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LINEAR MAPS ON BLOCK UPPER TRIANGULAR MATRIX ALGEBRAS BEHAVING LIKE JORDAN DERIVATIONS THROUGH COMMUTATIVE ZERO PRODUCTS

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Abstract. Let $\mathscr{T}=\mathscr{T}(n_1,n_2,\cdots,n_k)\subseteq M_n(\mathscr{C})$ be a block upper triangular matrix algebra and let $\mathscr{M}$ be a 2-torsion free unital $\mathscr{T}$-bimodule, where $\mathscr{C}$ is a commutative ring. Let $\Delta:\mathscr{T}\to\mathscr{M}$ be a $\mathscr{C}$-linear map. We show that if $\Delta(X)Y+X\Delta(Y)+\Delta(Y)X+Y\Delta(X)=0$ whenever $X,Y\in\mathscr{T}$ are such that $XY=YX=0$, then $\Delta(X)=D(X)+\alpha(X)+X\Delta(I)$, where $D:\mathscr{T}\to\mathscr{M}$ is a derivation, $\alpha:\mathscr{T}\to\mathscr{M}$ is an antiderivation, I is the identity matrix and $\Delta(I)X=X\Delta(I)$ for all $X\in\mathscr{T}$. We also prove that under some sufficient conditions on $\mathscr{T}$, we have $\alpha=0$. As a corollary, we show that under given sufficient conditions, each Jordan derivation $\Delta:\mathscr{T}\to\mathscr{M}$ is a derivation and this is an answer to the question raised in [9]. Some previous results are also generalized by our conclusions.

1. Introduction

In this paper, $\mathscr{C}$ will denote a commutative ring with unity and all algebras and modules will be unital over $\mathscr{C}$. Let $\mathscr{A}$ be an algebra. Recall that a $\mathscr{C}$-linear map Δ from $\mathscr{A}$ into an $\mathscr{A}$-bimodule $\mathscr{M}$ is a *Jordan derivation* if $\Delta(xy+yx)=\Delta(x)y+x\Delta(y)+\Delta(y)x+y\Delta(x)$ for all $x,y\in\mathscr{A}$. It is called a *derivation* if $\Delta(xy)=\Delta(x)y+x\Delta(y)$ for all $x,y\in\mathscr{A}$. Also, Δ is called an *antiderivation* if $\Delta(xy)=\Delta(y)x+y\Delta(x)$ for all $x,y\in\mathscr{A}$. If Δ is only additive, we say that Δ is an *additive (Jordan, anti) derivation*. For an element $m\in\mathscr{M}$, the mapping $I_m:\mathscr{A}\to\mathscr{M}$ given by $I_m(x)=xm-mx$ is a derivation which will be called an *inner derivation*. Clearly, each derivation or antiderivation is a Jordan derivation. The converse is, in general, not true (see [3]). The question of determining the structure of Jordan derivations and the conditions under which each Jordan derivation becomes a derivation attracted much attention of mathematicians. Herstein [12] proved that every additive Jordan derivation on a prime ring whose characteristic is not 2 is an additive derivation. Brešar [5] proved that Herstein's result is true for 2-torsion free semiprime rings. Sinclair [20] proved that every continuous Jordan derivation on a semisimple Banach algebra is a derivation. Johnson showed in [17] that any continuous Jordan derivation from a C^*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule is a derivation. Further, Jordan derivations were studied on other operator algebras (see [19, 21]). By a classical result of Jacobson and Rickart [14]

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every additive Jordan derivation on a full matrix ring over a 2-torsion free unital ring is an additive derivation. The first author has proved in [8] that any additive Jordan derivation from a full matrix ring over a unital ring into any 2-torsion free bimodule (not necessarily unital) is an additive derivation which is a generalization of a result in [14]. Also, in [4, 18], Jordan derivations of some rings (algebras) have been studied that these algebras are generalizations of full matrix rings (algebras). Zhang and Yu [22] showed that every Jordan derivation of triangular algebras is a derivation, and in [6] their result was generalized to trivial extensions. In [11], the author studied the Jordan derivations on a subring $\mathscr{S}$ of a full matrix ring that contains the all upper triangular matrices over a 2-torsion free ring and showed that in this case every Jordan derivation on $\mathscr{S}$ can be uniquely represented as the sum of a derivation and a special Jordan derivation. Benkovič [3] determined Jordan derivations on triangular matrices over commutative rings and proved that every Jordan derivation from the algebra of all upper triangular matrices over a commutative ring into an arbitrary unital bimodule over this algebra is the sum of a derivation and an antiderivation. In [9] the author generalized the main result of [3] to block upper triangular matrix algebras. In particular, it is shown that any Jordan derivation from the block upper triangular matrix algebra $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k) \subseteq M_n(\mathscr{C})$ into a 2-torsion free unital $\mathscr{T}$-bimodule is the sum of a derivation and an antiderivation, where $\mathscr{C}$ is a commutative ring; and at the end of the article, the following question raised: under what conditions each antiderivation of $\mathscr{T}$ is zero? In this paper we answer this question under some mild conditions.

There are many papers concerning the study of conditions under which Jordan derivations of rings or algebras can be completely determined by the action on some sets of points. We refer the reader to [1, 10, 13, 15, 16] and the references therein. In this paper we consider the subsequent condition on a $\mathscr{C}$-linear map Δ from an algebra $\mathscr{A}$ into an $\mathscr{A}$-bimodule $\mathscr{M}$:

$$x, y \in \mathscr{A}, \quad xy = yx = 0 \Rightarrow \Delta(x)y + x\Delta(y) + \Delta(y)x + y\Delta(x) = 0 \quad (\mathbb{P}).$$

It is clear that each Jordan derivation satisfies $(\mathbb{P})$, so the problem of determining the structure of maps satisfying $(\mathbb{P})$ is a generalization of the problem of determining the structure of Jordan derivations. In [1] the authors considered the Condition $(\mathbb{P})$ on a continuous linear map Δ from a C^*-algebra $\mathscr{A}$ into an essential Banach $\mathscr{A}$-bimodule $\mathscr{M}$, and they showed that there exist a derivation $D : \mathscr{A} \to \mathscr{M}$ and a bimodule homomorphism $\phi : \mathscr{A} \to \mathscr{M}$ such that $\Delta = D + \phi$. In [8], the author considered an additive map Δ from $M_n(\mathscr{R})$, the ring of all $n \times n$ matrices over a unital ring $\mathscr{R}$, into a 2-torsion free unital $M_n(\mathscr{R})$-bimodule $\mathscr{M}$ which satisfies $(\mathbb{P})$ and showed that $\Delta(X) = D(X) + X\Delta(I)$, where $D : M_n(\mathscr{R}) \to \mathscr{M}$ is a derivation, I is the identity matrix and $\Delta(I)X = X\Delta(I)$ for all $X \in M_n(\mathscr{R})$. In [10] additive maps satisfying $(\mathbb{P})$ on a triangular ring (of course, in a more general sense) are studied, and in [13] additive maps satisfying $(\mathbb{P})$ on a generalized matrix ring are considered. Also, in [7] (continuous) linear maps satisfying $(\mathbb{P})$ on some operator algebras are investigated. Note that each of the following conditions on a linear (or additive) map $\Delta : \mathscr{A} \to \mathscr{M}$ implies $(\mathbb{P})$ which have been considered by a number of authors (see, for instance, [2, 7, 13, 15, 16, 23] and the references therein):

$$x,y\in\mathscr{A},\quad xy=0\Rightarrow\Delta(x)y+x\Delta(y)=0;$$

$$x,y\in\mathscr{A},\quad xy=yx=0\Rightarrow\Delta(x)y+x\Delta(y)=0;$$

$$x,y\in\mathscr{A},\quad xy=0\Rightarrow\Delta(x)y+x\Delta(y)+\Delta(y)x+y\Delta(x)=0;$$

$$x,y\in\mathscr{A},\quad xy+yx=0\Rightarrow\Delta(x)y+x\Delta(y)+\Delta(y)x+y\Delta(x)=0.$$

Therefore, the results obtained for maps satisfying $(\mathbb{P})$ still hold if any of the above conditions is replaced by $(\mathbb{P})$.

In this paper we consider the problem of characterizing a $\mathscr{C}$-linear map Δ satisfying $(\mathbb{P})$ from $\mathscr{T}=\mathscr{T}(n_1,n_2,\cdots,n_k)$, a block upper triangular matrix algebra, into a 2-torsion free unital $\mathscr{T}$-bimodule $\mathscr{M}$. In Theorem 3.2, we show that there exists a unique derivation $D:\mathscr{T}\to\mathscr{M}$ and a unique antiderivation $\alpha:\mathscr{T}\to\mathscr{M}$ such that $\Delta(X)=D(X)+\alpha(X)+\Delta(I)X$ and $\Delta(I)X=X\Delta(I)$ for each $X\in\mathscr{T}$, where α has a certain property. This result generalizes [9, Theorem 3.2]. Also, Corollary 3.3 (and hence Theorem 3.2) is a generalization of [3, Theorem 1.1]. In Theorem 3.2, it is not necessarily true that $\alpha=0$ and this theorem doesn't determine when $\alpha=0$. In Theorem 3.6, we apply some sufficient conditions to the block upper trangular matrix algebra $\mathscr{T}=\mathscr{T}(n_1,n_2,\cdots,n_k)$, so that if the $\mathscr{C}$-linear map $\Delta:\mathscr{T}\to\mathscr{M}$ satisfies $(\mathbb{P})$ and $\mathscr{M}$ is a 2-torsion free unital $\mathscr{T}$-bimodule, then there exists a derivation $D:\mathscr{T}\to\mathscr{M}$ such that $\Delta(X)=D(X)+\Delta(I)X$, where $X\Delta(I)=\Delta(I)X$ for each $X\in\mathscr{T}$. Corollary 3.7 expresses the conditions under which each Jordan derivation $\Delta:\mathscr{T}\to\mathscr{M}$ is a derivation. Therefore, Corollary 3.7 is an answer to the question posed in [9].

2. Preliminaries

We denote the algebra of all $n\times n$ matrices over $\mathscr{C}$ by $M_n(\mathscr{C})$ $(n\geqslant 1)$, the subalgebra of all upper triangular matrices by $T_n(\mathscr{C})$, and the subalgebra of all diagonal matrices by $D_n(\mathscr{C})$. Let $n\geqslant 1$ and assume that $n=n_1+n_2+\cdots+n_k$, where $n_1,n_2,\cdots,n_k$ $(k\geqslant 1)$ is a finite sequence of positive integers. The *block upper triangular matrix algebra* $\mathscr{T}=\mathscr{T}(n_1,n_2,\cdots,n_k)$ is a subalgebra of $M_n(\mathscr{C})$ of all matrices of the form

$$X=\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1k}\\ 0 & X_{22} & \cdots & X_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & X_{kk}\end{bmatrix},$$

where X_{ij} is an $n_i\times n_j$ matrix over $\mathscr{C}$. Also, k is called the *number of summands* of $\mathscr{T}(n_1,n_2,\cdots,n_k)$. Note that if $k=1$ and $n_1=n$, then $M_n(\mathscr{C})=\mathscr{T}(n_1)$ is a block upper triangular matrix algebra. Also, when $k=n$ and $n_i=1$ for each $1\leqslant i\leqslant k$, then $\mathscr{T}(n_1,n_2,\cdots,n_k)=T_n(\mathscr{C})$.

We shall denote the identity matrix of $M_n(\mathscr{C})$ by I. Also, E_{ij} is the usual matrix unit and $x_{i,j}$ is the (ij)th entry of $X\in M_n(\mathscr{C})$ for $1\leqslant i,j\leqslant n$. Hence $E_{ij}XE_{jj}=x_{i,j}E_{ij}$ for $X\in M_n(\mathscr{C})$ and $1\leqslant i,j\leqslant n$.

Suppose that $F_1 = \sum_{i=1}^{n_1} E_i$ and $F_j = \sum_{i=1}^{n_j} E_{i+n_1+\cdots+n_{j-1}}$ for $2 \leqslant j \leqslant k$, where $E_l = E_{ll}$. Then $\{F_1, \cdots, F_k\}$ is a set of nontrivial idempotents of $\mathscr{T}(n_1, n_2, \cdots, n_k)$ such that $F_1 + \cdots + F_k = I$ and $F_iF_j = F_jF_i = 0$ for $1 \leqslant i, j \leqslant k$ with $i \neq j$. Moreover, we have $F_j\mathscr{T}(n_1, n_2, \cdots, n_k)F_j \cong M_{n_j}(\mathscr{C})$ for any $1 \leqslant j \leqslant k$. We use $\mathscr{D}(n_1, n_2, \cdots, n_k)$ for the subalgebra of $\mathscr{T}(n_1, n_2, \cdots, n_k)$ defined by

$$\mathscr{D}(n_1, n_2, \cdots, n_k) = F_1\mathscr{T}(n_1, n_2, \cdots, n_k)F_1 + \cdots + F_k\mathscr{T}(n_1, n_2, \cdots, n_k)F_k.$$

Note that, if $\mathscr{T}(n_1, n_2, \cdots, n_k) = T_n(\mathscr{C})$, then $\mathscr{D}(n_1, n_2, \cdots, n_k) = D_n(\mathscr{C})$.

By $[X, Y] = XY - YX$ we denote the commutator or the Lie product of the elements $X, Y \in M_n(\mathscr{C})$.

3. Main results

Let $\mathscr{M}$ be a unital $\mathscr{C}$-module. Given that $cm = mc$ ($c \in \mathscr{C}, m \in \mathscr{M}$), the zero map is the only linear derivation from $\mathscr{C}$ into $\mathscr{M}$ and each $\mathscr{C}$-linear mapping $T : \mathscr{C} \to \mathscr{M}$ is as follows: $T(c) = cT(1) = T(1)c$, where 1 is the unity of $\mathscr{C}$ and $c \in \mathscr{C}$, it follows that for each $\mathscr{C}$-linear mapping $\Delta : \mathscr{C} \to \mathscr{M}$ satisfying $(\mathbb{P})$ we have $\Delta(c) = \delta(c) + \Delta(1)c$ and $c\Delta(1) = \Delta(1)c$ for all $c \in \mathscr{C}$, where δ is the zero derivation. In view of this fact and [8, Theorem 2.1], we have the following lemma which will be needed in the proofs of our results.

LEMMA 3.1. *Let $M_n(\mathscr{C})$, for $n \geqslant 1$, be the algebra of all $n \times n$ matrices over $\mathscr{C}$ and $\mathscr{M}$ be a 2-torsion free unital $M_n(\mathscr{C})$-bimodule. Let $\Delta : M_n(\mathscr{C}) \to \mathscr{M}$ be a $\mathscr{C}$-linear map satisfying $(\mathbb{P})$. Then there exists a derivation $\delta : M_n(\mathscr{C}) \to \mathscr{M}$ such that $\Delta(X) = \delta(X) + X\Delta(I)$ and $\Delta(I)X = X\Delta(I)$ for each $X \in M_n(\mathscr{C})$.*

The following theorem is one of the main results of this paper.

THEOREM 3.2. *Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k)$ be the block upper triangular algebra in $M_n(\mathscr{C})$ $(n \geqslant 1)$ and $\mathscr{M}$ be a 2-torsion free unital $\mathscr{T}$-bimodule. Let $\Delta : \mathscr{T} \to \mathscr{M}$ be a $\mathscr{C}$-linear map satisfying $(\mathbb{P})$. Then there exist a derivation $D : \mathscr{T} \to \mathscr{M}$ and an antiderivation $\alpha : \mathscr{T} \to \mathscr{M}$ such that $\Delta(X) = D(X) + \alpha(X) + \Delta(I)X$ and $\Delta(I)X = X\Delta(I)$ for each $X \in \mathscr{T}$, and $\alpha(\mathscr{D}(n_1, n_2, \cdots, n_k)) = \{0\}$. Moreover, D and α are uniquely determined.*

Proof. We prove by induction on k, the number of summands of $\mathscr{T}$. When $k = 1$, $\mathscr{T} = M_n(\mathscr{C})$ and $\mathscr{D}(n_1) = M_n(\mathscr{C})$. By Lemma 3.1, there exists a derivation $D : M_n(\mathscr{C}) \to \mathscr{M}$ such that $\Delta(X) = D(X) + \Delta(I)X$ and $\Delta(I)X = X\Delta(I)$ for each $X \in M_n(\mathscr{C})$. In this case $\alpha = 0$ is the only antiderivation such that $\alpha(\mathscr{D}(n_1)) = \{0\}$. Hereon the result is correct.

Assume inductively that $k \geqslant 1$ and the result holds for each block upper triangular algebra $\mathscr{T}(n_1, n_2, \cdots, n_k)$ with k summands.

Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_{k+1}) \subseteq M_n(\mathscr{C})$ be a block upper triangular algebra with $k+1$ summands. Set $P = F_1$ and $Q = I - P = F_2 + \cdots + F_{k+1}$. Then P and Q are nontrivial idempotents of $\mathscr{T}$ such that $PQ = QP = 0$. Also, $Q\mathscr{T}P = \{0\}$, $P\mathscr{T}P$ and $Q\mathscr{T}Q$ are subalgebras of $\mathscr{T}$ with unity P and Q, respectively, and we have the decomposition $\mathscr{T} = P\mathscr{T}P \dotplus P\mathscr{T}Q \dotplus Q\mathscr{T}Q$ as a sum of $\mathscr{C}$-linear spaces. Moreover, $P\mathscr{T}P \cong M_{n_1}(\mathscr{C})$ and $Q\mathscr{T}Q \cong \mathscr{T}(n_2,n_3,\cdots,n_{k+1}) \subseteq M_{n-n_1}(\mathscr{C})$ ($\mathscr{C}$-algebra isomorphisms) is a block upper triangular algebra with k summands, where $\mathscr{D}(n_2,\cdots,n_{k+1}) \cong F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}$.

Suppose $\mathscr{M}$ is a 2-torsion free unital $\mathscr{T}$-bimodule and $\Delta : \mathscr{T} \to \mathscr{M}$ is a $\mathscr{C}$-linear map satisfying $(\mathbb{P})$. Define $\Lambda : \mathscr{T} \to \mathscr{M}$ by $\Lambda(X) = \Delta(X) - I_B(X)$, where $B = P\Delta(P)Q - Q\Delta(P)P$. So Λ is a $\mathscr{C}$-linear map which satisfies $(\mathbb{P})$ and $P\Lambda(P)Q = Q\Lambda(P)P = 0$. Also, $\Lambda(I) = \Delta(I)$. We establish the theorem for Λ.

The proof will proceed in several steps:

Step 1. $\Lambda(PXP) = P\Lambda(PXP)P$ and $\Lambda(QXQ) = Q\Lambda(QXQ)Q$ for all $X \in \mathscr{T}$.

Let $X \in \mathscr{T}$. Since $P(QXQ) = (QXQ)P = 0$, we have

$$\Lambda(P)QXQ + P\Lambda(QXQ) + \Lambda(QXQ)P + QXQ\Lambda(P) = 0. \tag{3.1}$$

Multiplying this identity by P on both sides, we have $2P\Lambda(QXQ)P = 0$. So $P\Lambda(QXQ)P = 0$. Multiplying (3.1) on the left by P and on the right by Q, and using the fact that $P\Lambda(P)Q = 0$, we arrive at $P\Lambda(QXQ)Q = 0$. Similarly, from (3.1) and the identity $Q\Lambda(P)P = 0$, we see that $Q\Lambda(QXQ)P = 0$. Therefore, from above conclusions we arrive at

$$\Lambda(QXQ) = Q\Lambda(QXQ)Q$$

for all $X \in \mathscr{T}$. Applying Λ to $(PXP)Q = Q(PXP) = 0$ we obtain

$$\Lambda(PXP)Q + PXP\Lambda(Q) + \Lambda(Q)(PXP) + X\Lambda(PXP) = 0. \tag{3.2}$$

From the identity $\Lambda(QXQ) = Q\Lambda(QXQ)Q$, (3.2) and using a similar method as above we get

$$\Lambda(PXP) = P\Lambda(PXP)P$$

for all $X \in \mathscr{T}$.

Step 2. $\Lambda(PXQ) = P\Lambda(PXQ)Q + Q\Lambda(PXQ)P$ for all $X \in \mathscr{T}$.

For all $X,Y \in \mathscr{T}$ we have $(PXQ)(PYQ) = (PYQ)(PXQ) = 0$. Applying Λ to this identity, we find that

$$\Lambda(PXQ)PYQ + PXQ\Lambda(PYQ) + \Lambda(PYQ)PXQ + PYQ\Lambda(PXQ) = 0.$$

Multiplying this equation on both sides by P and by Q, we get respectively

$$PXQ\Lambda(PYQ)P + PYQ\Lambda(PXQ)P = 0, \tag{3.3}$$

and

$$Q\Lambda(PXQ)PYQ + Q\Lambda(PYQ)PXQ = 0. \tag{3.4}$$

We have $(PXP+PXPYQ)(Q-PYQ)=(Q-PYQ)(PXP+PXPYQ)=0$ and so

$$\begin{aligned}&\Lambda(PXP+PXPYQ)(Q-PYQ)\\&+(PXP+PXPYQ)\Lambda(Q-PYQ)\\&+\Lambda(Q-PYQ)(PXP+PXPYQ)\\&+(Q-PYQ)\Lambda(PXP+PXPYQ)=0\end{aligned} \tag{3.5}$$

for all $X,Y\in\mathscr{T}$. Multiplying (3.5) by P on both sides, replacing X by P and then using Step 1 and (3.3), we get

$$P\Lambda(PYQ)P=0$$

for all $Y\in\mathscr{T}$. Also, $(QXQ+PYQXQ)(P-PYQ)=(P-PYQ)(QXQ+PYQXQ)=0$ for all $X,Y\in\mathscr{T}$. Applying Λ to this identity, we find that

$$\begin{aligned}&\Lambda(QXQ+PYQXQ)(P-PYQ)\\&+(QXQ+PYQXQ)\Lambda(P-PYQ)\\&+\Lambda(P-PYQ)(QXQ+PYQXQ)\\&+(P-PYQ)\Lambda(QXQ+PYQXQ)=0\end{aligned} \tag{3.6}$$

for all $X,Y\in\mathscr{T}$. Multiplying (3.6) by Q on both sides, replacing X by Q and then using Step 1 and (3.4), we arrive at

$$Q\Lambda(PYQ)Q=0$$

for all $Y\in\mathscr{T}$. Now from previous equations it follows that

$$\Lambda(PXQ)=P\Lambda(PXQ)Q+Q\Lambda(PXQ)P$$

for all $X\in\mathscr{T}$.

Step 3.

$$\begin{aligned}P\Lambda(PXPYQ)Q=PXP\Lambda(PYQ)Q+P\Lambda(PXP)PYQ\\-PXPYQ\Lambda(Q)Q,\end{aligned} \tag{3.7}$$

$$\begin{aligned}P\Lambda(PYQXQ)Q=PYQ\Lambda(QXQ)Q+P\Lambda(PYQ)QXQ\\-P\Lambda(P)PYQXQ,\end{aligned} \tag{3.8}$$

and

$$P\Lambda(P)PYQ=PYQ\Lambda(Q)Q \tag{3.9}$$

for all $X,Y\in\mathscr{T}$.

Multiplying (3.5) by P on the left and by Q on the right and using Steps 1 and 2, we get (3.7). Multiplying (3.6) by P on the left and by Q on the right and using Steps

2 and 3, we obtain (3.8). Replacing X by P in (3.7) we find (3.9).

Step 4.
$$PXQ\Lambda(PYQ)P=0 \quad \text{and} \quad Q\Lambda(PXQ)PYQ=0$$
for all $X,Y\in\mathscr{T}$.

Multiplying (3.5) and (3.6) on the left by Q and on the right by P, by Step 1, for all $X,Y\in\mathscr{T}$ we have

$$\begin{aligned} Q\Lambda(PXPYQ)P &= Q\Lambda(PYQ)PXP;\\ Q\Lambda(PXQYQ)P &= QYQ\Lambda(PXQ)P. \end{aligned} \tag{3.10}$$

Let $1\leqslant i,\ k\leqslant n_1$ and $n_1\leqslant j,\ l\leqslant n$ be arbitrary. By (3.10) and (3.3), we have

$$\begin{aligned} E_{ij}\Lambda(E_{kl})P &= E_{ij}\Lambda(E_{ki}E_{il})P = E_{ij}\Lambda(E_{il})E_{ki}\\ &= E_{ij}\Lambda(E_{ij}E_{jl})E_{ki} = E_{ij}E_{jl}\Lambda(E_{ij})E_{ki} = E_{il}\Lambda(E_{ij})E_{ki}\\ &= -E_{ij}\Lambda(E_{il})E_{ki} = -E_{ij}\Lambda(E_{ki}E_{il})P = -E_{ij}\Lambda(E_{kl})P, \end{aligned}$$

since $E_{ki}\in P\mathscr{T}P$, E_{ij}, E_{il}, $E_{kl}\in P\mathscr{T}Q$, and $E_{jl}\in Q\mathscr{T}Q$. So $E_{ij}\Lambda(E_{kl})P=0$. Also, by (3.10) and (3.4) we find that

$$\begin{aligned} Q\Lambda(E_{ij})E_{kl} &= Q\Lambda(E_{il}E_{lj})E_{kl} = E_{lj}\Lambda(E_{il})E_{kl}\\ &= E_{lj}\Lambda(E_{ik}E_{kl})E_{kl} = E_{lj}\Lambda(E_{kl})E_{ik}E_{kl} = E_{lj}\Lambda(E_{kl})E_{il}\\ &= -E_{lj}\Lambda(E_{il})E_{kl} = -Q\Lambda(E_{il}E_{lj})E_{kl} = -Q\Lambda(E_{ij})E_{kl}, \end{aligned}$$

since $E_{ik}\in P\mathscr{T}P$, E_{ij}, E_{il}, $E_{kl}\in P\mathscr{T}Q$, and $E_{lj}\in Q\mathscr{T}Q$. Hence,

$$Q\Lambda(E_{ij})E_{kl}=0.$$

For any $X,Y\in\mathscr{T}$, let

$$PXQ=\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}x_{i,j}E_{ij}$$

and

$$PYQ=\sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n}y_{k,l}E_{kl}.$$

From the equalities

$$E_{ij}\Lambda(E_{kl})P=0,\quad Q\Lambda(E_{ij})E_{kl}=0$$

and linearity of Λ, it follows that

$$\begin{aligned} PXQ\Lambda(PYQ)P &= \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}x_{i,j}E_{ij}\Lambda\left(\sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n}y_{k,l}E_{kl}\right)P\\ &= \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}\sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n}x_{i,j}y_{k,l}E_{ij}\Lambda(E_{kl})P=0, \end{aligned}$$

and

$$\begin{aligned} Q\Lambda(PXQ)PYQ &= \sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n} Q\Lambda\left(\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} x_{i,j}E_{ij}\right) y_{k,l}E_{kl} \\ &= \sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} x_{i,j}y_{k,l}Q\Lambda(E_{ij})E_{kl} = 0. \end{aligned}$$

Step 5.

$$\begin{aligned} P\Lambda(PXPYP)P = PXP\Lambda(PYP)P + P\Lambda(PXP)PYP \\ - PXPYP\Lambda(P)P, \end{aligned}$$

and

$$P\Lambda(P)PXP = PXP\Lambda(P)P$$

for all $X,Y \in \mathscr{T}$.

Define $J : P\mathscr{T}P \to P\mathscr{M}P$ by $J(PXP) = P\Lambda(PXP)P$. Clearly J is a well-defined $\mathscr{C}$-linear map. If $PXPYP = PYPXP = 0$ $(X,Y \in \mathscr{T})$, by hypothesis and Step 1, it follows that

$$PXPJ(PYP) + J(PXP)PYP + PYPJ(PXP) + J(PYP)PXP = 0.$$

So J satisfies $(\mathbb{P})$. Also, $P\mathscr{M}P$ is a 2-torsion free unital $P\mathscr{T}P$-bimodule. By Lemma 3.1 and the facts that $P\mathscr{T}P \cong M_{n_1}(\mathscr{C})$ and $P = F_1$ is the identity element of this algebra, there exists a derivation $\delta : P\mathscr{T}P \to P\mathscr{M}P$ such that $J(PXP) = \delta(PXP) + PXPJ(P)$ and $J(P)PXP = PXPJ(P)$ for each $X \in \mathscr{T}$. So, we have

$$\begin{aligned} J(PXPYP) &= \delta(PXPYP) + PXPYPJ(P) \\ &= \delta(PXP)PYP + PXP\delta(PYP) + PXPYPJ(P) \\ &= (J(PXP) - PXPJ(P))PYP \\ &\quad + PXP(J(PYP) - PYPJ(P)) \\ &\quad + PXPYPJ(P) \\ &= J(PXP)PYP + PXPJ(PYP) - PXPYPJ(P), \end{aligned}$$

for all $X,Y \in \mathscr{T}$. Now using the definition of J and the equality $J(P) = P\Lambda(P)P$, we conclude Step 5.

Step 6. There exist a derivation $g : Q\mathscr{T}Q \to Q\mathscr{M}Q$ and an antiderivation $\gamma : Q\mathscr{T}Q \to Q\mathscr{M}Q$ such that

$$Q\Lambda(QXQ)Q = g(QXQ) + \gamma(QXQ) + QXQ\Lambda(Q)Q,$$

and

$$QXQ\Lambda(Q)Q = Q\Lambda(Q)QXQ$$

for all $X \in \mathscr{T}$. Moreover, $\gamma(F_2\mathscr{T}F_2+\cdots+F_{k+1}\mathscr{T}F_{k+1})=\{0\}$, and

$$PXQ\gamma(QYQ)=0$$

for all $X,Y \in \mathscr{T}$.

QMQ is a 2-torsion free unital $Q\mathscr{T}Q$-bimodule. Define $G: Q\mathscr{T}Q \to Q\mathscr{M}Q$ by $G(QXQ)=Q\Lambda(QXQ)Q$. Clearly G is a well-defined $\mathscr{C}$-linear map. Let $QXQYQ = QYQXQ = 0$ $(X,Y \in \mathscr{T})$. From hypothesis and the definition of G, we see that

$$QXQG(QYQ)+G(QXQ)QYQ+QYQG(QXQ)+G(QYQ)QXQ=0.$$

Hence, G satisfies $(\mathbb{P})$. In view of the isomorphisms

$$Q\mathscr{T}Q \cong \mathscr{T}(n_2,n_3,\cdots,n_{k+1}) \subseteq M_{n-n_1}(\mathscr{C}),$$

$$\mathscr{D}(n_2,\ldots,n_{k+1}) \cong F_2\mathscr{T}F_2+\cdots+F_{k+1}\mathscr{T}F_{k+1}$$

and induction hypothesis, there exist a derivation $g: Q\mathscr{T}Q \to Q\mathscr{M}Q$ and an antiderivation $\gamma: Q\mathscr{T}Q \to Q\mathscr{M}Q$ such that $Q\Lambda(QXQ)Q = G(QXQ) = g(QXQ)+\gamma(QXQ)+QXQ\Lambda(Q)Q$ and $QXQ\Lambda(Q)Q = Q\Lambda(Q)QXQ$ for all $X \in \mathscr{T}$, and $\gamma(\mathscr{D}(n_2,\ldots,n_{k+1})) = \gamma(F_2\mathscr{T}F_2+\cdots+F_{k+1}\mathscr{T}F_{k+1}) = \{0\}$. We will show that $PXQ\gamma(QYQ)=0$ for all $X,Y \in \mathscr{T}$.

By (3.8) and (3.9), for all $X,Y,Z \in \mathscr{T}$ we have

$$\begin{aligned}P\Lambda(PXQYQZQ)Q &= P\Lambda((PXQ)(QYQZQ))Q\\ &= PXQ\Lambda(QYQZQ)Q+P\Lambda(PXQ)QYQZQ\\ &\quad -PXQYQZQ\Lambda(Q)Q.\end{aligned}$$

Replace X by YQZ in $Q\Lambda(QXQ)Q = g(QXQ)+\gamma(QXQ)+QXQ\Lambda(Q)Q$ and then multiply it by PXQ on the left. From above conclusion we obtain

$$\begin{aligned}P\Lambda(PXQYQZQ)Q &= PXQg(QYQZQ)+PXQ\gamma(QYQZQ)\\ &\quad +PXQYQZQ\Lambda(Q)Q+P\Lambda(PXQ)QYQZQ\\ &\quad -PXQYQZQ\Lambda(Q)Q\\ &= PXQg(QYQZQ)+PXQ\gamma(QYQZQ)\\ &\quad +P\Lambda(PXQ)QYQZQ.\end{aligned}$$

On the other hand, by (3.8) and (3.9), for all $X,Y,Z \in \mathscr{T}$ we have

$$\begin{aligned}P\Lambda(PXQYQZQ)Q &= P\Lambda((PXQYQ)(QZQ))Q\\ &= PXQYQ\Lambda(QZQ)Q+P\Lambda(PXQYQ)QZQ \qquad (3.11)\\ &\quad -PXQYQZQ\Lambda(Q)Q\end{aligned}$$

Using again (3.8) for $P\Lambda(PXQYQ)Q$ in the last equation and then replacing $Q\Lambda(QYQ)Q$ by $g(QYQ)+\gamma(QYQ)+QYQ\Lambda(Q)Q$ for all $Y\in\mathscr{T}$, we arrive at

$$\begin{aligned} P\Lambda(PXQYQZQ)Q &= PXQYQ\Lambda(QZQ)Q+P\Lambda(PXQYQ)QZQ \\ &\quad -PXQYQZQ\Lambda(Q)Q \\ &= PXQYQ\Lambda(QZQ)Q+PXQ\Lambda(QYQ)QZQ \\ &\quad +P\Lambda(PXQ)QYQZQ-PXQYQ\Lambda(Q)QZQ \\ &\quad -PXQYQZQ\Lambda(Q)Q \\ &= PXQYQg(QZQ)Q+PXQYQ\gamma(QZQ) \\ &\quad +PXQYQZQ\Lambda(Q)Q+PXQg(QYQ)QZQ \\ &\quad +PXQ\gamma(QYQ)QZQ+PXQYQ\Lambda(Q)QZQ \\ &\quad +P\Lambda(PXQ)QYQZQ-PXQYQ\Lambda(Q)QZQ \\ &\quad -PXQYQZQ\Lambda(Q)Q. \end{aligned} \tag{3.12}$$

Comparing the two expressions (3.11) and (3.12) for $P\Lambda(PXQYQZQ)Q$, using Step 3, the equality $Q\Lambda(Q)QXQ=QXQ\Lambda(Q)Q$ ($X\in\mathscr{T}$) and the facts that g is a derivation and γ is an antiderivation, we arrive at

$$PXQ\gamma([QYQ,QZQ])=0 \tag{3.13}$$

for all $X,Y,Z\in\mathscr{T}$. Now from the facts that $Q=F_2+\cdots+F_{k+1}$ and $F_jQ=QF_j=F_j$ for all $2\leqslant j\leqslant k+1$, we have

$$\begin{aligned} QXQ-\sum_{j=2}^{k+1}F_jXF_j &= \left(\sum_{j=2}^{k+1}F_j\right)QXQ-\sum_{j=2}^{k+1}F_jXF_j \\ &= \sum_{j=2}^{k+1}(F_jXQ-F_jXF_j) \\ &= \sum_{j=2}^{k+1}F_jX(Q-F_j) \\ &= \sum_{j=2}^{k+1}[F_j,F_jX(Q-F_j)] \end{aligned} \tag{3.14}$$

for all $X\in\mathscr{T}$. Note that $F_j,F_jX(Q-F_j)\in Q\mathscr{T}Q$. By (3.13), (3.14) and $\gamma(F_2\mathscr{T}F_2+\cdots+F_{k+1}\mathscr{T}F_{k+1})=0$, we get

$$\begin{aligned} PXQ\gamma(QYQ) &= PXQ\gamma\left(QYQ-\sum_{j=2}^{k+1}F_jYF_j+\sum_{j=2}^{k+1}F_jYF_j\right) \\ &= PXQ\gamma\left(QYQ-\sum_{j=2}^{k+1}F_jYF_j\right) \\ &= \sum_{j=2}^{k+1}PXQ\gamma([F_j,F_jY(Q-F_j)])=0 \end{aligned}$$

for all $X,Y \in \mathscr{T}$.

Step 7. $\Lambda(I)X = X\Lambda(I)$ for all $X \in \mathscr{T}$.

By Step 1, we have

$$\Lambda(I) = P\Lambda(P)P + Q\Lambda(Q)Q.$$

By Steps 5, 6 and (3.9), we arrive at

$$\begin{aligned}\Lambda(I)X &= (P\Lambda(P)P + Q\Lambda(Q)Q)(PXP + PXQ + QXQ)\\ &= P\Lambda(P)PXP + P\Lambda(P)PXQ + Q\Lambda(Q)QXQ\\ &= PXP\Lambda(P)P + PXQ\Lambda(Q)Q + QXQ\Lambda(Q)Q\\ &= X\Lambda(I)\end{aligned}$$

for all $X \in \mathscr{T}$.

Step 8. The mapping $\delta : \mathscr{T} \to \mathscr{M}$ defined by

$$\delta(X) = P\Lambda(PXP)P + P\Lambda(PXQ)Q + g(QXQ) + Q\Lambda(Q)QXQ - \Lambda(I)X$$

is a derivation and the mapping $\alpha : \mathscr{T} \to \mathscr{M}$ defined by

$$\alpha(X) = Q\Lambda(PXQ)P + \gamma(QXQ)$$

is an antiderivation such that $\alpha(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \{0\}$. Moreover,

$$\Lambda(X) = \delta(X) + \alpha(X) + \Lambda(I)X$$

for all $X \in \mathscr{T}$.

Obviously, δ is a $\mathscr{C}$-linear map. Since $Q\mathscr{T}P = \{0\}$, it follows that $PXYP = PXPYP$, $PXYQ = PXPYQ + PXQYQ$ and $QXYQ = QXQYQ$. So we have

$$\begin{aligned}\delta(XY) &= P\Lambda(PXYP)P + P\Lambda(PXYQ)Q\\ &\quad + g(QXYQ) + Q\Lambda(Q)QXYQ - \Lambda(I)XY\\ &= P\Lambda(PXPYP)P + P\Lambda(PXPYQ)Q + P\Lambda(PXQYQ)Q\\ &\quad + g(QXQYQ) + Q\Lambda(Q)QXQYQ - \Lambda(I)XY.\end{aligned}$$

Now, by Steps 3, 5 and the fact that g is a derivation, it is easy to see that

$$\begin{aligned}\delta(XY) &= PXP\Lambda(PYP)P + P\Lambda(PXP)PYP - PXPYP\Lambda(P)P\\ &\quad + PXP\Lambda(PYQ)Q + P\Lambda(PXP)PYQ - PXPYQ\Lambda(Q)Q\\ &\quad + PXQ\Lambda(QYQ)Q + P\Lambda(PXQ)QYQ - P\Lambda(P)PXQYQ\\ &\quad + g(QXQ)QYQ + QXQg(QYQ)\\ &\quad + Q\Lambda(Q)QXQYQ - \Lambda(I)XY.\end{aligned} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\delta(X)Y &= P\Lambda(PXP)PYP + P\Lambda(PXP)PYQ + P\Lambda(PXQ)QYQ \\ &\quad + g(QXQ)QYQ + Q\Lambda(Q)QXQYQ - \Lambda(I)XY.\end{aligned} \tag{3.16}$$

also, by Steps 6, 7 we have

$$\begin{aligned}X\delta(Y) &= PXP\Lambda(PYP)P + PXP\Lambda(PYQ)Q + PXQg(QYQ) \\ &\quad + QXQg(QYQ) + PXQ\Lambda(Q)QYQ + QXQ\Lambda(Q)QYQ \\ &\quad - P\Lambda(P)PXPYP - P\Lambda(P)PXPYQ - P\Lambda(P)PXQYQ \\ &\quad - Q\Lambda(Q)QXQYQ \\ &= PXP\Lambda(PYP)P + PXP\Lambda(PYQ)Q + PXQg(QYQ) \\ &\quad + QXQg(QYQ) + PXQ\Lambda(Q)QYQ - P\Lambda(P)PXPYP \\ &\quad - P\Lambda(P)PXPYQ - P\Lambda(P)PXQYQ.\end{aligned} \tag{3.17}$$

Hence, by the fact that $PXQg(QYQ) = PXQ(Q\Lambda(QYQ)Q - \gamma(QYQ) - QYQ\Lambda(Q)Q)$ and $PXQ\gamma(QYQ) = 0$ (for all $X,Y \in \mathscr{T}$) from Step 6 and comparing (3.15) to (3.16) and (3.17), we arrive at $\delta(XY) = \delta(X)Y + X\delta(Y)$. That is δ is a derivation.

It is clear that α is a linear map. On the other hand, for each $X,Y \in \mathscr{T}$, using (3.10) and since γ is an antiderivation, we have

$$\begin{aligned}\alpha(XY) &= Q\Lambda(PXPYQ)P + Q\Lambda(PXQYQ)P + \gamma(QXQYQ) \\ &= Q\Lambda(PYQ)PXP + QYQ\Lambda(PXQ)P + QYQ\gamma(QXQ) \\ &\quad + \gamma(QYQ)QXQ.\end{aligned}$$

Moreover, by Steps 4 and 6, $Q\Lambda(PYQ)PXQ = PYQ\Lambda(PXQ)P = 0$ and $PYQ\gamma(QXQ) = 0$ also, $\gamma(QYQ)QXP = 0$ by the fact that $Q\mathscr{T}P = \{0\}$. Hence, we see

$$\begin{aligned}\alpha(XY) &= Q\Lambda(PYQ)PXP + QYQ\Lambda(PXQ)P + QYQ\gamma(QXQ) \\ &\quad + \gamma(QYQ)QXQ + Q\Lambda(PYQ)PXQ + PYQ\Lambda(PXQ)P \\ &\quad + PYQ\gamma(QXQ) + \gamma(QYQ)QXP \\ &= Y\alpha(X) + \alpha(Y)X.\end{aligned}$$

Let $F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1}$ be an arbitrary element of $\mathscr{D}(n_1, n_2, \cdots, n_{k+1})$. Since $\gamma(F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}) = \{0\}$, $F_1Q = QF_1 = 0$, $PF_j = F_jP = 0$ and $F_jQ = QF_j = F_j$ for any $2 \leqslant j \leqslant k$, it follows that

$$\begin{aligned}&\alpha(F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1}) \\ =&Q\Lambda(P(F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1})Q)P \\ &+ \gamma(Q(F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1})Q) \\ =&\gamma(F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1}) = 0.\end{aligned}$$

Therefore, $\alpha(D(n_1,n_2,\cdots,n_{k+1}))=\{0\}$. Now by Steps 1, 2 and 6, for every $X\in\mathscr{T}$ we have

$$\begin{aligned}\Lambda(X)&=P\Lambda(PXP)P+P\Lambda(PXQ)Q+Q\Lambda(PXQ)P+Q\Lambda(QXQ)Q\\&=P\Lambda(PXP)P+P\Lambda(PXQ)Q+Q\Lambda(PXQ)P+g(QXQ)\\&\quad+\gamma(QXQ)+Q\Lambda(Q)QXQ-\Lambda(I)X+\Lambda(I)X\\&=\delta(X)+\alpha(X)+\Lambda(I)X.\end{aligned}$$

By the above results and the definition of Λ we obtain

$$\Delta(X)-I_B(X)=\Lambda(X)=\delta(X)+\alpha(X)+\Lambda(I)X$$

for all $X\in\mathscr{T}$, where $\delta:\mathscr{T}\to\mathscr{M}$ is a derivation, $\alpha:\mathscr{T}\to\mathscr{M}$ is an antiderivation and $\alpha(\mathscr{D}(n_1,n_2,\cdots,n_{k+1}))=\{0\}$. Also, $\Delta(I)=\Lambda(I)$ and so, $\Delta(I)X=X\Delta(I)$ for all $X\in\mathscr{T}$. Define $D:\mathscr{T}\to\mathscr{M}$ by $D(X)=\delta(X)+I_B(X)$. Hence D is a derivation and

$$\Delta(X)=D(X)+\alpha(X)+\Delta(I)X$$

for all $X\in\mathscr{T}$.

Finally, we will show that D and α are uniquely determined. Suppose that $\Delta(X)=D'(X)+\alpha'(X)+\Delta(I)X$ $(X\in\mathscr{T})$, where $D':\mathscr{T}\to\mathscr{M}$ is a derivation, $\alpha':\mathscr{T}\to\mathscr{M}$ is an antiderivation and $\alpha'(\mathscr{D}(n_1,n_2,\cdots,n_{k+1}))=\{0\}$. Hence, we have

$$D'(X)+\alpha'(X)+\Delta(I)X=D(X)+\alpha(X)+\Delta(I)X\quad(X\in\mathscr{T}).$$

So, $D'-D=\alpha'-\alpha$. Therefore, $\alpha'-\alpha:\mathscr{T}\to\mathscr{M}$ is both a derivation and an antiderivation. Hence

$$(\alpha'-\alpha)([X,Y])=0$$

for all $X,Y\in\mathscr{T}$. As in the proof of Step 6, it can be shown that

$$X-\sum_{j=1}^{k+1}F_jXF_j=\sum_{j=1}^{k+1}[F_j,F_jX(I-F_j)]\quad(k\geqslant 1,X\in\mathscr{T}).$$

Since $(\alpha'-\alpha)(\mathscr{D}(n_1,n_2,\cdots,n_{k+1}))=\{0\}$, it follows that

$$\begin{aligned}(\alpha'-\alpha)(X)&=(\alpha'-\alpha)\left(X-\sum_{j=1}^{k+1}F_jXF_j+\sum_{j=1}^{k+1}F_jXF_j\right)\\&=(\alpha'-\alpha)\left(X-\sum_{j=1}^{k+1}F_jXF_j\right)\\&=\sum_{j=1}^{k+1}(\alpha'-\alpha)([F_j,F_jX(I-F_j)])\\&=0.\end{aligned}$$

So $\alpha'=\alpha$ and hence $D'=D$. This completes the proof of the theorem.
The following corollary is immediate.

COROLLARY 3.3. *Let $T_n(\mathscr{C})$ $(n \geqslant 1)$ be an upper triangular matrix algebra and $\mathscr{M}$ be a 2-torsion free unital $T_n(\mathscr{C})$-bimodule. Let $\Delta : T_n(\mathscr{C}) \to \mathscr{M}$ be a $\mathscr{C}$-linear map satisfying $(\mathbb{P})$. Then there exist a derivation $D : T_n(\mathscr{C}) \to \mathscr{M}$ and an antiderivation $\alpha : T_n(\mathscr{C}) \to \mathscr{M}$ such that $\Delta(X) = D(X) + \alpha(X) + \Delta(I)X$ and $\alpha(D_n(\mathscr{C})) = \{0\}$, $X\Delta(I) = \Delta(I)X$ for all $X \in T_n(\mathscr{C})$. Moreover, D and α are uniquely determined.*

If $\mathscr{M}$ is a 2-torsion free unital $\mathscr{T}(n_1, n_2, \cdots, n_k)$-bimodule and $\Delta : \mathscr{T} \to \mathscr{M}$ is a Jordan derivation, then Δ satisfies $(\mathbb{P})$ and $\Delta(I) = 0$, and so we have the following corollary which is the main result of [9]. Thus Theorem 3.2 is a generalization of [9, Theorem 3.2].

COROLLARY 3.4. *Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k)$ be a block upper triangular matrix algebra in $M_n(\mathscr{C})$ $(n \geqslant 1)$ and $\mathscr{M}$ be a 2-torsion free unital $\mathscr{T}$-bimodule. Suppose that $\Delta : \mathscr{T} \to \mathscr{M}$ is a Jordan derivation. Then there exist a derivation $D : \mathscr{T} \to \mathscr{M}$ and an antiderivation $\alpha : \mathscr{T} \to \mathscr{M}$ such that $\Delta = D + \alpha$ and $\alpha(\mathscr{D}(n_1, n_2, \cdots, n_k)) = \{0\}$. Moreover, D and α are uniquely determined.*

By Corollary 3.4 (or Corollary 3.3) we have the following corollary, which is proved in [3]. So Theorem 3.2 (and Corollary 3.3) generalizes [3, Theorem 1.1].

COROLLARY 3.5. *Let $T_n(\mathscr{C})$ $(n \geqslant 1)$ be an upper triangular matrix algebra and $\mathscr{M}$ be a 2-torsion free unital $T_n(\mathscr{C})$-bimodule. Suppose that $\Delta : T_n(\mathscr{C}) \to \mathscr{M}$ is a Jordan derivation. Then there exist a derivation $D : T_n(\mathscr{C}) \to \mathscr{M}$ and an antiderivation $\alpha : T_n(\mathscr{C}) \to \mathscr{M}$ such that $\Delta = D + \alpha$, $\alpha(D_n(\mathscr{C})) = \{0\}$. Moreover, D and α are uniquely determined.*

In Theorem 3.2, it is possible that the antiderivation α be zero. But this theorem doesn't say when $\alpha = 0$. In the next theorem, we add some mild conditions to the block upper trangular matrix algebra $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k)$ so that $\alpha = 0$.

THEOREM 3.6. *Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k) \subseteq M_n(\mathscr{C})$ be a block upper trangular matrix algebra with $n \geqslant 2$. Let $n_i \geqslant 2$ for each $1 \leqslant i \leqslant k$ and $\mathscr{M}$ be a 2-torsion free unital $\mathscr{T}$-bimodule. If the $\mathscr{C}$-linear map $\Delta : \mathscr{T} \to \mathscr{M}$ satisfies $(\mathbb{P})$, then there exists a derivation $D : \mathscr{T} \to \mathscr{M}$ such that $\Delta(X) = D(X) + \Delta(I)X$, and $X\Delta(I) = \Delta(I)X$ for each $X \in \mathscr{T}$.*

Proof. The proof is by induction on k the number of summands of $\mathscr{T}$. When $k = 1$ then $\mathscr{T} = M_n(\mathscr{C})$ and the result is established by the Lemma 3.1.

Assume inductively that $k \geqslant 1$ and the result holds for any block upper triangular algebra $\mathscr{T}(n_1, n_2, \cdots, n_k)$ with k summands, where each $n_i \geqslant 2$ for $1 \leqslant i \leqslant k$. Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_{k+1})$ be a block upper triangular algebra with $k+1$ summands and each $n_i \geqslant 2$ for $1 \leqslant i \leqslant k+1$. We set $P = F_1$ and $Q = I - P = F_2 + \cdots + F_{k+1}$ as in the proof of Theorem 3.2. Let $\Delta : \mathscr{T} \to \mathscr{M}$ be a $\mathscr{C}$-linear map which satisfies $(\mathbb{P})$. Define $\Lambda : \mathscr{T} \to \mathscr{M}$ by $\Lambda(X) = \Delta(X) - I_B(X)$, where $B = P\Delta(P)Q - Q\Delta(P)P$. It is clear that Λ is a $\mathscr{C}$-linear map which satisfies $(\mathbb{P})$. Moreover, $P\Lambda(P)Q = Q\Lambda(P)P = 0$ and $\Lambda(I) = \Delta(I)$. We will show that the result is correct in this case.

All the results obtained for Λ in the Steps 1–3 and equations 3.10 of the Theorem 3.2 hold here too. We prove that $Q\Lambda(PXQ)P=0$ for all $X\in\mathscr{T}$. By (3.10), we have

$$\begin{aligned}Q\Lambda(PXPYPZQ)P &= Q\Lambda(PZQ)PXPYP\\ &= Q\Lambda(PXPZQ)PYP\\ &= Q\Lambda(PYPXPZQ)P\end{aligned}$$

for all $X,Y,Z\in\mathscr{T}$. Hence $Q\Lambda([PXP,PYP]ZQ)P=0$. So

$$Q\Lambda(PZQ)[PXP,PYP]=0$$

for all $X,Y,Z\in\mathscr{T}$. Thus

$$\begin{aligned}&Q\Lambda(PZQ)PW_1P[PXP,PYP]PW_2P\\ =&Q\Lambda(PW_1PZQ)[PXP,PYP]PW_2P=0\end{aligned}$$

for all $X,Y,Z,W_1,W_2\in\mathscr{T}$. Let $\mathscr{I}$ be the ideal generated by all commutators in $P\mathscr{T}P$; i.e., the ideal generated by $[P\mathscr{T}P,P\mathscr{T}P]$ in $P\mathscr{T}P$. Then

$$Q\Lambda(PZQ)\mathscr{I}=0$$

for all $Z\in\mathscr{T}$. Since $P\mathscr{T}P\cong M_{n_1}(\mathscr{C})$, $n_1\geqslant 2$, it follows that $\mathscr{I}=P\mathscr{T}P$. So

$$Q\Lambda(PZQ)P=0$$

for all $Z\in\mathscr{T}$.

On the other hand, by a proof similar to the proof given in the Step 5 of Theorem 3.2, we obtain

$$P\Lambda(PXPYP)P=PXP\Lambda(PYP)P+P\Lambda(PXP)PYP-PXPYP\Lambda(P)P,$$

and

$$P\Lambda(P)PXP=PXP\Lambda(P)P.$$

for all $X,Y\in\mathscr{T}$.

$Q\mathscr{M}Q$ is a 2-torsion free unital $Q\mathscr{T}Q$-bimodule. Define $G:Q\mathscr{T}Q\to Q\mathscr{M}Q$ by $G(QXQ)=Q\Lambda(QXQ)Q$. As in proof of the Step 6 of Theorem 3.2, we see that G is a well-defined $\mathscr{C}$-linear map satisfying $(\mathbb{P})$. According to the isomorphism $Q\mathscr{T}Q\cong\mathscr{T}(n_2,n_3,\cdots,n_{k+1})\subseteq M_{n-n_1}(\mathscr{C})$ and the fact that $n_i\geqslant 2$ for all $2\leqslant i\leqslant k+1$, by induction hypothesis, it follows that there exists a derivation $g:Q\mathscr{T}Q\to Q\mathscr{M}Q$ such that $G(QXQ)=g(QXQ)+QXQG(Q)$ and $QXQG(Q)=G(Q)QXQ$. By the definition of G, $G(Q)=Q\Lambda(Q)Q$. Hence we have

$$Q\Lambda(QXQ)Q=g(QXQ)+QXQ\Lambda(Q)Q,$$

and

$$Q\Lambda(Q)QXQ=QXQ\Lambda(Q)Q.$$

Moreover, by the fact that g is a derivation we have

$$Q\Lambda(QXQYQ)Q = QXQ\Lambda(QYQ)Q + Q\Lambda(QXQ)QYQ - Q\Lambda(Q)QXQYQ$$

for all $X, Y \in \mathscr{T}$.

Define the mapping $\delta : \mathscr{T} \to \mathscr{M}$ by $\delta(X) = \Lambda(X) + \Lambda(I)X$ for all $X \in \mathscr{T}$. From the above results one can check directly that δ is a derivation and $\Lambda(I)X = X\Lambda(I)$ for all $X \in \mathscr{T}$. (similar to the proof of Theorem 3.2). Now define $D : \mathscr{T} \to \mathscr{M}$ by $D(X) = \delta(X) + I_B(X)$. Clearly, D is a derivation, $\Delta(X) = D(X) + \Delta(I)X$, and $\Delta(I)X = X\Delta(I)$ for all $X \in \mathscr{T}$.

At the end of [9], it is asked under what conditions every Jordan derivation from the block upper triangular matrix algebra $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k) \subseteq M_n(\mathscr{C})$ into a 2-torsion free unital $\mathscr{T}$-bimodule is a derivation? The following corollary of Theorem 3.6 answers this question for the case when $n_i \geqslant 2$ for all $1 \leqslant i \leqslant k$.

COROLLARY 3.7. *Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k) \subseteq M_n(\mathscr{C})$ be a block upper trangular matrix algebra with $n \geqslant 2$. Let $n_i \geqslant 2$ for each $1 \leqslant i \leqslant k$ and $\mathscr{M}$ be a 2-torsion free unital $\mathscr{T}$-bimodule. Then every Jordan derivation $\Delta : \mathscr{T} \to \mathscr{M}$ is a derivation.*

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