\mathscr{H})$ are doubly commuting (n,m)-power D-normal operators, then TS is (k,j)-power D-normal for every $j \in \mathbb{N}$, where k is the least common multiple of n and m.*

Proof. Since T and S are doubly commuting (n,m)-power D-normal operators, it follows that TS is (n,m)-power D-normal operator (by [22, Theorem 2.2]. Since k is the least common multiple of n and m, by [22, Theroem 2.1], $(TS)^k$ is D-normal. On the other hand, we have $(TS)^k$ commutes with $(TS)^j$ for every $j \in \mathbf{N}$. By Fuglede's theorem, it holds $(TS)^{*j}\left((TS)^D\right)^k = \left((TS)^D\right)^k(TS)^{*j}$. Hence TS is (k,j)-power D-normal for every $j \in \mathbb{N}$.

Consider two normal operators T and S on a Hilbert space. It is known that, in general, TS is not normal. Kaplansky showed that it may be possible that TS is normal while ST is not. Indeed, he showed that if T and TS are normal, then ST is normal if and only if S commutes with TT^*. The study of operators satisfying Kaplansky theorem is of significant interest and is currently being done by a number of mathematicians around the world.

In the following two theorems we give sufficient conditions on two some operators defined on a Hilbert space, which make their product are (n,m)-power D-normal and (n,m)-power D-quasi-normal.

THEOREM 2.2. *Let $T,S \in D\mathscr{B}(\mathscr{H})$ such that T is normal and TS is (n,m)-power D-normal. Then*

$$TT^*S = STT^* \Longrightarrow ST \text{ is } (n,m)\text{-power } D\text{-normal.}$$

Proof. Under the assumption that T is normal it is well known that $T = UP = PU$ where U is unitary and P is positive operator.

$$TT^*S = STT^* \Longrightarrow UPU^*PS = SUPP^*U^* \Longrightarrow PS = SP.$$

On the other hand

$$U^*TSU = U^*UPSU = ST.$$

Form which it follows that ST is unitary equivalent to a (n,m)-power D-normal. In view of [22, Proposition 2.1] we obtain that ST is (n,m)-power D-normal as required.

REMARK 2.1. The reverse implication does not hold in the previous result as shown in the following example.

EXAMPLE 2.1. Consider the operators $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acting on the three dimensional Hilbert space $\mathbb{C}^3$. A simple calculation shows that $(TS)^D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $ST = (TS)^D$. Moreover TS and ST are $(1,1)$-power D-normal, however $TT^*S \neq STT^*$.

REMARK 2.2. If $m = 1$, then Theorem 2.2 coincides with Proposition 3.2 in [11].

In [22, Theorem 3.3] it was observed that the class $[(n,m)D\mathbf{QN}]$ is closed under unitary equivalence.

THEOREM 2.3. *Let $T,S \in D\mathscr{B}(\mathscr{H})$ such that T is normal and TS is (n,m)-power D-quasi-normal. Then*

$$TT^*S = STT^* \Longrightarrow ST \text{ is } (n,m)-\text{power } D-\text{quasi-normal.}$$

Proof. A similar arguments as in proof of Theorem 2.2 show that ST is unitary equivalent to a (n,m)-power D-quasi-normal and the conclusion of this theorem follows from [22, Theorem 3.3].

It was proved in [13] that TS and ST are normal if and only if $S^*ST = TSS^*$ and $T^*TS = STT^*$. In the following theorem we generalize this result to the class of (n,m)-power D-normal operators.

THEOREM 2.4. *Let $S,T \in D\mathscr{B}(\mathscr{H})$ such that $(TS)^m = T^mS^m$ and $(ST)^m = S^mT^m$. the following hold:*

(1) *If ST and TS are of class $[(n,m)D\mathbf{N}]$, then*

$$S^{*m}((ST)^D)^{nm} = ((TS)^D)^{nm}S^{*m} \text{ and } ((ST)^D)^{nm}T^{*m} = T^{*m}((TS)^D)^{nm}.$$

(2) *If* $S^{*m}((ST)^D)^{nm} = ((TS)^D)^{nm}S^{*m}$ *and* $((ST)^D)^{nm}T^{*m} = T^{*m}((TS)^D)^{nm}$, *then* TS *and* ST *are of class* $[(nm,m)D\mathbf{N}]$.

Proof. (1) Since ST and TS are of class $[(n,m)D\mathbf{N}]$, it follows that $(TS)^{nm})$ and $(ST)^{nm})$ are of class $[D\mathbf{N}]$ (by [22, Theorem 2.1]). By using [8, Corollary 1.1] it is easily seen that

$$S^m\big((S^mT^m)^D\big)^n = \big((T^mS^m)^D\big)^nS^m \text{ or equivalently } S^m\big((ST)^D\big)^{nm} = \big((TS)^D\big)^{nm}S^m$$

and

$$T^m\big((S^mT^m)^D\big)^n = \big((T^mS^m)^D\big)^nT^m \text{ or equivalently } T^m\big((ST)^D\big)^{nm} = \big((TS)^D\big)^{nm}T^m.$$

By Fuglede Putnam theorem we get

$$S^{*m}\big((ST)^D\big)^{nm} = \big((TS)^D\big)^{nm}S^{*m}$$

and

$$T^{*m}\big((ST)^D\big)^{nm} = \big((TS)^D\big)^{nm}T^{*m}.$$

(2) If $S^{*m}((ST)^D)^{nm} = ((TS)^D)^{nm}S^{*m}$, then multiplying this equation from the left by T^{*m} we get

$$T^{*m}S^{*m}((ST)^D)^{nm} = T^{*m}((TS)^D)^{nm}S^{*m} = ((ST)^D)^{nm}T^{*m}S^{*m}$$

and so that

$$(ST)^{*m}((ST)^D)^{nm} = ((ST)^D)^{nm}(ST)^{*m}.$$

Similarly, if $((ST)^D)^{nm}T^{*m} = T^{*m}((TS)^D)^{nm}$, then multiplying this equation from the right by S^{*m} we get

$$((TS)^D)^{nm}(TS)^{*m} = (TS)^{*m}((TS)^D)^{nm}.$$

Hence, TS and ST are of class $[(nm,m]D\mathbf{N}]$.

REMARK 2.3. If $m = 1$, then the statements (1) and (2) are equivalent. In this case Theorem 2.3 coincides with [11, Theorem 2.4].

The following Theorem extended [11, Theorem 3.5].

THEOREM 2.5. *Let* $T,S \in D\mathscr{R}(\mathscr{H})$ *and* $A \in \mathscr{B}(\mathscr{H})$. *If* T *and* S *are of class* $[(n,m)D\mathbf{N}]$ *and* $TA = AS$, *then the following statement hold:*

(1) $\big(T^D\big)^{*j}A = A\big(S^D\big)^{*j}$ *where* j *is the least common multiple of* n *and* m.

(2) $\big(T^D\big)^{*nm}A = A\big(S^D\big)^{*nm}$.

Proof. (1) Since T and S are of class $[(n,m)D\mathbf{N}]$, it follows that T^j, S^j, T^{nm} and S^{nm} are of class $[D\mathbf{N}]$ (by [22, Theorem 2.1]. On the other hand, under the assumption that $TA = AS$, we obtain in view of [11, Lemma 2.6] that $T^DA = AS^D$. From which we deduce that $\big(T^D\big)^jA = A\big(S^D\big)^j$. It is well known that $\big(T^D\big)^j$ and $\big(S^D\big)^j$ are normal operators. Thanks to Fuglede-Putnam theorem we obtain that

$$\big(T^D\big)^{*j}A = A\big(S^D\big)^{*j}.$$

(2) The proof of this statement is similar to the statement (1) since we omit it.

We have the following corollary from this theorem.

COROLLARY 2.2. *Let $T \in D\mathscr{R}(\mathscr{H})$ and $A \in \mathscr{B}(\mathscr{H})$. If T is of class $[(n,m)D\mathbf{N}]$ and $TA = AT$, then the following statement hold:*

(1) $\big(T^D\big)^{*j}A = A\big(T^D\big)^{*j}$ *where j is the least common multiple of n and m.*

(2) $\big(T^D\big)^{*nm}A = A\big(T^D\big)^{*nm}$.

For $T \in \mathscr{B}(\mathscr{H})$ it is well know that the ascent $p(T)$ of an operator T is the smallest non-negative integer r such that $\mathscr{N}(T^r) = \mathscr{N}(T^{r+1})$ and if such an integer does not exist then we put $p(T) = \infty$. Analogously, descent $q(T)$ of the operator T is the smallest non-negative integer s such that $\mathscr{R}(T^s) = \mathscr{R}(T^{s+1})$ and if such an integer does not exist then we put $q(T) = \infty$. If $p(T)$ and $q(T)$ are finite then $p(T) = q(T)$ [17, Proposition 38.3].

THEOREM 2.6. *([7]) For an operator $T \in \mathscr{B}(\mathscr{H})$ we have the following:*

(1) *If $\mathscr{N}(T^k) = \mathscr{N}(T^{k+1})$ for some k, then $\mathscr{N}(T^n) = \mathscr{N}(T^k)$ for all $n \geqslant k$.*

(2) *If $\mathscr{R}(T^k) = \mathscr{R}(T^{k+1})$ for some k, then $\mathscr{R}(T^n) = \mathscr{R}(T^k)$ for all $n \geqslant k$.*

The following proposition shows that the ascent and descent of T^D are finite if $T \in [(n,m)D\mathbf{N}]$.

THEOREM 2.7. *For any operator $T \in D\mathscr{R}(\mathscr{H})$ with T is of class $[(n,m)D\mathbf{N}]$, the following assertions hold:*

(1) $p(T^D) = q(T^D) \leqslant j$ *, where j is the least common multiple of n and m.*

(2) $\mathscr{N}^\infty(T^D) = \mathscr{N}\big(\big(T^D\big)^j\big)$ *and* $\big(T^D\big)^\infty(\mathscr{H}) = \mathscr{R}\big(\big(T^D\big)^j\big)$*, where*

$$\mathscr{N}^\infty(T^D) = \bigcup_{k\in\mathbb{N}} \mathscr{N}\big((T^D)^k\big) \text{ and } \big(T^D\big)^\infty(\mathscr{H}) = \bigcap_{k\in\mathbb{N}} \big((T^D)^k\big)(\mathscr{H})$$

are the hyper kernel and hyper range respectively.

Proof. It is well known that, for any normal operator S, $\mathcal{N}(S^2)=\mathcal{N}(S)$ and $\mathcal{R}(S^2)=\mathcal{R}(S)$.

Since T is of class $[(n,m)D\mathbf{N}]$, it follows that T^j is of class $[D\mathbf{N}]$, then $\left(T^D\right)^j$ is normal. This implies that

$$\mathcal{N}\left(\left(T^D\right)^{2j}\right)=\mathcal{N}\left(\left(T^D\right)^j\right) \text{ and } \mathcal{R}\left(\left(T^D\right)^{2j}\right)=\mathcal{R}\left(\left(T^D\right)^j\right).$$

From the following subspace inclusions

$$\begin{aligned}\mathcal{N}\left(T^D\right)\subset\mathcal{N}\left(\left(T^D\right)^{2j}\right)\subset\cdots\subset\mathcal{N}\left(\left(T^D\right)\right)^j\subset\mathcal{N}\left(\left(T^D\right)\right)^{j+1}\subset\cdots\\ \subset\mathcal{N}\left(\left(T^D\right)\right)^{2j}=\mathcal{N}\left(\left(T^D\right)^j\right)\subset\mathcal{N}\left(\left(T^D\right)^{2j+1}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}\left(\left(T^D\right)\right)^j=\mathcal{R}\left(\left(T^D\right)^{2j}\right)\subset\mathcal{R}\left(\left(T^D\right)^{2j-1}\right)\subset\cdots\subset\mathcal{R}\left(\left(T^D\right)^{j+1}\right)\subset\mathcal{R}\left(\left(T^D\right)^j\right)\subset\\ \subset\mathcal{R}\left(\left(T^D\right)^{j-1}\right)\subset\cdots\subset\mathcal{R}\left(T^D\right),\end{aligned}$$

it follows by applying Theorem 2.6 that

$$\mathcal{N}\left(\left(T^D\right)\right)^j=\mathcal{N}\left(\left(T^D\right)\right)^{j+1} \text{ and } \mathcal{R}\left(\left(T^D\right)\right)^j=\mathcal{R}\left(\left(T^D\right)\right)^{j+1}.$$

Hence this implies that the ascent and descent of the Drazin inverse T^D of T are less than or equal to j i.e;. $p(T^D)\leqslant j$ and $q(T^D)\leqslant j$.

Since both are finite $p(T^D)=q(T^D)$ ([17]).

(2) Also

$$\mathcal{N}^{\infty}\left(T^D\right)=\bigcup_{k\in\mathbb{N}}\mathcal{N}\left((T^D)^k\right)=\mathcal{N}\left(\left(T^D\right)^j\right)$$

and

$$\left(T^D\right)^{\infty}(\mathcal{H})=\bigcap_{k\in\mathbb{N}}\left((T^D)^k\right)(\mathcal{H})=\mathcal{R}\left(\left(T^D\right)^j\right).$$

3. D-m-quasi-normal operators

In this section, the class of D-m-quasi-normal operators as a generalization of the classes of D-quasi-normal operators is introduced. We make several observations about members from this class.

DEFINITION 3.1. Let $T\in D\mathcal{R}(\mathcal{H})$. We said that T is D-m-quasi-normal if

$$T^D\left(T^*T\right)^m=\left(T^*T\right)^mT^D \tag{3.1}$$

for some positive integer m. This class of operators will denoted by $[D(\mathbf{QN})^m]$.

REMARK 3.1. (i) If $m = 1$, then D-m-quasi-normal becomes D-quasi-normal.

(ii) $T \in [mD\mathbf{QN}] \Longleftrightarrow [T^D, (T^*T)^m] = 0$.

(iii) $T \in [mD\mathbf{QN}] \Longleftrightarrow T^D|T|^{2m} = |T|^{2m}T^D$.

REMARK 3.2. Obviously, that the class of D-m-quasi-normal operators includes classes of quasi-normal and D-quasi-normal operators,ie., the following inclusions holds

$$[\mathbf{QN}] \subset [D\mathbf{QN}] \subset [(D\mathbf{QN})^m].$$

THEOREM 3.1. *Let $T \in D\mathscr{R}(\mathscr{H})$. Then T is of class $[D(\mathbf{QN})^m]$ if and only if C commutes with ReT^D and ImT^D.*

Proof. Let T be D-m-quasi-normal, i.e.,

$$T^D\big(T^*T\big)^m = \big(T^*T\big)^m T^D,$$

it follows that $C^2ReT^D = ReT^DC^2$. Since C is non-negative definite, it follows that $CReT^D = ReT^DC$. In Similar way we can prove that $CImT^D = ImT^DC$.

Conversely, Assume that $CReT^D = ReT^DC$ and $CImT^D = ImT^DC$. Then

$$C^2ReT^D = ReT^DC^2 \quad \text{and} \quad C^2ImT^D = ImT^DC^2.$$

Hence

$$C^2\big(ReT^D + iImT^D\big) = \big(ReT^D + iImT^D\big)C^2$$

and we have $C^2T^D = T^DC^2$. Consequently, $T^D\big(T^*T\big)^m = \big(T^*T\big)^m T^D$.

THEOREM 3.2. *Let $T \in D\mathscr{R}(\mathscr{H})$ such that T satisfied the following conditions*

(i) B commutes with ReT^D and ImT^D

(ii) $C^2T^D = T^DB^2$,

then T is of class $[D(\mathbf{QN})^m]$.

Proof. Since $BReT^D = ReT^DB$ and $BImT^D = ImT^DB$, it follows that

$$\begin{cases} B^2T^D + B^2\big(T^D\big)^* = T^DB^2 + \big(T^D\big)^*B^2 \\ B^2T^D - B^2\big(T^D\big)^* = T^DB^2 - \big(T^D\big)^*B^2 \end{cases}.$$

This gives $B^2T^D = T^DB^2 = C^2T^D$. Hence T is D-m-quasi-normal or $T \in [D(\mathbf{QN})^m]$.

THEOREM 3.3. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T is normal and D-m-quasi-normal, then T^D is m-quasi-normal.*

Proof. Since T is m-D-quasi-normal, it follows in view of Lemma 1.1 that

$$\begin{aligned} & T^D\big(T^*T\big)^m = \big(T^*T\big)^m T^D \\ \Rightarrow\ & T^D\big(\big(T^*T\big)^D\big)^m = \big(\big(T^*T\big)^D\big)^m T^D \\ \Rightarrow\ & T^D\big(((T)^D)^*T^D\big)^m = \big(((T)^D)^*T^D\big)^m T^D \quad (\text{since } T^*T = TT^*). \end{aligned}$$

Hence T^D is m-quasi-normal operator.

4. n-power D-m-quasi-normal operators

In this section, the class of n-power D-m-quasi-normal operators as a generalization of the classes of D-m-quasi-normal and n-power m-quasi-normal operators is introduced. In addition, we make several observations about members from this class.

DEFINITION 4.1. Let $T \in D\mathscr{R}(\mathscr{H})$. We said that T is n-power D-m-quasi normal if

$$\big(T^D\big)^n\big(T^*T\big)^m = \big(T^*T\big)^m\big(T^D\big)^n \tag{4.1}$$

for some positive integers n and m. This class of operators will denoted by $[nD(\mathbf{QN})^{\mathbf{m}}]$.

REMARK 4.1. We make the following observations

(1) $[D\mathbf{QN}]$ is the class of D-quasi-normal operator i.e., $[(1,1)D\mathbf{QN}] = [D\mathbf{QN}]$.

(2) $[(nD\mathbf{QN}]$ is the class of n-power D-quasi normal operators:

(3) n-power m-quasi-normal operator is an n-power D-m-quasi-normal.

(4) Every n-power-D-quasi-normal operator is an n-power D-m-quasi-normal:

$$[nD\mathbf{QN}] \subset [nD(\mathbf{QN})^m].$$

(5) Every D-m-quasi-normal operator is an n-power-D-m-quasi-normal: $[nD(\mathbf{QN}] \subset [nD(\mathbf{QN})^m]$.

REMARK 4.2. (i) $T \in [nD(\mathbf{QN})^m] \Longleftrightarrow [\big(T^D\big)^n, \big(T^*T\big)^m] = 0$.

(ii) $T \in [nD(\mathbf{QN})^m] \Longleftrightarrow \big(T^D\big)^n|T|^{2m} = |T|^{2m}\big(T^D\big)^n$.

REMARK 4.3. Clearly, that the class of n-power D-m-quasi-normal operators includes classes of n-power quasi-normal, n-power D-quasi-normal and D-m-quasi-normal operators,ie., the following inclusions holds

$$[n\mathbf{QN}] \subset [nD\mathbf{QN}] \subset [nD(\mathbf{QN})^m], \quad [D(\mathbf{QN})^m] \subset [nD(\mathbf{QN})^m].$$

REMARK 4.4. If T is an n-power D-m-quasi normal, then T is

- $2n$-power D-m-quasinormal operator.
- n-power D-$2m$-quasi-normal operator.
- $2n$-power D-$2m$-quasinormal quasi-normal operator.

The following proposition gives a characterization of an n-power D-m-quasi-normal operators.

PROPOSITION 4.1. *Let $T \in D\mathscr{R}(\mathscr{H})$. If $A = \left(T^D\right)^n + (T^*T)^m$ and $B = \left(T^D\right)^n - (T^*T)^m$, then T is of class $[nD(\mathbf{QN})^m]$ if and only if A commutes with B.*

Proof. Commutativity of A and B is equivalent to $\left(T^D\right)^n (T^*T)^m = (T^*T)^m \left(T^D\right)^n$.

PROPOSITION 4.2. *Let T, A, B be as in Proposition 4.1. If T is of class $[nD(\mathbf{QN})^m]$, then $\left(T^D\right)^n (T^*T)^m$ commutes with A and B.*

Proof. By (4.1) we have that

$$(T^D)^n (T^*T)^m \Big((T^D)^n \pm (T^*T)^m \Big) = \Big((T^D)^n \pm (T^*T)^m \Big) (T^D)^n (T^*T)^m.$$

In general, the two classes $[nD(\mathbf{QN})^m]$ and $[(n+1))D(\mathbf{QN})^m]$ are not the same (see [23]).

PROPOSITION 4.3. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T is both of class $[nD(\mathbf{QN})^m]$ and $[(n+1)D(QN)^m]$, then it is of class $[(2n+1)D(\mathbf{QN})^m]$. i.e.,*

$$[nD(QN)^m] \cap [(n+1)D(\mathbf{QN})^m] \subset [(2n+1)D(\mathbf{QN})^m].$$

Proof. Since T is both of class $[nD(\mathbf{QN})^m]$ and $[(n+1D(QN)^m]$, it follows that

$$(T^D)^{n+1} (T^*T)^m = (T^*T)^m (T^D)^{n+1}, \tag{4.2}$$

and

$$(T^D)^{2n+1} (T^*T)^m = (T^D)^n (T^*T)^m (T^D)^{n+1} = (T^*T)^m (T^D)^{2n+1},$$

so that $(T^D)^{2n+1} (T^*T)^m$ may be transformed into $(T^*T)^m (T^D)^{2n+1}$.

PROPOSITION 4.4. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T is of class $[nD(\mathbf{QN})^{\mathbf{m}}]$ such that T is a partial isometry, then T is of class $[(n+1)D(\mathbf{QN})^m]$.*

Proof. Since T is a partial isometry, therefore

$$TT^*T = T. \tag{4.3}$$

and hence $(T^*T)^m = T^*T$. From which we deduce that if is of class $[nD(\mathbf{QN})^m]$, then T is of class $[nD\mathbf{QN}]$. By applying [10, Theorem 5.18] we deduce that T is of class $[(n+1)D\mathbf{QN}]$. Hence T is of class $[(n+1)D(\mathbf{QN})^m]$ in view of the statement (4) of Remark 4.1.

The class $[nD(\mathbf{QN})^m]$ has the following properties.

THEOREM 4.1. *The class $[nD(\mathbf{QN})^m]$ is closed under unitary equivalence.*

Proof. Let $S \in D\mathscr{B}(\mathscr{H})$ be unitary equivalent to T. Then there is a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that $T = U^*SU$ which implies that $T^* = U^*S^*U$. Noting that $T^n = U^*S^nU$, $(T^*T)^m = U^*(S^*S)^mU$ and $\left(U^*TU\right)^D = U^*T^DU$. Inserting $I = UU^*$ suitably, then if T is of class $[nD(\mathbf{QN})^m]$ we deduce that

$$U^*(S^D)^n(S^*S)^mU = (T^D)^n(T^*T)^m = (T^*T)^m(T^D)^n = U^*(S^*S)^m(S^D)^nU.$$

Therefore S is of class $[nD(\mathbf{QN})^m]$.

The following example shows that the unitarily equivalence in Theorem 4.1 is replaced by similarity then the result is need not be true.

EXAMPLE 4.1. Consider the two operators $T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acting on the two dimensional Hilbert space $\mathbb{C}^2$, then T is D-2-quasi-normal operator, but $S = XTX^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ is not D-2-quasi-normal operator.

PROPOSITION 4.5. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T is both of class $[nD(\mathbf{QN})^m]$ and $[(n+1)D(\mathbf{QN})^m]$ such that T is injective, then T^D is D-m-quasi-normal i.e. $T^D \in [D(\mathbf{QN})^m]$.*

Proof. Since T is of class $[nD(\mathbf{QN})^m] \cap [(n+1)D(\mathbf{QN})^m]$, it follows that

$$(T^D)^n\left(T^D(T^*T)^m - (T^*T)^mT^D\right) = 0.$$

If T^D is injective, then so is $(T^D)^n$ and we have $T^D(T^*T)^m - (T^*T)^mT^D = 0$, whence T^D is D-m-quasi-normal.

PROPOSITION 4.6. *Let $T \in D\mathscr{R}(\mathscr{H})$ such that T is of class $[2D(\mathbf{QN})^m]$ $\bigcap[3D(\mathbf{QN})^m]$, then T is of class $[nD(\mathbf{QN})^m]$ for all positive integers $n \geqslant 4$ and $m \geqslant 1$.*

Proof. We proof the assertion by using the mathematical induction. For $n = 4$ it is a consequence of Remark 3.2 .

We prove this for $n = 5$. Since $T \in [2D(\mathbf{QN})^m]$,

$$(T^D)^2(T^*T)^m = (T^*T)^m(T^D)^2, \tag{4.4}$$

multiplying (4.4) to the left by $(T^D)^3$ we get

$$(T^D)^5(T^*T)^m = (T^D)^3(T^*T)^m(T^D)^2.$$

Thus implies

$$(T^D)^5(T^*T)^m = (T^*T)^m(T^D)^5.$$

Now assume that the result is true for $n \geqslant 5$ that is

$$(T^D)^n(T^*T)^m = (T^*T)^m(T^D)^n,$$

then

$$\begin{aligned}(T^D)^{n+1}(T^*T)^m &= T^D(T^*T)^mT^n = T^D(T^*T)^mT^2T^{n-2} = (T^D)^3(T^*T)^m(T^D)^{n-2}\\ &= (T^*T)^m(T^D)^{n+1}.\end{aligned}$$

Thus T is of class $[(n+1)D(\mathbf{QN})^m]$. The proof is complete.

PROPOSITION 4.7. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T is of class $[nD(\mathbf{QN})^2]\bigcap[nD(\mathbf{QN})^3]$, then T is of class $[nD(\mathbf{QN})^m]$ for all positive integer $n \geqslant 1$ and $m \geqslant 4$.*

Proof. Since T is in $[nD(\mathbf{QN})^2]$ and in $[nD(\mathbf{QN})^3]$ we have

$$(T^D)^n(T^*T)^2 = (T^*T)^2(T^D)^n \text{ and } (T^D)^n(T^*T)^3 = (T^*T)^3(T^D)^n$$

from which if follows that

$$(T^D)^n(T^*T)^4 = (T^*T)^4(T^D)^n \text{ and } (T^D)^n(T^*T)^5 = (T^*T)^{\propto}5(T^D)^n.$$

Hence T is in $[nD(\mathbf{QN})^4]$ and in $[nD(\mathbf{QN})^5]$. Now assume that T is in $[nD(\mathbf{QN})^m]$ for some positive integer $m \geqslant 5$.

We have to distinguish two cases: First case: if m is even we have

$$\begin{aligned}(T^D)^n(T^*T)^{m+1} &= (T^D)^n(T^*T)^2(T^*T)^{m-1} = (T^*T)^2(T^D)^n(T^*T)^2(T^*T)^{m-3}\\ &= (T^*T)^4(T^D)^n(T^*T)^{m-3}\\ &= (T^*T)^4(T^D)^n(T^*T)^2(T^*T)^{n-5} = (T^*T)^6(T^*T)^3(T^D)^n(T^*T)^{m-5}\\ &= \vdots\\ &= (T^*T)^{m-2}(T^D)^n(T^*T)^2 = (T^*T)^{m+1}(T^D)^n.\end{aligned}$$

Second case: if m is odd, then we have

$$\begin{aligned}(T^D)^n(T^*T)^{m+1} &= (T^D)^n(T^*T)^2(T^*T)^{m-1} = (T^*T)^2(T^D)^n(T^*T)^2(T^*T)^{m-3}\\ &= (T^*T)^4(T^D)^n(T^*T)^{m-3} = (T^*T)^4(T^D)^n(T^*T)^2(T^*T)^{n-5}\\ &= (T^*T)^6(T^*T)^3(T^D)^n(T^*T)^{k-5}\\ &= \vdots\\ &= (T^*T)^{m-2}(T^D)^n(T^*T)^3 = (T^*T)^{m+1}(T^D)^n.\end{aligned}$$

So that T is of class $[nD(\mathbf{QN})^{m+1}]$.

PROPOSITION 4.8. *$T \in D\mathscr{R}(\mathscr{H})$ such that T is of class $[nD(\mathbf{QN})^m]$ and of class $[nD(\mathbf{QN})^{m+1}]$. If T is injective, then T is of class $[nD\mathbf{QN}]$.*

Proof. Since $T \in [nD(\mathbf{QN})^m] \cap [nD(\mathbf{QN})^{m+1}]$, we have

$$\begin{aligned}&\big(T^D\big)^n\big(T^*T\big)^{m+1} = \big(T^*T)\big)^{m+1}\big(T^D\big)^n\\ \Longrightarrow\ &\big(T^*T\big)^m\big(T^D\big)^n\big(T^*T\big) = \big(T^*T\big)^{m+1}\big(T^D\big)^n\\ \Longrightarrow\ &\big(T^*T\big)^m\big[\big(T^D\big)^nT^*T - T^*T\big(T^D\big)^n\big] = 0\\ \Longrightarrow\ &\big(T^*T\big)^{m-1}\big[\big(T^D\big)^nT^*T - T^*T\big(T^D\big)^n\big] = 0 \quad \big(\text{by } \mathscr{N}(T^*T) = \mathscr{N}(T)\big)\\ \Longrightarrow\ &\big(T^*T\big)^{m-2}\big[\big(T^D\big)^nT^*T - T^*T\big(T^D\big)^n\big] = 0 \quad \big(\text{by } \mathscr{N}(T^*T) = \mathscr{N}(T)\big)\\ \Longrightarrow\ &\vdots\\ \Longrightarrow\ &\big(T^*T\big)\big[\big(T^D\big)^nT^*T - T^*T\big(T^D\big)^n\big] = 0 \quad \big(\text{by } \mathscr{N}(T^*T) = \mathscr{N}(T)\big)\\ \Longrightarrow\ &\big[\big(T^D\big)^nT^*T - T^*T\big(T^D\big)^n\big] = 0 \quad \big(\text{by } \mathscr{N}(T^*T) = \mathscr{N}(T)\big).\end{aligned}$$

Therefore T is of class $[nD\mathbf{QN}]$.

THEOREM 4.2. *Let $T \in D\mathscr{R}(\mathscr{H})^D$. If T is normal and n-power D-m-quasi-normal, then T^D is n-power m-quasi-normal.*

Proof. Since T is n-power D-m-quasi-normal, it follows in view of Lemma 1.1 that

$$\begin{aligned}&\big(T^D\big)^n\big(T^*T\big)^m = \big(T^*T\big)^m\big(T^D\big)^n\\ \Rightarrow\ &\big(T^D\big)^n\big(\big(T^*T\big)^D\big)^m = \big(\big(T^*T\big)^D\big)^m\big(T^D\big)^n\\ \Rightarrow\ &\big(T^D\big)^n\big(((T)^D)^*T^D\big)^m = \big(((T)^D)^*T^D\big)^m\big(T^D\big)^n \quad (\text{since } T^*T = TT^*).\end{aligned}$$

Hence T^D is n-power m-quasi-normal operator as required.

THEOREM 4.3. *Let $T \in D\mathscr{R}(\mathscr{H})$. If T is of class $[nD(\mathbf{QN})^m]$, then the following statements hold:*

(1) *T is of class $[jD(\mathbf{QN})^j]$ where j is the least common multiple of n and m.*

(2) *T is of class $[nmD(\mathbf{QN})^{nm}]$.*

Proof. (1) Since T is n-power D-m-quasi-normal we have

$$\left(T^D\right)^n\left(T^*T\right)^m=\left(T^*T\right)^m\left(T^D\right)^n.$$

Let $j=rn$ and $j=sm$, it is easy to see that

$$\begin{aligned}\left(T^D\right)^j\left(T^*T\right)^j &= \left(T^D\right)^{rn}\left(T^*T\right)^{sm}=\left[\left(T^D\right)^n\right]^r\left[\left(T^*T\right)^m\right]^s\\ &= \underbrace{\left(T^D\right)^n.\cdots.\left(T^D\right)^n}_{r-\text{times}}\underbrace{\left(T^*T\right)^m\cdots\left(T^*T\right)^m}_{s-\text{times}}\\ &= \underbrace{\left(T^*T\right)^m\cdots\left(T^*T\right)^m}_{s-\text{times}}\underbrace{\left(T^D\right)^n.\cdots.\left(T^D\right)^n}_{r-\text{times}}\\ &= \left(T^*T\right)^{sm}\left(T^D\right)^{rn}=\left(T^*T\right)^{sm}\left(T^D\right)^{rn}=\left(T^*T\right)^j\left(T^D\right)^j,\end{aligned}$$

which means that T^j is j-power D-j-quasi-normal.

(ii) By similar way.

The following proposition gives a sufficient condition for which T^* is of $[nD(\mathbf{QN})^m]$ whenever T is of class $[nD(\mathbf{QN})^m]$. We declare its proof obvious.

PROPOSITION 4.9. *Let $T\in D\mathscr{R}(\mathscr{H})$ such that T is of class $[nD(\mathbf{QN})^m]$. If T is normal, then T^* is of class $[nD(\mathbf{QN})^m]$.*

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