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# SHORTED OPERATORS WITH RESPECT TO A PARTIAL ORDER IN A DUAL MODULE

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*Abstract.* The purpose of this paper is to determine exactly the shorted operators in the sense of linear functionals under the direct sum partial order.

## 1. Introduction

Throughout this paper $R$ denotes an associative ring with identity $1_R$ and modules are unitary right $R$-modules. For a right $R$-module $M_R = M$, $S = \mathrm{End}_R(M)$ denotes the ring of all right $R$-module endomorphisms of $M$. It is well known that $M$ is a left $S$ and right $R$-bimodule. Then $M^* := \mathrm{Hom}_R(M,R)$ is a left $R$-right $S$-bimodule and $M^*$ is called the *dual module* of $M$. Elements of $M^*$ are called *linear functionals*.

An element $m \in M$ is called *a* (*Zelmanowitz*) *regular element* if

$$m = m\varphi(m) \equiv \; m\varphi m$$

for some $\varphi \in M^*$. A module $M$ is called *regular* (in the sense of Zelmanowitz) if every element of $M$ is regular. For a ring $R$, let $a \in R$ be a regular element (in the sense of von Neumann). Then there exists $a^- \in R$ such that $a = aa^-a$. It is well-known that $a \in R$ is regular (in the sense of von Neumann) if and only if $a$ is regular in $R_R$ (or, similarly, in the left $R$-module ${}_RR$) (in the sense of Zelmanowitz).

Let $M$ be a module. It is known that $\mathrm{Hom}_R(R,M) \cong M$. Let $m \in M$ be regular, say $m = m\varphi m$ where $\varphi \in M^*$. Then for the map $m\varphi\colon M \to M$, defined by

$$(m\varphi)(x) = m\varphi(x) \equiv m\varphi x, \quad x \in M,$$

we may conclude that $m\varphi \in S$ and that $m\varphi$ is an idempotent in $S$.

In [5], Blackwood et al. defined a relation $\leqslant^{\oplus}$ on a ring $R$ in the following way: For $a,b \in R$,

$$a \leqslant^{\oplus} b \quad \text{if} \quad bR = aR \oplus (b-a)R.$$

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They proved that the relation $\leqslant^{\oplus}$ is a partial order on the von Neumann regular rings and called it the *direct sum partial order*.

Motivated by the concept of the direct sum partial order on von Neumann regular rings, we extend this concept to the dual modules. In this direction, we introduce a direct sum partial order on dual modules. We investigate in detail when linear functionals are correlated according to the direct sum partial order.

The shorted operator arises in various contexts and contributes as a tool for solving many important issues (see [1, 2, 3, 4, 6, 7, 8]). In [1], Anderson introduced the shorted operator in finite dimensional linear spaces in terms of an explicit matrix construction. It was shown to be the solution to a maximization problem. In [2], the same maximization problem was used to furnish the definition, and the definition is extended to an arbitrary Hilbert space (not necessarily finite dimensional). The definition of the shorted operator is given in [2] as follows: Given a closed subspace $S$ of a Hilbert space $H$, the *shorted operator* $A_S$ of a positive operator $A$ on $S$ is defined as

$$A_S = \max\{D \mid 0 \leqslant D \leqslant A, \mathrm{ran} D \subset S\},$$

for the partial ordering

$$A \leqslant B \Leftrightarrow A, B \text{ are self-adjoint and } B - A \text{ is positive.}$$

The existence of such a maximum is guaranteed in [2]. In [8], Raïssouli noticed that the assumed condition "$S$ is closed" is not necessary to define the shorted operator. If $H$ is a finite dimensional space, a physical interpretation of the shorted operator in terms of electrical circuits can be found in [2]. Mitra and Puri obtained two explicit representations for the shorted operator (in finite dimensional linear spaces), one in terms of the $g$-inverse, the other in terms of the least squares inverse of a complex matrix in [6]. Extension of shorted operator to convex functionals has been studied in [8] by Raïssouli.

For a shorted operator, it is natural to ask the following question: what should be the analogue of $A_S$ when the variable operator $A$ is a linear functional? The answer of this question is given in Theorem 2.21. As an application of shorted operators in dual module theory, in this paper, our main aim is to determine maximal elements in a subset of a dual module via the direct sum partial order.

## 2. Shorted operators in a dual module

Let $M$ be a left $S$-right $R$-bimodule where $S = \mathrm{End}_R(M)$. For the sake of brevity, in the sequel, $S$ will stand for the endomorphism ring of the module $M$ considered. We will denote the identity map in $S$ by $1_S$.

Let $R$ be a ring and $a \in R$. In [5], an element $b \in R$ is said to be a *von Neumann inverse* of $a$ if $aba = a$. An element $b \in R$ is said to be *weak von Neumann inverse* of $a$ if $bab = b$. If $aba = a$ and $bab = b$, then $b$ is called a *strong von Neumann inverse* of $a$. In the following, we extend the aforementioned inverses to the general module theoretic setting.

DEFINITION 2.1. Let $M$ be a module and $m \in M$.

(1) If $m = m\varphi m$ for some $\varphi \in M^*$, then $\varphi$ is called a *regular support* of $m$ and it will be denoted by $m^{(1)}$. Also, $\{m^{(1)}\}$ denotes the set of all regular supports of $m \in M$.

(2) If $\varphi = \varphi m\varphi$ for some $\varphi \in M^*$, then $\varphi$ is called a *weak regular support* of $m$ and it will be denoted by $m^{(2)}$. Also, $\{m^{(2)}\}$ denotes the set of all weak regular supports of $m \in M$.

(3) If $m = m\varphi m$ and $\varphi = \varphi m\varphi$ for some $\varphi \in M^*$, then $\varphi$ is called a *strong regular support* of $m$ and it will be denoted by $m^{(1,2)}$. Also, $\{m^{(1,2)}\}$ denotes the set of all strong regular supports of $m \in M$.

Let $M$ be a module and $\varphi \in M^*$. Set $H_\varphi = \{m \in M \mid \varphi = \varphi m\varphi\}$. If $m \in M$ is regular, then there exists $\varphi \in M^*$ with $m = m\varphi m$. It can be shown that $\varphi m\varphi \in \{m^{(1,2)}\}$. Hence $m \in H_{\varphi m\varphi}$.

Consider the canonical map $\theta : M \to M^{**}$ from a module $M$ into its double dual $M^{**} := \mathrm{Hom}_R(M^*, R)$ defined by

$$\begin{aligned} \theta:\ M &\longrightarrow \quad M^{**} \\ m &\longrightarrow \theta(m):\ M^* \longrightarrow R \\ &\qquad\qquad\quad \varphi \ \longrightarrow (\varphi)(\theta(m)) = \varphi m. \end{aligned}$$

It is known that $\theta$ is injective if and only if $M$ is torsionless (i.e., $M$ can be embedded into some direct products of $R$). For any module $M$, its dual module $M^*$ is always torsionless. The module $M$ is also said to be *reflexive* if $\theta$ is bijective. Finitely generated torsion-free modules over a Dedekind domain, finitely generated free modules and the direct sum of countably many copies of the integers as a module over the integers are some examples of reflexive modules. In the next result, we investigate under what conditions the set $H_\varphi$ is nonempty for any $\varphi$ in a dual module.

PROPOSITION 2.2. *Let $M$ be a module and $\varphi \in M^*$. If $H_\varphi \neq \emptyset$, then $\varphi$ is regular. The converse holds if the canonical map $\theta : M \to M^{**}$ is surjective.*

*Proof.* Assume that $H_\varphi \neq \emptyset$. Then $\varphi = \varphi m\varphi$ for some $m \in M$. Hence $\theta(m) \in M^{**}$. By the definition of $\theta$, we obtain $\varphi(\theta(m))\varphi = \varphi m\varphi = \varphi$. Thus $\varphi$ is regular. For the converse, suppose that $\varphi$ is regular and $\theta$ is surjective. Then there exists $\alpha \in M^{**}$ such that $\varphi = \varphi\alpha\varphi$. The surjectivity of $\theta$ yields $\alpha = \theta(m)$ for some $m \in M$. It follows that $\varphi = \varphi\alpha\varphi = \varphi(\theta(m))\varphi = \varphi m\varphi$ in the light of the definition of $\theta$. Therefore $m \in H_\varphi$, and so $H_\varphi \neq \emptyset$.

COROLLARY 2.3. *Let $M$ be a module. If $H_\varphi \neq \emptyset$ for every $\varphi \in M^*$, then $M^*$ is regular. The converse holds if the canonical map $\theta : M \to M^{**}$ is surjective.*

Some decompositions of a dual module $M^*$ as a left $R$-module and as a right $S$-module are obtained as follows.

THEOREM 2.4. *Let $M$ be a module, $\varphi \in M^*$ and $m \in H_\varphi$. Then $M^*$ has the following decompositions.*

(1) *$M^* = R\varphi \oplus K$ where $K = \{\alpha \in M^* \mid \alpha m\varphi = 0\}$ as a left $R$-module.*

(2) *$M^* = \varphi S \oplus L$ where $L = \{\alpha \in M^* \mid \varphi m\alpha = 0\}$ as a right $S$-module.*

*Proof.* (1) Let $\alpha \in M^*$. Then $\alpha = \alpha m\varphi + \alpha(1_S - m\varphi)$. Note that $\alpha m\varphi \in R\varphi$. Being $\varphi = \varphi m\varphi$ implies $\alpha(1_S - m\varphi)m\varphi = 0$ entailing that $\alpha(1_S - m\varphi) \in K$. Hence $M^* \subseteq R\varphi + K$. The reverse inclusion is obvious, and so $M^* = R\varphi + K$. In order to see that this sum is direct, let $r\varphi = k \in R\varphi \cap K$ where $r \in R$ and $k \in K$. Since $m \in H_\varphi$, we have $r\varphi = r\varphi m\varphi = km\varphi = 0$. Thus $R\varphi \cap K = \{0\}$. Therefore $M^* = R\varphi \oplus K$.
(2) Similar to the proof of (1).

We now characterize the set $H_\varphi$ for any linear functional $\varphi$.

LEMMA 2.5. *Let $M$ be a module, $\varphi \in M^*$ and $m \in H_\varphi$. Then*

$$H_\varphi = m + (1_S - m\varphi)M + M(1_R - \varphi m).$$

*Proof.* Let $m \in H_\varphi$ and $K = m + (1_S - m\varphi)M + M(1_R - \varphi m)$. For any

$$m + (1_S - m\varphi)m_1 + m_2(1_R - \varphi m) \in K$$

where $m_1, m_2 \in M$, we have

$$\varphi(m + (1_S - m\varphi)m_1 + m_2(1_R - \varphi m))\varphi = \varphi m\varphi = \varphi.$$

This means $m + (1_S - m\varphi)m_1 + m_2(1_R - \varphi m) \in H_\varphi$, and so $K \subseteq H_\varphi$. For the reverse inclusion, let $n \in H_\varphi$. For $n - m, m\varphi n \in M$, we have

$$n = m + (1_S - m\varphi)(n - m) + m\varphi n(1_R - \varphi m) \in K.$$

Hence $H_\varphi \subseteq K$. Thus $H_\varphi = K$ as claimed.

Inspired by the notion of the direct sum order on rings, we now introduce the direct sum order in the setting of dual modules.

DEFINITION 2.6. Let $M$ be a right $R$-module with $S = \mathrm{End}_R(M)$ and $\varphi_1, \varphi_2 \in M^*$. We write

$$\varphi_1 \leqslant^{\oplus} \varphi_2 \quad if \quad \varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1)S.$$

In the next results, we give some characterizations of the direct sum order for linear functionals from different perspectives.

THEOREM 2.7. *Let $M$ be a module and $\varphi_1, \varphi_2 \in M^*$ with $H_{\varphi_2} \neq \emptyset$. Then the following statements are equivalent.*

(1) $\varphi_1 \leqslant^{\oplus} \varphi_2$;

(2) $R\varphi_2 = R\varphi_1 \oplus R(\varphi_2 - \varphi_1)$.

*Proof.* (1) $\Rightarrow$ (2) Let $\varphi_1 \leqslant^{\oplus} \varphi_2$. Then $\varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1)S$. By the hypothesis, $\varphi_2 = \varphi_2 m \varphi_2$ for some $m \in M$. Since $\varphi_1 S \subseteq \varphi_2 S$, there exists $f \in S$ such that $\varphi_1 = \varphi_2 f$. Hence

$$\varphi_2 m \varphi_1 = \varphi_2 m \varphi_2 f = \varphi_2 f = \varphi_1$$

and so

$$\varphi_2 m(\varphi_2 - \varphi_1) = \varphi_2 m \varphi_2 - \varphi_2 m \varphi_1 = \varphi_2 - \varphi_1.$$

From $\varphi_2 m \varphi_1 = \varphi_1$, we have $\varphi_1 = (\varphi_1 + (\varphi_2 - \varphi_1))m\varphi_1 = \varphi_1 m \varphi_1 + (\varphi_2 - \varphi_1)m\varphi_1$. It follows

$$\varphi_1(1_S - m\varphi_1) = (\varphi_2 - \varphi_1)m\varphi_1 \in \varphi_1 S \cap (\varphi_2 - \varphi_1)S.$$

Since $\varphi_1 S \cap (\varphi_2 - \varphi_1)S = \{0\}$, we obtain $\varphi_1 m \varphi_1 = \varphi_1$. On the other hand, from $\varphi_2 m(\varphi_2 - \varphi_1) = \varphi_2 - \varphi_1$, we have $\varphi_2 - \varphi_1 = (\varphi_1 + (\varphi_2 - \varphi_1))m(\varphi_2 - \varphi_1) = \varphi_1 m(\varphi_2 - \varphi_1) + (\varphi_2 - \varphi_1)m(\varphi_2 - \varphi_1)$. This yields

$$\varphi_1 m(\varphi_2 - \varphi_1) = (\varphi_2 - \varphi_1)(1_S - m(\varphi_2 - \varphi_1)) \in \varphi_1 S \cap (\varphi_2 - \varphi_1)S = \{0\}.$$

Thus $\varphi_1 m(\varphi_2 - \varphi_1) = 0$. So we have $\varphi_1 m \varphi_2 = \varphi_1 m \varphi_1 = \varphi_1$. Therefore $\varphi_1 \in R\varphi_2$, and so $R\varphi_1 + R(\varphi_2 - \varphi_1) \subseteq R\varphi_2$. The reverse inclusion is obvious. Then $R\varphi_2 = R\varphi_1 + R(\varphi_2 - \varphi_1)$. Let $r_1\varphi_1 = r_2(\varphi_2 - \varphi_1) \in R\varphi_1 \cap R(\varphi_2 - \varphi_1)$ for some $r_1, r_2 \in R$. Multiplying the equality by $m\varphi_1$ from the right, we obtain $r_1\varphi_1 = r_1\varphi_1 m \varphi_1 = r_2(\varphi_2 - \varphi_1)m\varphi_1 = 0$. Therefore $R\varphi_2 = R\varphi_1 \oplus R(\varphi_2 - \varphi_1)$.
(2) $\Rightarrow$ (1) Similar to the proof of (1) $\Rightarrow$ (2).

THEOREM 2.8. *Let $M$ be a module and $\varphi_1, \varphi_2 \in M^*$ with $H_{\varphi_2} \neq \emptyset$. Then the following statements are equivalent.*

(1) $\varphi_1 \leqslant^{\oplus} \varphi_2$.

(2) $\varphi_1 S \cap (\varphi_2 - \varphi_1)S = R\varphi_1 \cap R(\varphi_2 - \varphi_1) = \{0\}$.

*Proof.* (1) $\Rightarrow$ (2) Clear by the definition of $\leqslant^{\oplus}$ and Theorem 2.7.
(2) $\Rightarrow$ (1) By hypothesis, $\varphi_2 = \varphi_2 m \varphi_2$ for some $m \in M$. Then

$$\varphi_2 m(\varphi_1 + (\varphi_2 - \varphi_1)) = \varphi_1 + (\varphi_2 - \varphi_1).$$

Hence $\varphi_2 m \varphi_1 - \varphi_1 = (\varphi_2 - \varphi_1) - \varphi_2 m(\varphi_2 - \varphi_1)$, and so

$$(\varphi_2 m - 1_R)\varphi_1 = (1_R - \varphi_2 m)(\varphi_2 - \varphi_1) \in R\varphi_1 \cap R(\varphi_2 - \varphi_1).$$

By (2), $\varphi_2 m \varphi_1 = \varphi_1$. It follows $\varphi_1 \in \varphi_2 S$. This yields $\varphi_1 S + (\varphi_2 - \varphi_1)S \subseteq \varphi_2 S$. The reverse inclusion is obvious. Also by (2), $\varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1)S$. Therefore $\varphi_1 \leqslant^{\oplus} \varphi_2$.

We now prove that the relation $\leqslant^{\oplus}$ is a partial order on a dual module $M^*$ when the sets $H_\varphi$ are nonempty for any linear functional $\varphi \in M^*$.

THEOREM 2.9. *Let $M$ be a module. If $H_\varphi \neq \emptyset$ for any $\varphi \in M^*$, then the relation $\leqslant^{\oplus}$ is a partial order on $M^*$.*

*Proof.* *Reflexivity*: Let $\varphi \in M^*$. From $\varphi S = \varphi S \oplus 0$, we have $\varphi \leqslant^{\oplus} \varphi$.
*Antisymmetry:* Let $\varphi_1, \varphi_2 \in M^*$ with $\varphi_1 \leqslant^{\oplus} \varphi_2$ and $\varphi_2 \leqslant^{\oplus} \varphi_1$. Then $\varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1)S$ and $\varphi_1 S = \varphi_2 S \oplus (\varphi_1 - \varphi_2)S$. So $\varphi_1 S \subseteq \varphi_2 S$ and $\varphi_2 S \subseteq \varphi_1 S$. Hence $\varphi_1 S = \varphi_2 S$. The decomposition of $\varphi_1 S$ implies $(\varphi_1 - \varphi_2)S = 0$. Since $S$ has an identity, $\varphi_1 = \varphi_2$.
*Transitivity:* Let $\varphi_1, \varphi_2, \varphi_3 \in M^*$ with $\varphi_1 \leqslant^{\oplus} \varphi_2$ and $\varphi_2 \leqslant^{\oplus} \varphi_3$. Then $\varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1)S$ and $\varphi_3 S = \varphi_2 S \oplus (\varphi_3 - \varphi_2)S$. In the light of these decompositions, $\varphi_1 S \subseteq \varphi_2 S \subseteq \varphi_3 S$ and $\varphi_1 S \cap (\varphi_2 - \varphi_1)S = \varphi_2 S \cap (\varphi_3 - \varphi_2)S = \{0\}$. We claim that $\varphi_1 S \cap (\varphi_3 - \varphi_1)S = \{0\}$. Let $\varphi_1 f = (\varphi_3 - \varphi_1)g$ for some $f, g \in S$. So $\varphi_1 f = (\varphi_3 - \varphi_2)g + (\varphi_2 - \varphi_1)g$. It follows $\varphi_1 f - (\varphi_2 - \varphi_1)g = (\varphi_3 - \varphi_2)g \in \varphi_2 S \cap (\varphi_3 - \varphi_2)S = \{0\}$. This shows $\varphi_1 f = (\varphi_2 - \varphi_1)g \in \varphi_1 S \cap (\varphi_2 - \varphi_1)S = \{0\}$. Thus $\varphi_1 f = 0$, and so $\varphi_1 S \cap (\varphi_3 - \varphi_1)S = \{0\}$ as claimed. On the other hand, by Theorem 2.7, $R\varphi_2 = R\varphi_1 \oplus R(\varphi_2 - \varphi_1)$ and $R\varphi_3 = R\varphi_2 \oplus R(\varphi_3 - \varphi_2)$. Similarly, these decompositions yield $R\varphi_1 \cap R(\varphi_3 - \varphi_1) = \{0\}$. Therefore $\varphi_1 \leqslant^{\oplus} \varphi_3$ by Theorem 2.8.

Let $M$ be a module and $m_1, m_2 \in M$. We write

$$m_1 \leqslant^{\oplus} m_2 \quad if \quad m_2 R = m_1 R \oplus (m_2 - m_1)R.$$

By a similar discussion in the proof of Theorem 2.9, the relation $\leqslant^{\oplus}$ is a partial order on a regular module. In the following, we obtain a connection between direct sum partial orders for a module and its dual module.

THEOREM 2.10. *Let $M$ be a module, $m_1, m_2 \in M$ with $m_2$ regular. If $m_1 \leqslant^{\oplus} m_2$, then for every $\varphi_2 \in \{m_2^{(1,2)}\}$, there exists $\varphi_1 \in \{m_1^{(1,2)}\}$ such that $\varphi_1 \leqslant^{\oplus} \varphi_2$.*

*Proof.* Let $m_1 \leqslant^{\oplus} m_2$. Since $m_2$ is regular, $\{m_2^{(1)}\} \neq \emptyset$. This implies $\{m_2^{(1,2)}\} \neq \emptyset$ by the fact that if $\varphi \in \{m_2^{(1)}\}$, then $\varphi m_2 \varphi \in \{m_2^{(1,2)}\}$. Let $\varphi_2 \in \{m_2^{(1,2)}\}$. Define $\varphi_1 = \varphi_2 m_1 \varphi_2 \in M^*$. We firstly claim that $\varphi_1 \in \{m_1^{(1,2)}\}$. Since $m_1 \leqslant^{\oplus} m_2$, $m_1 R \subseteq m_2 R$, and so $m_1 = m_2 r$ for some $r \in R$. Then $m_1 = m_2 r = m_2 \varphi_2 m_2 r = m_2 \varphi_2 m_1$. Hence $m_1 \varphi_2 m_1 = m_2 \varphi_2 m_1 - (m_2 - m_1)\varphi_2 m_1 = m_1 - (m_2 - m_1)\varphi_2 m_1$. This implies $m_1(\varphi_2 m_1 - 1_R) = -(m_2 - m_1)\varphi_2 m_1 \in m_1 R \cap (m_2 - m_1)R$. Being $m_1 \leqslant^{\oplus} m_2$ yields $m_1 R \cap (m_2 - m_1)R = \{0\}$, and so $m_1 = m_1 \varphi_2 m_1$. On the one hand, $m_1 \varphi_1 m_1 = m_1 \varphi_2 m_1 \varphi_2 m_1 = m_1 \varphi_2 m_1 = m_1$. This means $\varphi_1 \in \{m_1^{(1)}\}$. On the other hand, using $m_1 = m_1 \varphi_2 m_1$, we have $\varphi_1 m_1 \varphi_1 = \varphi_1$ entailing that $\varphi_1 \in \{m_1^{(2)}\}$. Thus $\varphi_1 \in \{m_1^{(1,2)}\}$ as claimed. Note that $m_1 \varphi_1 = m_1 \varphi_2$ and $\varphi_1 m_1 = \varphi_2 m_1$ by the fact $m_1 = m_1 \varphi_2 m_1$. We now show that $\varphi_1 \leqslant^{\oplus} \varphi_2$. On the one hand, let $\varphi_1 s_1 = (\varphi_2 - \varphi_1)s_2 \in \varphi_1 S \cap (\varphi_2 - \varphi_1)S$ for some $s_1, s_2 \in S$. Then $m_1 \varphi_1 s_1 = m_1 \varphi_2 s_2 - m_1 \varphi_1 s_2$. Since $m_1 \varphi_1 = m_1 \varphi_2$, we have $m_1 \varphi_1 s_1 = 0$ entailing $\varphi_1 m_1 \varphi_1 s_1 = 0$, and so $\varphi_1 s_1 = 0$. This implies $\varphi_1 S \cap (\varphi_2 - \varphi_1)S = \{0\}$. On the other hand, let $r_1 \varphi_1 = r_2(\varphi_2 - \varphi_1) \in R\varphi_1 \cap R(\varphi_2 - \varphi_1)$ for some $r_1, r_2 \in R$. Similarly, we obtain $r_1 \varphi_1 m_1 = 0$ using $\varphi_1 m_1 = \varphi_2 m_1$. This entails $r_1 \varphi_1 m_1 \varphi_1 = 0$.

Then $r_1\varphi_1 = 0$. It follows $R\varphi_1 \cap R(\varphi_2 - \varphi_1) = \{0\}$. Therefore $\varphi_1 \leqslant^{\oplus} \varphi_2$ by Theorem 2.8.

We now give more characterizations of the direct sum order for linear functionals as can be seen below.

PROPOSITION 2.11. *Let $M$ be a module and $\varphi_1, \varphi_2 \in M^*$. Then the following are equivalent.*

(1) $\varphi_1 \leqslant^{\oplus} \varphi_2$ *and* $H_{\varphi_2} \neq \emptyset$.

(2) $H_{\varphi_1} \cap H_{\varphi_2} \neq \emptyset$, $R\varphi_1 \subseteq R\varphi_2$ *and* $\varphi_1 S \subseteq \varphi_2 S$.

*Proof.* (1) $\Rightarrow$ (2) Let $\varphi_1 \leqslant^{\oplus} \varphi_2$ and $H_{\varphi_2} \neq \emptyset$. Then, by Definition 2.6, $\varphi_1 S \subseteq \varphi_2 S$, so $\varphi_1 = \varphi_2 f$ for some $f \in S$. Also, $R\varphi_1 \subseteq R\varphi_2$ by Theorem 2.7. Since $H_{\varphi_2} \neq \emptyset$, $\varphi_2 = \varphi_2 m \varphi_2$ for some $m \in M$. Hence $\varphi_1 = \varphi_2 f = \varphi_2 m \varphi_2 f = \varphi_2 m \varphi_1$. Thus

$$\varphi_1 m \varphi_1 = \varphi_2 m \varphi_1 - (\varphi_2 - \varphi_1) m \varphi_1 = \varphi_1 - (\varphi_2 - \varphi_1) m \varphi_1.$$

It follows

$$\varphi_1 (m\varphi_1 - 1_S) = \varphi_1 m \varphi_1 - \varphi_1 = -(\varphi_2 - \varphi_1) m \varphi_1 \in \varphi_1 S \cap (\varphi_2 - \varphi_1) S.$$

Since $\varphi_1 S \cap (\varphi_2 - \varphi_1) S = \{0\}$, we have $\varphi_1 = \varphi_1 m \varphi_1$. This means $m \in H_{\varphi_1}$. Therefore $m \in H_{\varphi_1} \cap H_{\varphi_2}$.

(2) $\Rightarrow$ (1) Let $H_{\varphi_1} \cap H_{\varphi_2} \neq \emptyset$, $R\varphi_1 \subseteq R\varphi_2$ and $\varphi_1 S \subseteq \varphi_2 S$. Being $H_{\varphi_1} \cap H_{\varphi_2} \subseteq H_{\varphi_2}$ yields $H_{\varphi_2} \neq \emptyset$. Since $\varphi_1 S \subseteq \varphi_2 S$, clearly, $\varphi_1 S + (\varphi_2 - \varphi_1) S \subseteq \varphi_2 S$. The reverse inclusion is obvious. So $\varphi_2 S = \varphi_1 S + (\varphi_2 - \varphi_1) S$. In order to see that this sum is direct, let $\varphi_1 f = (\varphi_2 - \varphi_1) g \in \varphi_1 S \cap (\varphi_2 - \varphi_1) S$ for some $f, g \in S$. By (2), there exists $m \in H_{\varphi_1} \cap H_{\varphi_2}$. Multiplying the equality by $\varphi_1 m$ from the left, we have

$$\begin{aligned} \varphi_1 m \varphi_1 f &= \varphi_1 m (\varphi_2 - \varphi_1) g \\ \varphi_1 f &= \varphi_1 m \varphi_2 g - \varphi_1 m \varphi_1 g \\ &= \varphi_1 m \varphi_2 g - \varphi_1 g. \end{aligned}$$

Since $R\varphi_1 \subseteq R\varphi_2$, $\varphi_1 = r\varphi_2$ for some $r \in R$. Then $\varphi_1 f = r\varphi_2 m \varphi_2 g - \varphi_1 g = r\varphi_2 g - \varphi_1 g = \varphi_1 g - \varphi_1 g = 0$. Hence $\varphi_1 S \cap (\varphi_2 - \varphi_1) S = \{0\}$. Thus $\varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1) S$. Therefore $\varphi_1 \leqslant^{\oplus} \varphi_2$.

THEOREM 2.12. *Let $M$ be a module and $\varphi_1, \varphi_2 \in M^*$ with $H_{\varphi_2} \neq \emptyset$. Then the following are equivalent.*

(1) $\varphi_1 \leqslant^{\oplus} \varphi_2$.

(2) *For each* $m_2 \in H_{\varphi_2}$, $\varphi_1 = \varphi_1 m_2 \varphi_2 = \varphi_2 m_2 \varphi_1 = \varphi_1 m_2 \varphi_1$.

(3) *For each* $m_2 \in H_{\varphi_2}$, *there exists* $m_1 \in H_{\varphi_1}$ *such that* $\varphi_1 m_1 = \varphi_2 m_1 = \varphi_1 m_2$ *and* $m_1 \varphi_1 = m_1 \varphi_2 = m_2 \varphi_1$.

(4) *There exists* $m \in H_{\varphi_1}$ *such that* $\varphi_1 m = \varphi_2 m$ *and* $m\varphi_1 = m\varphi_2$.

*Proof.* (1) $\Rightarrow$ (2) Assume that $\varphi_1 \leqslant^{\oplus} \varphi_2$. This means $\varphi_2 S = \varphi_1 S \oplus (\varphi_2 - \varphi_1)S$, and so $\varphi_1 S \subseteq \varphi_2 S$. Then $\varphi_1 = \varphi_2 f$ for some $f \in S$. Let $m_2 \in H_{\varphi_2}$. Hence $\varphi_1 = \varphi_2 f = \varphi_2 m_2 \varphi_2 f = \varphi_2 m_2 \varphi_1$. This yields $\varphi_1 m_2 \varphi_1 = \varphi_2 m_2 \varphi_1 - (\varphi_2 - \varphi_1) m_2 \varphi_1 = \varphi_1 - (\varphi_2 - \varphi_1) m_2 \varphi_1$, and so

$$\varphi_1 m_2 \varphi_1 - \varphi_1 = -(\varphi_2 - \varphi_1) m_2 \varphi_1 \in \varphi_1 S \cap (\varphi_2 - \varphi_1) S = \{0\}$$

entailing $\varphi_1 = \varphi_1 m_2 \varphi_1$. Note that $\varphi_2 - \varphi_1 = \varphi_2 m_2 (\varphi_2 - \varphi_1)$. Then

$$(\varphi_2 - \varphi_1) m_2 (\varphi_2 - \varphi_1) = (\varphi_2 - \varphi_1) - \varphi_1 m_2 (\varphi_2 - \varphi_1).$$

Hence $(\varphi_2 - \varphi_1)(m_2(\varphi_2 - \varphi_1) - 1_S) = -\varphi_1 m_2 (\varphi_2 - \varphi_1) \in \varphi_1 S \cap (\varphi_2 - \varphi_1) S = \{0\}$. It follows $\varphi_1 m_2 \varphi_2 = \varphi_1 m_2 \varphi_1$ entailing $\varphi_1 = \varphi_1 m_2 \varphi_2$.

(2) $\Rightarrow$ (3) Let $m_2 \in H_{\varphi_2}$ and consider the element $m_1 = m_2 \varphi_1 m_2 \in M$. Using $\varphi_1 = \varphi_1 m_2 \varphi_1$, we have $\varphi_1 m_1 \varphi_1 = \varphi_1$. This means $m_1 \in H_{\varphi_1}$. Since $\varphi_1 = \varphi_1 m_2 \varphi_1$ and $\varphi_1 = \varphi_2 m_2 \varphi_1$, we obtain $\varphi_1 m_1 = \varphi_1 m_2$ and $\varphi_2 m_1 = \varphi_1 m_2$, respectively. Hence $\varphi_1 m_1 = \varphi_2 m_1 = \varphi_1 m_2$. Similarly, being $\varphi_1 = \varphi_1 m_2 \varphi_1$ and $\varphi_1 = \varphi_1 m_2 \varphi_2$ imply $m_1 \varphi_1 = m_2 \varphi_1$ and $m_1 \varphi_2 = m_2 \varphi_1$, respectively. Thus $m_1 \varphi_1 = m_1 \varphi_2 = m_2 \varphi_1$.

(3) $\Rightarrow$ (4) Since $H_{\varphi_2} \neq \emptyset$, assume that $m_2 \in H_{\varphi_2}$. By (3), there exists $m_1 \in H_{\varphi_1}$ such that $\varphi_1 m_1 = \varphi_2 m_1$ and $m_1 \varphi_1 = m_1 \varphi_2$.

(4) $\Rightarrow$ (1) Since $H_{\varphi_2} \neq \emptyset$, let $m_2 \in H_{\varphi_2}$. Then by (4), there exists $m_1 \in H_{\varphi_1}$ with $\varphi_1 m_1 = \varphi_2 m_1$ and $m_1 \varphi_1 = m_1 \varphi_2$. Hence

$$\begin{aligned} \varphi_1 m_2 \varphi_1 &= \varphi_1 m_1 \varphi_1 m_2 \varphi_1 m_1 \varphi_1 \\ &= \varphi_1 m_1 \varphi_2 m_2 \varphi_2 m_1 \varphi_1 \\ &= \varphi_1 m_1 \varphi_2 m_1 \varphi_1 \\ &= \varphi_1 m_1 \varphi_1 m_1 \varphi_1 \\ &= \varphi_1 m_1 \varphi_1 \\ &= \varphi_1. \end{aligned}$$

This yields $m_2 \in H_{\varphi_1} \cap H_{\varphi_2}$, and so $H_{\varphi_1} \cap H_{\varphi_2} \neq \emptyset$. On the one hand, $\varphi_1 = \varphi_1 m_1 \varphi_1 = \varphi_1 m_1 \varphi_2 \in R\varphi_2$ entails $R\varphi_1 \subseteq R\varphi_2$. On the other hand, $\varphi_1 = \varphi_1 m_1 \varphi_1 = \varphi_2 m_1 \varphi_1 \in \varphi_2 S$ yields $\varphi_1 S \subseteq \varphi_2 S$. By Proposition 2.11, $\varphi_1 \leqslant^{\oplus} \varphi_2$.

PROPOSITION 2.13. *Let $M$ be a module, $\varphi_1, \varphi_2 \in M^*$ with $H_{\varphi_2} \neq \emptyset$. If $\varphi_1 \leqslant^{\oplus} \varphi_2$, then $H_{\varphi_2} \subseteq H_{\varphi_1}$. The converse holds if $H_{(1_R - \varphi_2 m)\varphi_1} \neq \emptyset$ and $H_{\varphi_1(1_S - m\varphi_2)} \neq \emptyset$ for some $m \in H_{\varphi_2}$.*

*Proof.* Let $\varphi_1 \leqslant^{\oplus} \varphi_2$. By Proposition 2.11, there exist $f \in S$ and $r \in R$ such that $\varphi_1 = \varphi_2 f = r\varphi_2$, also $m \in H_{\varphi_1} \cap H_{\varphi_2}$. Let $x \in H_{\varphi_2}$. Then we have $\varphi_1 = \varphi_1 m \varphi_1$, $\varphi_2 = \varphi_2 m \varphi_2 = \varphi_2 x \varphi_2$. Hence

$$\varphi_1 x \varphi_1 = r\varphi_2 x \varphi_2 f = r\varphi_2 f = r\varphi_1$$

and

$$\varphi_1 = \varphi_1 m\varphi_1 = r\varphi_2 m\varphi_2 f = r\varphi_2 f = r\varphi_1.$$

Thus $\varphi_1 = \varphi_1 x\varphi_1$, and so $x \in H_{\varphi_1}$. Therefore $H_{\varphi_2} \subseteq H_{\varphi_1}$.

Conversely, let $m \in H_{\varphi_2}$ with $H_{(1_R-\varphi_2 m)\varphi_1} \neq \emptyset$ and $H_{\varphi_1(1_S-m\varphi_2)} \neq \emptyset$. By Lemma 2.5, $H_{\varphi_2} = m+(1_S-m\varphi_2)M+M(1_R-\varphi_2 m)$. Then for every $m_1,m_2 \in M$, $n := m+(1_S-m\varphi_2)m_1+m_2(1_R-\varphi_2 m) \in H_{\varphi_2}$. By (2), $m,n \in H_{\varphi_1}$. Multiplying this equality by $\varphi_1$ from the left and the right, we obtain

$$\varphi_1(1_S-m\varphi_2)m_1\varphi_1+\varphi_1 m_2(1_R-\varphi_2 m)\varphi_1 = 0$$

for every $m_1,m_2 \in M$, i.e., $\varphi_1(1_S-m\varphi_2)M\varphi_1+\varphi_1 M(1_R-\varphi_2 m)\varphi_1 = \{0\}$. It follows that $\varphi_1(1_S-m\varphi_2)M\varphi_1 = \{0\}$ and $\varphi_1 M(1_R-\varphi_2 m)\varphi_1 = \{0\}$. Hence $\varphi_1(1_S-m\varphi_2)M\varphi_1(1_S-m\varphi_2) = \{0\}$ and $(1_R-\varphi_2 m)\varphi_1 M(1_R-\varphi_2 m)\varphi_1 = \{0\}$. By assumption, there exist $x \in H_{(1_R-\varphi_2 m)\varphi_1}$ and $y \in H_{\varphi_1(1_S-m\varphi_2)}$. These mean $(1_R-\varphi_2 m)\varphi_1 x(1_R-\varphi_2 m)\varphi_1 = (1_R-\varphi_2 m)\varphi_1$ and $\varphi_1(1_S-m\varphi_2)y\varphi_1(1_S-m\varphi_2) = \varphi_1(1_S-m\varphi_2)$. Thus $(1_R-\varphi_2 m)\varphi_1 = 0$ and $\varphi_1(1_S-m\varphi_2) = 0$. This yields $\varphi_1 = \varphi_2 m\varphi_1 = \varphi_1 m\varphi_2$. So $\varphi_1 S \subseteq \varphi_2 S$ and $R\varphi_1 \subseteq R\varphi_2$. Therefore $\varphi_1 \leqslant^{\oplus} \varphi_2$ by Proposition 2.11.

In the next result, we characterize weak regular supports of a regular element in a module in terms of the direct sum order on the dual module.

PROPOSITION 2.14. *Let $M$ be a module, $m$ be a regular element of $M$ and $\varphi_1 \in M^*$. Then the following are equivalent.*

(1) $\varphi_1 \in \{m^{(2)}\}$.

(2) *There exists $\varphi \in \{m^{(1,2)}\}$ such that $\varphi_1 \leqslant^{\oplus} \varphi$.*

*Proof.* (1) $\Rightarrow$ (2) Let $\varphi_1 \in \{m^{(2)}\}$. This means that $\varphi_1 m\varphi_1 = \varphi_1$. Since $m \in M$ is regular, there exists $\varphi_2 \in \{m^{(1)}\}$. Then $m\varphi_2 m = m$. Set

$$\varphi = \varphi_1 + \varphi_2(m - m\varphi_1 m)\varphi_2.$$

Then

$$m\varphi m = m\varphi_1 m + m\varphi_2 m\varphi_2 m + m\varphi_2 m\varphi_1 m\varphi_2 m = m$$

since $m\varphi_2 m = m$. Similarly, by making use of $\varphi_1 m\varphi_1 = \varphi_1$ and $m\varphi_2 m = m$, it is easily checked that $\varphi m\varphi = \varphi$. Hence $\varphi \in \{m^{(1,2)}\}$. Next we show $\varphi_1 \leqslant^{\oplus} \varphi$. We use $(\varphi_1 m)^2 = \varphi_1 m$ and $m\varphi_2 m = m$ to have the following equalities:

$$\varphi m = \varphi_1 m + \varphi_2(m - m\varphi_1 m)\varphi_2 m = \varphi_1 m + \varphi_2 m - \varphi_2 m\varphi_1 m,$$

$$(\varphi m)(\varphi_1 m) = \varphi_1 m\varphi_1 m + \varphi_2 m\varphi_1 m - \varphi_2 m\varphi_1 m\varphi_1 m = \varphi_1 m,$$

$$(\varphi_1 m)(\varphi m) = \varphi_1 m\varphi_1 m + \varphi_1 m\varphi_2 m - \varphi_1 m\varphi_2 m\varphi_1 m = \varphi_1 m.$$

Multiplying $\varphi m\varphi_1 m = \varphi_1 m$ by $\varphi_1$ from the right, we have $\varphi m\varphi_1 = \varphi_1$. This implies $\varphi_1 S \subseteq \varphi S$. It follows that $\varphi S = \varphi_1 S + (\varphi - \varphi_1)S$. Also,

$$\begin{aligned}\varphi_1 m\varphi &= \varphi_1 m(\varphi_1 + \varphi_2(m - m\varphi_1 m)\varphi_2)\\ &= \varphi_1 m\varphi_1 + \varphi_1 m\varphi_2 m\varphi_2 - \varphi_1 m\varphi_2 m\varphi_1 m\varphi_2\\ &= \varphi_1 + \varphi_1 m\varphi_2 - \varphi_1 m\varphi_2\\ &= \varphi_1.\end{aligned}$$

To prove $\varphi_1 S \cap (\varphi - \varphi_1)S = \{0\}$, let $\varphi_1 f = (\varphi - \varphi_1)g \in \varphi_1 S \cap (\varphi - \varphi_1)S$ where $f, g \in S$. Multiplying the latter from the left by $\varphi_1 m$, we have $\varphi_1 m\varphi_1 f = (\varphi_1 m\varphi - \varphi_1 m\varphi_1)g$. By making use of $\varphi_1 m\varphi_1 = \varphi_1$ and $\varphi_1 m\varphi = \varphi_1$, we have $\varphi_1 f = 0$. Hence $\varphi_1 S \cap (\varphi - \varphi_1)S = \{0\}$. Therefore $\varphi_1 \leqslant^{\oplus} \varphi$.
(2) $\Rightarrow$ (1) Assume that $\varphi \in \{m^{(1,2)}\}$ with $\varphi_1 \leqslant^{\oplus} \varphi$. Since $\varphi = \varphi m\varphi$, $m \in H_\varphi$. So $H_\varphi \neq \emptyset$. By Proposition 2.13, $\varphi_1 \leqslant^{\oplus} \varphi$ implies $H_\varphi \subseteq H_{\varphi_1}$. Since $m \in H_\varphi$, $m \in H_{\varphi_1}$. Therefore $\varphi_1 m\varphi_1 = \varphi_1$, that is, $\varphi_1 \in \{m^{(2)}\}$. This completes the proof.

Let $M$ be a module, $m \in M$ be regular and $\varphi \in \{m^{(1,2)}\}$. Set $e = \varphi m \in R$ and $f = m\varphi \in S$. Then $e^2 = e$, $f^2 = f$ and $m \in fMe$. Consider the subset $eM^*f$ of $M^*$. Set $\mathscr{C}_\varphi = \{\beta \in eM^*f \mid \beta \leqslant^{\oplus} \varphi\}$. We define a maximal element in $\mathscr{C}_\varphi$ as follows: $\alpha \in \mathscr{C}_\varphi$ is maximal in $\mathscr{C}_\varphi$ if $\alpha \leqslant^{\oplus} \beta \leqslant^{\oplus} \varphi$ implies $\alpha = \beta$ or $\beta = \varphi$. In the sequel, we investigate the maximal elements in $eM^*f$.

LEMMA 2.15. *Let $M$ be a module, $e^2 = e \in R$, $f^2 = f \in S$ and $m \in fMe$ be regular. If $\varphi_1 \in \{m^{(2)}\}$ and $\varphi_2 \in \{m^{(1,2)}\}$ such that $\varphi_1 \leqslant^{\oplus} \varphi_2$, then $e\varphi_1 f \leqslant^{\oplus} e\varphi_2 f$.*

*Proof.* By definition, $\varphi_2 \in \{m^{(1,2)}\}$ implies $m \in H_{\varphi_2}$, so $H_{\varphi_2} \neq \emptyset$. By Proposition 2.11, being $\varphi_1 \leqslant^{\oplus} \varphi_2$ implies $R\varphi_1 \subseteq R\varphi_2$ and $\varphi_1 S \subseteq \varphi_2 S$. Then $\varphi_1 = \varphi_2 g = r\varphi_2$ for some $g \in S$ and $r \in R$. Hence

$$\varphi_1 m\varphi_2 = r\varphi_2 m\varphi_2 = r\varphi_2 = \varphi_1$$

and

$$\varphi_2 m\varphi_1 = \varphi_2 m\varphi_2 g = \varphi_2 g = \varphi_1.$$

Note that $m = fme$. On the one hand,

$$(e\varphi_1 f)m(e\varphi_2 f) = e\varphi_1(fme)\varphi_2 f = e\varphi_1 m\varphi_2 f = e\varphi_1 f.$$

This yields that $R(e\varphi_1 f) \subseteq R(e\varphi_2 f)$ since $e\varphi_1 fm \in R$. On the other hand,

$$(e\varphi_2 f)m(e\varphi_1 f) = e\varphi_2(fme)\varphi_1 f = e\varphi_2 m\varphi_1 f = e\varphi_1 f.$$

It follows that $(e\varphi_1 f)S \subseteq (e\varphi_2 f)S$ since $me\varphi_1 f \in S$. Since $\varphi_1 \in \{m^{(2)}\}$, we obtain $(e\varphi_1 f)m(e\varphi_1 f) = e\varphi_1 m\varphi_1 f = e\varphi_1 f$, this implies $m \in H_{e\varphi_1 f}$. Also, being $\varphi_2 \in \{m^{(2)}\}$ entails $(e\varphi_2 f)m(e\varphi_2 f) = e\varphi_2 m\varphi_2 f = e\varphi_2 f$, and this implies $m \in H_{e\varphi_2 f}$.
Hence $m \in H_{e\varphi_1 f} \cap H_{e\varphi_2 f}$ and so $H_{e\varphi_1 f} \cap H_{e\varphi_2 f} \neq \emptyset$. By Proposition 2.11, we get $e\varphi_1 f \leqslant^{\oplus} e\varphi_2 f$. This completes the proof.

REMARK 2.16. Let $M$ be a module, $m \in M$ and $\varphi_1 \in \{m^{(2)}\}$ and $\varphi_2 \in \{m^{(1,2)}\}$ such that $\varphi_1 \leqslant^{\oplus} \varphi_2$. Set $e_1 = \varphi_1 m$, $f_1 = m\varphi_1$ and $e_2 = \varphi_2 m$, $f_2 = m\varphi_2$. Being $\varphi_1 \leqslant^{\oplus} \varphi_2$ implies $R\varphi_1 \subseteq R\varphi_2$ and $\varphi_1 S \subseteq \varphi_2 S$.

Let $R$ be a ring and $e^2 = e$, $f^2 = f \in R$. Kaplansky defines $e \leqslant f$ if $e = ef = fe$.

LEMMA 2.17. *By the notation as given in Remark 2.16,* $\varphi_1 \leqslant^{\oplus} \varphi_2$ *implies* $e_1 \leqslant e_2$ *in* $R$ *and* $f_1 \leqslant f_2$ *in* $S$.

*Proof.* Assume that $\varphi_1 \leqslant^{\oplus} \varphi_2$. Then $R\varphi_1 \subseteq R\varphi_2$ and $\varphi_1 S \subseteq \varphi_2 S$. Multiplying $R\varphi_1 \subseteq R\varphi_2$ from the right by $m$, we get $Re_1 \subseteq Re_2$. Hence $e_1 = e_1 e_2$. Since $\varphi_1 S \subseteq \varphi_2 S$, $\varphi_1 = \varphi_2 g$ for some $g \in S$. Thus $e_2 e_1 = \varphi_2 m \varphi_1 m = \varphi_2 m \varphi_2 g m = \varphi_2 g m = \varphi_1 m = e_1$. Therefore $e_1 \leqslant e_2$ in $R$. Similarly, multiplying $\varphi_1 S \subseteq \varphi_2 S$ from the left by $m$, we get $f_1 S \subseteq f_2 S$. This implies $f_1 = f_2 f_1$. Being $R\varphi_1 \subseteq R\varphi_2$ entails $\varphi_1 = r\varphi_2$ where $r \in R$, so $f_1 f_2 = m\varphi_1 m \varphi_2 = m r \varphi_2 m \varphi_2 = m r \varphi_2 = m\varphi_1 = f_1$. Therefore $f_1 \leqslant f_2$.

Let $M$ be a module, $m \in M$ be regular and $\varphi \in \{m^{(1,2)}\}$. Set $e = \varphi m \in R$ and $f = m\varphi \in S$. In the following, we determine the maximal elements in the subset $eM^*f$.

THEOREM 2.18. *Let* $M$ *be a module,* $m \in M$ *and* $\varphi \in \{m^{(1,2)}\}$. *Then* $\alpha \in \mathscr{C}_\varphi$ *is maximal if and only if for any* $\beta \in M^*$ *with* $\beta \leqslant^{\oplus} \varphi$ *such that* $R\alpha \subseteq R\beta \subseteq M^*f$ *and* $\alpha S \subseteq \beta S \subseteq eM^*$*, we have* $\alpha = \beta$ *or* $\beta = \varphi$.

*Proof.* For the sufficiency, let $\alpha \in \mathscr{C}_\varphi$. Take $\beta \in \mathscr{C}_\varphi$ with $\alpha \leqslant^{\oplus} \beta \leqslant^{\oplus} \varphi$. Note that $\beta = e\beta f$. Since $m \in H_\varphi$ and $\beta \leqslant^{\oplus} \varphi$, by Proposition 2.13, $H_\varphi \subseteq H_\beta$, and so $H_\beta \neq \emptyset$. Being $\alpha \leqslant^{\oplus} \beta$ and Proposition 2.11 imply $\alpha S \subseteq \beta S = e\beta S \subseteq eM^*$ and $R\alpha \subseteq R\beta = R\beta f \subseteq M^*f$. By hypothesis, $\alpha = \beta$ or $\beta = \varphi$. Therefore $\alpha$ is maximal in $\mathscr{C}_\varphi$.

For the necessity, let $\alpha \in \mathscr{C}_\varphi$ be a maximal element in $\mathscr{C}_\varphi$. Let $\beta \in M^*$ such that $\beta \leqslant^{\oplus} \varphi$ and $R\alpha \subseteq R\beta \subseteq M^*f$ and $\alpha S \subseteq \beta S \subseteq eM^*$. Since $e \in R$ and $f \in S$ are idempotent elements, $\beta = \beta f$ and $\beta = e\beta$. So $\beta = e\beta f \in eM^*f$. It implies $\beta \in \mathscr{C}_\varphi$. Since $\alpha \leqslant^{\oplus} \varphi$, $\beta \leqslant^{\oplus} \varphi$ and $H_\varphi \neq \emptyset$, by Proposition 2.13, $H_\varphi \subseteq H_\alpha$ and $H_\varphi \subseteq H_\beta$. This entails $H_\varphi \subseteq H_\alpha \cap H_\beta$, so $H_\alpha \cap H_\beta \neq \emptyset$. By Proposition 2.11, $\alpha \leqslant^{\oplus} \beta$. By the maximality of $\alpha$, we have $\alpha = \beta$ or $\alpha = \varphi$.

REMARK 2.19. Let $M$ be a module, $m \in M$ be regular and $\varphi \in \{m^{(1,2)}\}$. Set $e = \varphi m \in R$ and $f = m\varphi \in S$. Then $e^2 = e$, $f^2 = f$ and $m \in fMe$. Note that $m = fm = me = fme$.

PROPOSITION 2.20. *By the notation as given in Remark 2.19,* $\mathscr{C}_\varphi = eM^*f \cap \{m^{(2)}\}$.

*Proof.* Let $\alpha \in \mathscr{C}_\varphi$. This means $\alpha = e\alpha f$ and $\alpha \leqslant^{\oplus} \varphi$. Since $m \in H_\varphi$, $H_\varphi \neq \emptyset$. By Proposition 2.13, $H_\varphi \subseteq H_\alpha$. It follows $m \in H_\alpha$, i.e., $\alpha = \alpha m \alpha$. This implies $\alpha \in \{m^{(2)}\}$. Then $\alpha \in eM^*f \cap \{m^{(2)}\}$, and so $\mathscr{C}_\varphi \subseteq eM^*f \cap \{m^{(2)}\}$. For the reverse inclusion, let $\alpha \in eM^* f \cap \{m^{(2)}\}$. Since $\alpha \in eM^* f$, it is enough to show $\alpha \leqslant^{\oplus} \varphi$. Note that $m \in H_\alpha \cap H_\varphi$, so $H_\alpha \cap H_\varphi \neq \emptyset$. From $\alpha = e\alpha f$, we have $\alpha = e\alpha = \varphi m\alpha \in \varphi S$ and $\alpha = \alpha f = \alpha m\varphi \in R\varphi$. Then $\alpha S \subseteq \varphi S$ and $R\alpha \subseteq R\varphi$. By Proposition 2.11, $\alpha \leqslant^{\oplus} \varphi$. Hence $\alpha \in \mathscr{C}_\varphi$. Thus $eM^* f \cap \{m^{(2)}\} \subseteq \mathscr{C}_\varphi$. Therefore $\mathscr{C}_\varphi = eM^* f \cap \{m^{(2)}\}$, as asserted.

Let $\max \mathscr{C}_\varphi$ denote the set of all maximal elements of $\mathscr{C}_\varphi$.

THEOREM 2.21. *By the notation as given in Remark 2.19, if $\max \mathscr{C}_\varphi \neq \emptyset$, then*

$$\max \mathscr{C}_\varphi = \{e\beta f \in eM^* f \mid \beta \in \{m^{(1,2)}\}\}.$$

*Proof.* If $\varphi \in eM^* f$, then $\max \mathscr{C}_\varphi = \emptyset$. So we must assume that $\varphi \notin eM^* f$. Let $\alpha \in \max \mathscr{C}_\varphi$. By Proposition 2.20, $\alpha \in \{m^{(2)}\}$. Then there exists $\beta \in \{m^{(1,2)}\}$ such that $\alpha \leqslant^{\oplus} \beta$ according to Proposition 2.14. Also Lemma 2.15 yields $\alpha = e\alpha f \leqslant^{\oplus} e\beta f$. We claim that $e\beta f \leqslant^{\oplus} \varphi$. Note that $(e\beta f)m(e\beta f) = e\beta(fme)\beta f = e\beta m\beta f = e\beta f$. This shows $m \in H_{e\beta f}$, and so $m \in H_\varphi \cap H_{e\beta f}$. Hence $H_\varphi \cap H_{e\beta f} \neq \emptyset$. On the one hand, $e\beta f = \varphi m\beta f \in \varphi S$, so $(e\beta f)S \subseteq \varphi S$. On the other hand, $e\beta f = e\beta m\varphi \in R\varphi$, so $R(e\beta f) \subseteq R\varphi$. By Proposition 2.11, we have $e\beta f \leqslant^{\oplus} \varphi$, as claimed. Being $\alpha \leqslant^{\oplus} e\beta f \leqslant^{\oplus} \varphi$ and maximality of $\alpha$ in $\mathscr{C}_\varphi$ entail $\alpha = e\beta f$ or $e\beta f = \varphi$. The second case cannot be because of $\varphi \notin eM^* f$. Thus $\alpha = e\beta f$. Therefore $\max \mathscr{C}_\varphi \subseteq \{e\beta f \mid \beta \in \{m^{(1,2)}\}\}$.

For the reverse inclusion, let $e\alpha f \in eM^* f$ where $\alpha \in \{m^{(1,2)}\}$. Since

$$(e\alpha f)m(e\alpha f) = e\alpha(fme)\alpha f = e\alpha m\alpha f = e\alpha f,$$

we have $e\alpha f \in \{m^{(2)}\}$. By Proposition 2.20, $e\alpha f \in \mathscr{C}_\varphi$. Now assume that $e\alpha f \leqslant^{\oplus} \beta \leqslant^{\oplus} \varphi$ where $\beta \in \mathscr{C}_\varphi$. Again by Proposition 2.20, $\beta = e\beta f \in \{m^{(2)}\}$. Then $\beta \neq \varphi$ from the fact that $\varphi \notin eM^* f$. We assert that $e\alpha f = e\beta f$. Since $e\alpha f \leqslant^{\oplus} e\beta f$, we have the decomposition $e\beta fS = e\alpha fS \oplus (e\beta f - e\alpha f)S$. Also, by Proposition 2.13, $e\beta f \leqslant^{\oplus} \varphi$ implies $H_\varphi \subseteq H_{e\beta f}$. Then $m \in H_{e\beta f}$, i.e., $(e\beta f)m(e\beta f) = e\beta f$. On the other hand, the inclusions $e\alpha fS \subseteq e\beta fS$ and $Re\alpha f \subseteq Re\beta f$ yield $e\alpha f = e\beta fg = re\beta f$ for some $g \in S$ and $r \in R$. Hence

$$(e\beta f)m(e\alpha f) = e\beta(fme)\beta fg = e\beta m\beta fg = e\beta fg = e\alpha f.$$

Since $f = m\varphi$, $\alpha \in \{m^{(1)}\}$ and $e\alpha f = (e\beta f)m(e\alpha f)$, we have

$$\begin{aligned} e\beta f - e\alpha f &= e\beta f - (e\beta f)m(e\alpha f) \\ &= e\beta f - e\beta m\alpha f \\ &= e\beta f - e\beta m\alpha m\varphi \\ &= e\beta f - e\beta m\varphi \\ &= e\beta f - e\beta f \\ &= 0. \end{aligned}$$

Thus $e\alpha f = e\beta f$, as asserted. This means $e\alpha f \in \max \mathscr{C}_\varphi$, so $\{e\beta f \mid \beta \in \{m^{(1,2)}\}\} \subseteq \max \mathscr{C}_\varphi$. Therefore $\max \mathscr{C}_\varphi = \{e\beta f \mid \beta \in \{m^{(1,2)}\}\}$.

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